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ON THE ROSENTHAL INEQUALITY FOR MIXING FIELDS

ПРО НЕРІВНІСТЬ РОЗЕНТАЛЯ ДЛЯ ПОЛІВ, ЩО ЗАДОВОЛЬНЯЮТЬ УМОВИ ПЕРЕМІШУВАННЯ

A proof of the Rosenthal inequality for α-mixing random fields is given. The statements and proofs are modifications of the ones presented in the papers of Doukhan and Utev.

Доведено нерівність Розенталя для випадкових полів, що задовольняють умови α-перемішування. Твердження та доведення є модифікаціями теорем та доведень, наведених в роботах Дукхана та Утєва.

1. Introduction and results. Rosenthal's inequalities are important tools to prove consistency of some estimators for weakly dependent random processes and fields (see e. g. Fazekas and Kukush [1]). The first version of such inequalities was proved in Rosenthal [2] for independent random variables. Rosenthal's inequalities for mixing sequences were presented in Utev [3] and for mixing fields in Doukhan [4]. However, Doukhan remarks that the proof of the interpolation lemma in Utev [3] is "not clear" (see Doukhan [4, p. 27]). Actually, the first inequality in the expression preceding (4.4) in Utev [3] seems to be not valid. Therefore, one can not use Lemma 4.1 of Utev, so the extension of Rosenthal's inequality from positive even integer exponents to arbitrary positive real exponents is an open problem. On the other hand, Doukhan [4] presents Rosenthal's inequalities for α-mixing and for φ-mixing fields. However, by the opinion of the authors of the present paper there is a gap in the proof of Theorem 1 in Doukhan [4, p. 29].

The aim of this paper is to give a version of Rosenthal's inequality for α-mixing fields. The results and proofs here are slight modifications of the ones in Doukhan [4] and Utev [3]. The authors want to summarize what is clear in the above mentioned papers concerning the topic. Similar considerations can be made in the φ-mixing case (see also Remark 4 in Doukhan [4, p. 32]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Random variables are supposed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras in $\mathcal{F}$. The α-mixing coefficient is defined as follows,

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup \{ |\mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(AB)| : A \in \mathcal{A},\ B \in \mathcal{B}\}.$$

The covariance inequality in the α-mixing case is the following (see, e. g. Doukhan [4, p. 9])

$$|\operatorname{cov}(X, Y)| \le 8\,[\alpha(\sigma(X), \sigma(Y))]^{1/r} \|X\|_p \|Y\|_q,$$

$$r, p, q \ge 1, \quad \frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1.$$

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Let I be the set of integer lattice points in $\mathbb{R}^d$, $d \geq 1$. $\mathbb{R}^d$ will be considered with maximum norm and the distance generated by that norm. Let $\{Y_t\colon t \in I\}$ be a set of random variables. The α-mixing coefficient of Y is

$$\alpha_Y(r, u, v) = \sup\{\alpha(\mathcal{F}_{I_1}, \mathcal{F}_{I_2})\colon$$

$$\text{distance}(I_1, I_2) \geq r, \quad \text{card}(I_1) \leq u, \quad \text{card}(I_2) \leq v\},$$

where I_1 and I_2 are finite subsets in I, $\mathcal{F}_{I_i} = \sigma\{Y_t\colon t \in I_i\}$, $i = 1, 2$.

Let T be a finite set in I. Introduce the following notation

$$L(\mu, \varepsilon, T) = \sum_{t \in T} (\mathbb{E}|Y_t|^{\mu+\varepsilon})^{\mu/(\mu+\varepsilon)} = \sum_{t \in T} \|Y_t\|_{\mu+\varepsilon}^{\mu},$$

$$D(h, \varepsilon, T) = \begin{cases} L(h, 0, T), & \text{if } 0 < h \leq 1, \ \varepsilon \geq 0; \\ L(h, \varepsilon, T), & \text{if } 1 < h \leq 2, \ \varepsilon \geq 0; \\ \max\{L(h, \varepsilon, T), [L(2, \varepsilon, T)]^{h/2}\}, & \text{if } 2 < h, \ \varepsilon \geq 0. \end{cases}$$

Let s_r and b_r denote the number of points of I in a sphere with radius r and center in I and in a ball with radius r and center in I, respectively: $s_r = \text{card}(\{t\colon \|t\| = r\} \cap I)$, $b_r = \text{card}(\{t\colon \|t\| \leq r\} \cap I)$. Let

$$c_{u,h-u}^{(\alpha)} = 8u!\,(h-u-1)!\,(h-1)! \sum_{r=1}^{\infty} [\alpha_Y(r, u, h-u)]^{\varepsilon/(h+\varepsilon)} s_r b_r^{h-2}.$$

The following theorem is a version of Theorem 1 in Doukhan [4, p. 26]. Assumptions here are stronger than those of Doukhan [4]. Explicit formulae for the constants are given.

Theorem. *Let $l > 1$ and $\varepsilon > 0$. Let Y_t, $t \in I$, be centered random variables with $\mathbb{E}|Y_t|^{l+\varepsilon} < \infty$, $t \in I$. Let h be the smallest even integer with $h \geq\geq l$. Assume that $c_{u,h-u}^{(\alpha)} < \infty$ for $u = 1, \ldots, h-1$. Then there is a constant $K_{(\alpha)}$ such that*

$$\mathbb{E}\left|\sum_{t \in T} Y_t\right|^l \leq K_{(\alpha)} D(l, \varepsilon, T), \tag{1}$$

for any finite subset T of I.

***Remark* 1.** $K_{(\alpha)}$ does not depend on T but it depends on the mixing coefficients and l: $K_{(\alpha)} = H_h^{(\alpha)} C_l$, where

$$H_h^{(\alpha)} = 1 + \sum_{u=1}^{h-1} c_{u,h-u}^{(\alpha)} + \sum_{u=2}^{h-2} \binom{h}{u} H_u^{(\alpha)} H_{h-u}^{(\alpha)},$$

$$C_l = 2^{(h-l+\varepsilon)(2h+2l-1)/\varepsilon},$$

where we suppose that $0 < \varepsilon < l/2$. If l is an even integer then one can put $C_l = 1$.

***Remark* 2.** Inequality (1) is always satisfied for $0 < l \leq 1$ if we replace $K_{(\alpha)}$ with 1.

***Remark* 3.** The above result is valid in the following slightly more general setting. If I is a regular pattern in $\mathbb{R}^d$ then s_r should be replaced by $\tilde{s}_r = \text{card}(\{t\colon r-1 <$

$< \|t\| \le r\} \cap I)$, i. e. $\tilde{s}_r$ denotes the number of points of I in a ring with radius r, thickness 1 and center in I.

***Remark* 4.** For $d = 1$, i. e. for mixing sequences see Doukhan [4, p. 26].

2. Auxiliary results and interpolation lemma.

Lemma 1. *Let L be a finite subset in a metric space (M, ρ). Suppose that the minimal distance of two non-empty complementary subsets of L is r. Then one can choose two non-empty complementary subsets A and B in L such that the distance of A and B is r and there exists a connected graph with edges not longer than r and with set of vertices A, and the same for B.*

Proof. Let $s, t \in U \subseteq L$. We shall say that s is r-connected with t in U if there exists a connected graph with edges not longer than r and with vertices in U, moreover s and t are vertices of this graph. Let S_1 and S_2 be two non-empty complementary subsets of L such that $\rho(S_1, S_2) = r$. Let $t_1 \in S_1$, $t_2 \in S_2$ such that $\rho(t_1, t_2) = r$. Let $S_i^{(1)} \subseteq S_i$ be the set of points r-connected with t_i in S_i, $i = 1, 2$. Now, $\rho\left(\{S_1^{(1)} \cup S_2^{(1)}\}, \{(S_1 - S_1^{(1)}) \cup (S_2 - S_2^{(1)})\}\right) \ge r$. But r is the maximal distance between subsets of L, therefore either the second subset is empty or the distance is r. In the first case we are ready. In the second case let $\tilde{S}_1^{(1)} \subseteq S_1 - S_1^{(1)}$ be the set of points r-connected with $S_2^{(1)}$ in $(S_1 - S_1^{(1)}) \cup S_2^{(1)}$. The definition of $\tilde{S}_2^{(1)}$ is similar. Obviously $\tilde{S}_1^{(1)} \cup \tilde{S}_2^{(1)} \ne \varnothing$. Now, consider $(S_1 - \tilde{S}_1^{(1)}) \cup \tilde{S}_2^{(1)}$ and $(S_2 - \tilde{S}_2^{(1)}) \cup \tilde{S}_1^{(1)}$. The distance of these two sets is r. Moreover, in these sets the number of points r-connected with t_1 in $(S_1 - \tilde{S}_1^{(1)}) \cup \tilde{S}_2^{(1)}$ or the number of points r-connected with t_2 in $(S_2 - \tilde{S}_2^{(1)}) \cup \tilde{S}_1^{(1)}$ is greater than at the starting situation. Repeating the above procedure we obtain the result.

The following lemma is a version of Lemma 2 in Doukhan [4, p. 29]. There the lemma is stated for even integer $(a + b)$ with $(a + b) \ge 2$.

Lemma 2. *If $\delta \ge 0$, $a \ge 2$ and $b \ge 2$ are real numbers then*

$$D(a, \delta, T) D(b, \delta, T) \le D(a + b, \delta, T).$$

The proof will be based on Hölder's inequality:

1. Let X and Y be real random variables. If $p > 1$ and $q = p/(p-1)$ then

$$\mathbb{E}|XY| \le \|X\|_p \|Y\|_q. \tag{2}$$

2. If $a_i, b_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, $p > 1$ and $q = p/(p-1)$ then

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}. \tag{3}$$

Proof. Set

$$L_\nu = L(\nu, \delta, T),$$
$$D_\nu = D(\nu, \delta, T),$$
$$X_t = Y_t L_2^{-1/2} \quad \text{for} \quad t \in T,$$
$$L_\nu^* = \sum_{t \in T} \|X_t\|_{\nu+\delta}^{\nu},$$
$$D_\nu^* = L_\nu^* \vee (L_2^*)^{\nu/2} \quad \text{if} \quad \nu \ge 2,$$
$$c = a + b.$$

Then

$$L_\nu^* = \sum_{t \in T} \left(\mathbb{E} |Y_t L_2^{-1/2}|^{\nu+\delta} \right)^{\nu/(\nu+\delta)} = L_2^{-\nu/2} L_\nu,$$

thus we get

$$D_\nu^* = L_2^{-\nu/2} L_\nu \vee L_2^{-\nu/2} L_2^{\nu/2} = L_2^{-\nu/2} D_\nu, \quad \text{when} \quad \nu \geq 2, \tag{4}$$

and

$$L_2^* = 1. \tag{5}$$

Using (5) we have

$$D_\nu^* = L_\nu^* \vee (L_2^*)^{\nu/2} = L_\nu^* \vee 1, \quad \text{when} \quad \nu \geq 2. \tag{6}$$

This equality implies for any $a \geq 2$ and $b \geq 2$

$$D_a^* D_b^* = L_a^* L_b^* \vee L_a^* \vee L_b^* \vee 1. \tag{7}$$

(a) First we assume that $a > 2$. Set

$$u = \frac{(c+\delta)(a-2)}{c-2} \quad \text{and} \quad v = \frac{(2+\delta)(c-a)}{c-2}.$$

Then $u + v = a + \delta$, hence, using (2) with $p = \dfrac{c+\delta}{u}$ and $q = \dfrac{2+\delta}{v}$ we obtain

$$\mathbb{E}|X_t|^{a+\delta} = \mathbb{E}|X_t|^{u+v} \leq \left\| |X_t|^u \right\|_{(c+\delta)/u} \left\| |X_t|^v \right\|_{(2+\delta)/v}.$$

This inequality implies

$$L_a^* \leq \sum_{t \in T} \|X_t\|_{c+\delta}^{rc} \|X_t\|_{2+\delta}^{2s}, \tag{8}$$

where

$$r = \frac{ua}{c(a+\delta)}, \quad s = \frac{av}{2(a+\delta)}.$$

As $0 < r < a/c < 1$, we can apply (3) with $p = 1/r$, $q = 1/(1-r)$ to get from (8) that $L_a^* \leq (L_c^*)^r A^{1-r}$, where $A = \sum_{t \in T} \|X_t\|_{2+\delta}^{2s/(1-r)}$. As $s/(1-r) \geq 1$, from (5) we obtain $A \leq 1$, therefore $L_a^* \leq (L_c^*)^r$. Hence, if $L_a^* \geq 1$, then $L_c^* \geq 1$. Therefore, as $0 < r < a/c < 1$,

$$L_a^* \leq (L_c^*)^r \leq (L_c^*)^{a/c} \leq L_c^*, \quad \text{if} \quad L_a^* \geq 1. \tag{9}$$

(a′) Now, we concentrate on the case $a > 2$, $b > 2$. Then (9) is valid for b:

$$L_b^* \leq (L_c^*)^{b/c} \leq L_c^*, \quad \text{if} \quad L_b^* \geq 1. \tag{10}$$

These inequalities imply $L_b^* L_b^* \leq L_c^* \vee 1$. Therefore, using (7), (9), (10) and (6) we have

$$D_a^* D_b^* \leq (L_c^* \vee 1) \vee L_a^* \vee L_b^* = L_c^* \vee 1 = D_c^*.$$

Hence, using (4), we get the statement.

(b) Now, we assume that $a = b = 2$. Then using (7), (5) and (6), we get

$$D_2^* D_2^* = 1 \leq 1 \vee L_4^* = D_4^*.$$

Hence, using (4), we obtain the statement.

(c) If $a > 2$ and $b = 2$ then (5), (6) and (9) imply

$$D_a^* D_2^* = D_a^* \le D_c^*.$$

Hence, using (4), we have the statement.

(d) Finally, if $b > 2$ and $a = 2$ then the proof is the same as in (c).

This completes the proof of Lemma 2.

The following interpolation lemma is a version of Lemma 4.4 of Utev [3] and Lemma 1 in Doukhan [4, p. 27].

Let B be a separable Banach space with norm $\|\ \|$. Let $F = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ be a family of sub σ-algebras of the σ-algebra $\mathcal{F}$, and $\eta = \{\eta_1, \ldots, \eta_n\}$ be a family of centered random variables. The family η is said to be (F, B)-adapted if η_i is B-valued and $\mathcal{F}_i$-mesurable. We shall use the following notation:

$$M(\nu, \delta, \eta) = \sum_{i=1}^{n} \left(\mathbb{E}\|\eta_i\|^{\nu+\delta}\right)^{\nu/(\nu+\delta)} = \sum_{i=1}^{n} \|\eta_i\|_{\nu+\delta}^{\nu},$$

$$Q(\nu, \delta, \eta) = \begin{cases} M(\nu, \delta, \eta), & \text{if } 1 \le \nu \le 2; \\ M(\nu, \delta, \eta) \vee M^{\nu/2}(2, \delta, \eta), & \text{if } \nu > 2, \end{cases}$$

where $a \vee b = \max\{a, b\}$. $\mathrm{I}\{A\}$ denotes the indicator function of the set A.

Lemma 3. *Assume that for some fixed real constants $\nu \ge 1$, $\delta > 0$ and $c \ge 1$ any (F, B)-adapted centered family $\eta = \{\eta_1, \ldots, \eta_n\}$ satisfies*

$$\mathbb{E}\left\|\sum_{i=1}^{n} \eta_i\right\|^{\nu} \le c\, Q(\nu, \delta, \eta). \tag{11}$$

Set $t_0 = 1 \vee (\nu/2) \vee (\nu - \delta)$. Then for any t with $t_0 \le t \le \nu$ and any (F, B)-adapted centered family $\varphi = \{\varphi_1, \ldots, \varphi_n)$ satisfies

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varphi_i\right\|^{t} \le c\, 2^{4\nu-1} Q(t, \delta, \varphi).$$

We remark that $c \ge 1$ is the consequence of the other assumptions. In order to prove the lemma we require the following known inequalities.

1. (C_p-inequality.) If $x, y \in B$ and $p \ge 1$ then

$$\|x+y\|^p \le 2^{p-1}\left(\|x\|^p + \|y\|^p\right), \tag{12}$$

if $0 < p \le 1$ then

$$\|x+y\|^p \le \|x\|^p + \|y\|^p. \tag{13}$$

2. Let X be a B-valued random variable. If $p \ge 1$ then

$$(\mathbb{E}\,\|X\|)^p \le \mathbb{E}\,(\|X\|^p) \tag{14}$$

and

$$\mathbb{E}\,\|X - \mathbb{E}X\|^p \le 2^p\,\mathbb{E}\,\|X\|^p. \tag{15}$$

If $0 < p \le 1$ then

$$(\mathbb{E}\,\|X\|)^p \ge \mathbb{E}\,\|X\|^p \tag{16}$$

and

$$\mathbb{E}\,\|X - \mathbb{E}X\|^p \le 2\,(\mathbb{E}\,\|X\|)^p, \tag{17}$$

where $\mathbb{E}X$ is the Bochner integral.

3. If X is a B-valued random variable and $0 < q \le p$ then

$$\|X\|_q \le \|X\|_p \tag{18}$$

where $\|X\|_q = (\mathbb{E}\,\|X\|^q)^{1/q}$.

4. If $a_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $p \ge 1$ then

$$\sum_{i=1}^{n} |a_i|^p \le \left(\sum_{i=1}^{n} |a_i|\right)^p. \tag{19}$$

***Proof of Lemma* 3.** Let $\varphi = \{\varphi_1, \ldots, \varphi_n\}$ be a centered (F, B)-adapted family of random variables and $t_0 \le t \le \nu$ be a fixed constant. Set

$$Q = Q(t, \delta, \varphi),$$

$$y = Q^{1/t},$$

$$T_i = \varphi_i \mathrm{I}\{\|\varphi_i\| \le y\}, \quad i = 1, \ldots, n,$$

$$Y_i = \varphi_i \mathrm{I}\{\|\varphi_i\| > y\}, \quad i = 1, \ldots, n,$$

$$\eta_i = Y_i - \mathbb{E}Y_i, \quad i = 1, \ldots, n,$$

$$\eta = \{\eta_1, \ldots, \eta_n\},$$

$$\psi_i = T_i - \mathbb{E}T_i, \quad i = 1, \ldots, n,$$

$$\psi = \{\psi_1, \ldots, \psi_n\},$$

$$\xi_i = z\,(\|\eta_i\|^{t/\nu} - \mathbb{E}\,\|\eta_i\|^{t/\nu}), \text{ where } z \in B, \quad \|z\| = 1, \quad i = 1, \ldots, n,$$

$$\xi = \{\xi_1, \ldots, \xi_n\}.$$

Then $\eta_i + \psi_i = \varphi_i$ and $t \ge 1$ so (12) implies

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varphi_i\right\|^t \le 2^{t-1}\left(\mathbb{E}\left\|\sum_{i=1}^{n} \eta_i\right\|^t + \mathbb{E}\left\|\sum_{i=1}^{n} \psi_i\right\|^t\right), \tag{20}$$

$\dfrac{t}{\nu} \le 1$ and $\nu \ge 1$ so (13) and (12) imply

$$\mathbb{E}\left\|\sum_{i=1}^{n} \eta_i\right\|^t = \mathbb{E}\left(\left\|\sum_{i=1}^{n} \eta_i\right\|^{t/\nu}\right)^{\nu} \le$$

$$\le \mathbb{E}\left(\sum_{i=1}^{n} \|\eta_i\|^{t/\nu}\right)^{\nu} = \mathbb{E}\left(\sum_{i=1}^{n} (\|\eta_i\|^{t/\nu} - \mathbb{E}\|\eta_i\|^{t/\nu}) + \sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu}\right)^{\nu} \le$$

$$\le 2^{\nu-1}\left(\mathbb{E}\left\|\sum_{i=1}^{n} \xi_i\right\|^{\nu} + \left(\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu}\right)^{\nu}\right).$$

Set $V = c\,Q(\nu, \delta, \xi)$, $W = \left(\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu}\right)^{\nu}$. As ξ is centered and (F, B)-adapted, so this inequality and (11) imply

$$\mathbb{E}\left\|\sum_{i=1}^{n} \eta_i\right\|^{t} \le 2^{\nu-1}(V+W). \tag{21}$$

As ψ is centered, (F, B)-adapted and $\nu/t \ge 1$, thus using (14) and (11) we have

$$\mathbb{E}\left\|\sum_{i=1}^{n} \psi_i\right\|^{t} \le \left(\mathbb{E}\left\|\sum_{i=1}^{n} \psi_i\right\|^{\nu}\right)^{t/\nu} \le U, \tag{22}$$

where $U = (c\,Q(\nu, \delta, \psi))^{t/\nu}$. Then (20), (21) and (22) imply

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varphi_i\right\|^{t} \le 2^{t-1}U + 2^{t+\nu-2}V + 2^{t+\nu-2}W. \tag{23}$$

We thus have to estimate terms U, V and W.

U) Set $u = \nu(t+\delta)/t(\nu+\delta)$. Then $u \ge 1$, furthermore $\nu+\delta \ge 1$, hence (15) and (14) imply

$$\mathbb{E}\|\psi_i\|^{\nu+\delta} \le 2^{\nu+\delta}\mathbb{E}\|T_i\|^{\nu+\delta} \le 2^{\nu+\delta}(\mathbb{E}\|T_i\|^{u(\nu+\delta)})^{1/u}.$$

Thus

$$M(\nu, \delta, \psi) \le \sum_{i=1}^{n}\left(2^{\nu+\delta}(\mathbb{E}\|T_i\|^{u(\nu+\delta)})^{1/u}\right)^{\nu/(\nu+\delta)} =$$

$$= 2^{\nu}\sum_{i=1}^{n}\left(\mathbb{E}\|T_i\|^{u(\nu+\delta)}\right)^{\nu/u(\nu+\delta)}. \tag{24}$$

$\mathrm{I}\{\|\varphi_i\| \le y\}\|\varphi_i\|^{u(\nu+\delta)-(t+\delta)} \le y^{u(\nu+\delta)-(t+\delta)}$, because $u(\nu+\delta) - (t+\delta) \ge 0$. Hence

$$\mathbb{E}\|T_i\|^{u(\nu+\delta)} = \mathbb{E}(\|\varphi_i\|^{t+\delta}\mathrm{I}\{\|\varphi_i\| \le y\}\|\varphi_i\|^{u(\nu+\delta)-(t+\delta)}) \le$$

$$\le Q^{(t+\delta)(\nu/t-1)/t}\mathbb{E}\|\varphi_i\|^{t+\delta}.$$

Using this inequality and (24) we get

$$M(\nu, \delta, \psi) \le 2^{\nu}Q^{\nu/t-1}M(t, \delta, \varphi) \le 2^{\nu}Q^{\nu/t}. \tag{25}$$

(a) Assume that $\nu \le 2$. Then (25) implies

$$Q(\nu, \delta, \psi) = M(\nu, \delta, \psi) \le 2^{\nu}Q^{\nu/t}.$$

(b) If $t \le 2 \le \nu$, then (25) implies

$$M^{\nu/2}(2, \delta, \psi) \le (2^2 Q^{2/t})^{\nu/2} = 2^{\nu}Q^{\nu/t},$$

hence

$$Q(\nu, \delta, \psi) = \max\begin{cases} M(\nu, \delta, \psi) \le 2^{\nu}Q^{\nu/t}; \\ M^{\nu/2}(2, \delta, \psi) \le 2^{\nu}Q^{\nu/t}. \end{cases}$$

(c) Assume that $2 \le t$. Then because of (15) and $\| T_i \| \le \| \varphi_i \|$

$$M(2, \delta, \psi) = \sum_{i=1}^{n} \left(\mathbb{E}\|T_i - \mathbb{E}T_i\|^{2+\delta} \right)^{2/(2+\delta)} \le$$

$$\le 4 \sum_{i=1}^{n} \left(\mathbb{E}\|T_i\|^{2+\delta} \right)^{2/(2+\delta)} \le 4\, M(2, \delta, \varphi) \le 4\, Q^{2/t}.$$

This inequality and (25) imply

$$Q(\nu, \delta, \psi) = \max \begin{cases} M(\nu, \delta, \psi) \le 2^{\nu} Q^{\nu/t}; \\ M^{\nu/2}(2, \delta, \psi) \le (4 Q^{2/t})^{\nu/2} = 2^{\nu} Q^{\nu/t}. \end{cases}$$

Cases (a), (b) and (c) imply that

$$Q(\nu, \delta, \psi) \le 2^{\nu} Q^{\nu/t}$$

for every $1 \le t \le \nu$, hence

$$U \le (c\, 2^{\nu} Q^{\nu/t})^{t/\nu} = c^{t/\nu} 2^{t} Q. \tag{26}$$

V) Inequality (15) involves

$$\mathbb{E}\,\| \xi_i \|^{\nu+\delta} = \mathbb{E}\left| \|\eta_i\|^{t/\nu} - \mathbb{E}\|\eta_i\|^{t/\nu} \right|^{\nu+\delta} \le 2^{\nu+\delta}\, \mathbb{E}\,\| \eta_i \|^{t(\nu+\delta)/\nu}. \tag{27}$$

This inequality, (18), (15) and $\| Y_i \| \le \| \varphi_i \|$ imply the next inequality

$$M(\nu, \delta, \xi) \le 2^{\nu} \sum_{i=1}^{n} \|\eta_i\|^{t}_{t(\nu+\delta)/\nu} \le 2^{\nu} \sum_{i=1}^{n} \|\eta_i\|^{t}_{t+\delta} \le$$

$$\le 2^{\nu+t} \sum_{i=1}^{n} (\mathbb{E}\|Y_i\|^{t+\delta})^{t/(t+\delta)} \le 2^{\nu+t} M(t, \delta, \varphi) \le 2^{\nu+t} Q. \tag{28}$$

(a) Assume that $\nu \le 2$. Then (28) implies

$$Q(\nu, \delta, \xi) = M(\nu, \delta, \xi) \le 2^{\nu+t} Q.$$

(b) Assume that $t \le 2 \le \nu$. By (27) and (15), we have

$$M(2, \delta, \xi) \le 4 \sum_{i=1}^{n} (\mathbb{E}\|\eta_i\|^{t(2+\delta)/\nu})^{2/(2+\delta)} \le$$

$$\le 4^{1+t/\nu} \sum_{i=1}^{n} (\mathbb{E}\|Y_i\|^{t(2+\delta)/\nu})^{2/(2+\delta)}, \tag{29}$$

where we used that $t \ge \nu/2$. Now $\mathrm{I}\{ \|\varphi_i\| > y \} \|\varphi_i\|^{t(2+\delta)/\nu - (t+\delta)} \le$ $\le y^{t(2+\delta)/\nu - (t+\delta)}$, because $t(2+\delta)/\nu - (t+\delta) \le 0$. So (29) implies

$$M(2, \delta, \xi) \le$$

$$\le 4^{1+t/\nu} Q^{2((2+\delta)/\nu - (t+\delta)/t)/(2+\delta)} \sum_{i=1}^{n} \left((\mathbb{E}\|\varphi_i\|^{t+\delta})^{t/(t+\delta)} \right)^{2(t+\delta)/t(2+\delta)}.$$

Hence, using (19), we have

$$M(2, \delta, \xi) \le 4^{1+t/\nu} Q^{2((2+\delta)/\nu - (t+\delta)/t)/(2+\delta)} (M(t, \delta, \varphi))^{2(t+\delta)/t(2+\delta)} \le$$

$$\le 4^{1+t/\nu} Q^{2/\nu}.$$

Using this inequality and (28) we have

$$Q(\nu, \delta, \xi) = \max \begin{cases} M(\nu, \delta, \xi) \le 2^{\nu+t} Q; \\ M^{\nu/2}(2, \delta, \xi) \le (4^{1+t/\nu} Q^{2/\nu})^{\nu/2} = 2^{\nu+t} Q. \end{cases}$$

(c) Assume that $2 \le t$. We remark that (29) is valid in this case, hence

$$M(2, \delta, \xi) \le 4^{1+t/\nu} Q^{2/\nu - 2/t} M(2, \delta, \varphi) \le$$
$$\le 4^{1+t/\nu} Q^{2/\nu-2/t} Q^{2/t} = 4^{1+t/\nu} Q^{2/\nu}.$$

(Here we used that $\mathrm{I}\{\|\varphi_i\| > y\} \|\varphi_i\|^{t(2+\delta)/\nu-(2+\delta)} \le y^{t(2+\delta)/\nu-(2+\delta)}$ and the definition of Q.) Using the previous inequality and (28), we get

$$Q(\nu, \delta, \xi) = \max \begin{cases} M(\nu, \delta, \xi) \le 2^{\nu+t} Q; \\ M^{\nu/2}(2, \delta, \xi) \le (4^{1+t/\nu} Q^{2/\nu})^{\nu/2} = 2^{\nu+t} Q. \end{cases}$$

Cases (a), (b) and (c) imply that

$$Q(\nu, \delta, \xi) \le 2^{\nu+t} Q$$

for every $1 \le t \le \nu$, hence

$$V \le c\, 2^{\nu+t} Q. \tag{30}$$

W) Using (16) and (17) we have

$$\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu} \le \sum_{i=1}^{n} (\mathbb{E}\|Y_i - \mathbb{E}Y_i\|)^{t/\nu} \le 2^{t/\nu} \sum_{i=1}^{n} (\mathbb{E}\|Y_i\|)^{t/\nu}.$$

Furthermore, $\mathrm{I}\{\|\varphi_i\| > y\}\|\varphi_i\|^{1-\nu} \le y^{1-\nu}$, hence

$$\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu} \le 2^{t/\nu} Q^{1/\nu-1} \sum_{i=1}^{n} \|\varphi_i\|_\nu^t.$$

As $t + \delta \ge \nu$, we can apply (18)

$$\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu} \le 2^{t/\nu} Q^{1/\nu-1} M(t, \delta, \varphi) \le 2^{t/\nu} Q^{1/\nu}.$$

Thus

$$W \le 2^t Q. \tag{31}$$

Finally, (23), (26), (30) and (31) imply

$$\mathbb{E}\left\| \sum_{i=1}^{n} \varphi_i \right\|^t \le 2^{t-1}(c^{t/\nu} 2^t Q + 2^{\nu-1} c\, 2^{\nu+t} Q + 2^{\nu-1} 2^t Q) \le c\, 2^{4\nu-1} Q.$$

This completes the proof of Lemma 3.

***Corollary* 1.** *Assume that for some fixed real constants $\nu \ge 1$, $\delta > 0$ and $c \ge 1$ and for any (F, B)-adapted centered family $\eta = \{\eta_1, \ldots, \eta_n\}$ relation (11) is satisfied. Then for any t with $1 \le t \le \nu$ any (F, B)-adapted centered family $\varphi = \{\varphi_1, \ldots, \varphi_n\}$ satisfies*

$$\mathbb{E}\left\|\sum_{i=1}^{n}\varphi_i\right\|^t \leq CQ(t,\delta,\varphi),$$

where $C = c\,2^{(\nu-t+\delta)(2\nu+2t-1)/\delta}$ *if* $t \geq 2\delta$.

Proof. According to Lemma 3, in each step we can decrease the exponent by δ.

3. Proof of the Theorem.

Lemma 4. *Let* T *be a finite subset in* I, *let* h *be a fixed positive integer,* $\varepsilon > 0$. *Let* Y_t, $t \in T$, *be centered random variables with* $\mathbb{E}|Y_t|^{h+\varepsilon} < \infty$, $t \in T$. *Let*

$$A_h(T) = \sum_{\tau\in T^h}|\mathbb{E}(Y_{t_1}\ldots Y_{t_h})|,$$

where $\tau = \{t_1,\ldots,t_h\} \in T^h$. *Then*

$$A_h(T) \leq H_h^{(\alpha)}D(h,\varepsilon,T). \tag{32}$$

Proof. We omit superscript (α). We shall prove that for any positive integer h

$$A_h(T) \leq \left(1+\sum_{u=1}^{h-1}c_{u,h-u}\right)L(h,\varepsilon,T)+\sum_{u=2}^{h-2}\binom{h}{u}A_u(T)A_{h-u}(T). \tag{33}$$

Here $\sum_{u=1}^{h-1}(\cdot) = 0$ for $h = 1$ and $\sum_{u=2}^{h-2}(\cdot) = 0$ for $h = 1, 2, 3$. Random variables Y_t have zero expectation, therefore $A_1(T) = 0$. Moreover, we shall prove

$$A_2(T) \leq (1+c_{1,1})L(2,\varepsilon,T). \tag{34}$$

We have

$$A_h(T) \leq \sum_{t\in T}|\mathbb{E}Y_t^h| + \sum_{u=1}^{h-1}\sum_{r=1}^{\infty}\sum_{\xi}\sum_{\eta}|\mathbb{E}Y_\xi Y_\eta|, \tag{35}$$

where $\xi = \{t_1,\ldots,t_u\} \in T^u$, $\eta = \{t_{u+1},\ldots,t_h\} \in T^{h-u}$, $Y_\xi = Y_{t_1}\ldots Y_{t_u}$, $Y_\eta = Y_{t_{u+1}}\ldots Y_{t_h}$, moreover $\sum_\xi\sum_\eta$ means summation for all $\xi = \{t_1,\ldots,t_u\} \in T^u$, $\eta = \{t_{u+1},\ldots,t_h\} \in T^{h-u}$ so that the distance of sets $\{t_1,\ldots,t_u\}$ and $\{t_{u+1},\ldots,t_h\}$ is r which is the maximal distance between complementary pairs of non-empty subsets of $\{t_1,\ldots,t_h\}$. Remark that each $\{t_1,\ldots,t_h\} \in T^h$ should appear on the right hand side of (35), i.e. we take into account the order of components of τ. Using covariance inequality, we get

$$|\mathbb{E}Y_\xi Y_\eta| \leq |\mathbb{E}Y_\xi||\mathbb{E}Y_\eta| + 8[\alpha_Y(r,u,h-u)]^\rho\|Y_\xi\|_\nu\|Y_\eta\|_\mu, \tag{36}$$

where $\rho = \dfrac{\varepsilon}{h+\varepsilon}$, $\nu = \dfrac{h+\varepsilon}{u}$, $\mu = \dfrac{h+\varepsilon}{h-u}$. Using Hölder's inequality, we obtain

$$\|Y_\xi\|_\nu = \left(\mathbb{E}|Y_{t_1}\ldots Y_{t_u}|^{(h+\varepsilon)/u}\right)^{u/(h+\varepsilon)} \leq$$

$$\leq \left[\left(\prod_{i=1}^{u}\mathbb{E}|Y_{t_i}|^{h+\varepsilon}\right)^{1/u}\right]^{u/(h+\varepsilon)} = \prod_{i=1}^{u}\|Y_{t_i}\|_{h+\varepsilon}. \tag{37}$$

Now, by (37), the inequality of arithmetical and geometrical means, and Lemma 1, we get

$$\sum_{\xi}\sum_{\eta}\|Y_\xi\|_\nu\|Y_\eta\|_\mu \le \sum_{\xi}\sum_{\eta}\prod_{i=1}^{u}(\|Y_{t_i}\|_{h+\varepsilon}^{h})^{1/h}\prod_{i=u+1}^{h}(\|Y_{t_i}\|_{h+\varepsilon}^{h})^{1/h} \le$$

$$\le \frac{1}{h}\sum_{\xi}\sum_{\eta}\left(\sum_{i=1}^{u}\|Y_{t_i}\|_{h+\varepsilon}^{h} + \sum_{i=u+1}^{h}(\|Y_{t_i}\|_{h+\varepsilon}^{h}\right) \le$$

$$\le \sum_{t\in T} s_r b_r^{h-2} u!(h-u-1)!(h-1)!\|Y_s\|_{h+\varepsilon}^{h}. \qquad (38)$$

To explain the last inequality we remark that for any fixed $s \in T$ we can choose the other $u-1$ members of ξ at most $(u-1)!\,b_r^{u-1}$ ways, the point closest to η at most u ways, a point in distance r from that point at most s_r ways, and the other $h-u-1$ points in η at most $(h-u-1)!\,b_r^{h-u-1}$ ways. Moreover, $(h-1)! = \frac{h!}{h}$ stands because of the different orders of h elements. On the other hand

$$\sum_{r=1}^{\infty}\sum_{\xi}\sum_{\eta}|\mathbb{E}Y_\xi||\mathbb{E}Y_\eta| \le \binom{h}{u}A_u(T)A_{h-u}(T). \qquad (39)$$

Now, by (35), (36), (39), and (38), we get

$$A_h(T) \le \sum_{t\in T}|\mathbb{E}Y_t^h| + \sum_{u=1}^{h-1}\binom{h}{u}A_u(T)A_{h-u}(T) + \sum_{u=1}^{h-1}\sum_{s\in T}c_{u,h-u}\|Y_s\|_{h+\varepsilon}^{h} \le$$

$$\le \sum_{u=1}^{h-1}\binom{h}{u}A_u(T)A_{h-u}(T) + \left(1+\sum_{u=1}^{h-1}c_{u,h-u}\right)L(h,\varepsilon,T),$$

which gives (33). The above arguments in the simple case of $h = 2$ imply (34). Using Lemma 2 (33) gives (32).

Proof of Theorem. If h is an even positive integer, then

$$\mathbb{E}\left(\sum_{t\in T}Y_t\right)^h \le A_h(T).$$

This and Lemma 4 imply (1) for even l. For arbitrary l one can use Corollary 1.

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