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A PRIORI ESTIMATES FOR SOLUTIONS OF THE FIRST INITIAL BOUNDARY-VALUE PROBLEM FOR SYSTEMS OF FULLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

We prove a priori estimates for a solution of the first initial boundary-value problem for a system of fully nonlinear partial differential equations (PDE) in a bounded domain. In the proof, we reduce the initial boundary-value problem to a problem on a manifold without boundary and then reduce the resulting system on the manifold to a scalar equation on the total space of the corresponding bundle over the manifold.

Доведено апріорні оцінки для розв'язків першої початково-граничної задачі для системи повністю нелінійних диференціальних рівнянь з частинними похідними в обмеженій області. При цьому початково-гранична задача зводиться до задачі на многовиді без границі, яка в свою чергу зводиться до скалярного рівняння на тотальному просторі відповідного розшарування над многовидом.

1. Introduction. In this paper, we prove *a priori* estimates for a solution of the Dirichlet problem for a system of fully nonlinear elliptic equations in a smooth bounded domain $G \subset R^n$

$$F_l(x, u, D_x u, D_{xx} u_l) = 0, \qquad u_l(x) = 0, \quad x \in \partial G,$$
$$l = 1, 2, \ldots, m, \quad u \equiv (u_1, \ldots, u_m).$$

Problems of this type appear in various fields of mathematics and sciences such as differential geometry, the theory of optimal control, the theory of differential games, and others. For last three decades, numerous papers and some monographs were devoted to this topic [1–4]. Some of them were motivated by diffusion theory, and our approach follows and extends the way suggested there. Namely, we combine the ideas of the papers of Yu. L. Daletskii and the author [5–8], where the probabilistic representations of the Cauchy problem for systems of fully nonlinear parabolic equations on linear spaces and vector bundles were used to study a solution of the problem, and the approach due to Krylov [9], which shows the way to reduce the first initial boundary-value problem in G to a Cauchy problem on an auxiliary manifold $U \subset R^{n+4}$. In this paper, we extend the results due to Krylov to the case of systems. We assume here that F_l smoothly depends on its arguments. In the other case, we will need some special assumptions which allow us to derive an equation for ∇u_l. Note that, throughout the paper, we assume that for $F_l[u] = \mathcal{F}(x, u, p, r)$, $p \in R^n \otimes R^m$, $r \in R^n \otimes R^n$, we have $D_r \mathcal{F} \geq 0$ either everywhere or at least on solutions of (1).

The approach presented here is based on two reductions described in Sec. 2.

First, we use the observation due to Yu. L. Daletskii [10] according to which a system of m second-order parabolic equations in the phase space R^n can be reduced to a scalar second-order parabolic equation in the phase space R^{n+m}. Note that this observation, though quite natural from the probabilistic point of view, appears to be very useful for deriving *a priori* estimates of solutions of parabolic and elliptic equations [9] because it gives the way to use the maximum principle in the context of systems of equations.

Next, following [9], we reduce the initial boundary-value problem in the domain $G \times R^m$, $G \subset R^n$, to the initial problem on a total space $\mathcal{E}$ of a vector bundle over U.

The probabilistic background of the results and technique developed here are exposed in Sec. 3. In fact, we need them to provide the natural explanation for rather bulky calculations and conditions on the coefficients of the problem.

Section 4 deals with *a priori* èstimates of a solution of the equation on U derived in Sec. 2. In the last section, we prove the required *a priori* estiamates for a solution of the system under consideration.

2. Reduction of a system of equations in a bounded domain to a scalar equation on a bundle over a manifold without boundary. In this section, we first recall that a certain class of systems of parabolic (or elliptic) equations defined on a linear space or a maniforld can be interpreted as a scalar equation defined on a new linear space or a vector bundle over a manifold [7, 10]. Next, following Krylov [9], we reduce the first initial boundary-value problem for a system of nonlinear parabolic equations to a Cauchy problem on a maniforld without boundary.

To explain the approach, we consider the simple case of a system of linear parabolic equations and start with introducing necessary notation and definitions.

Let $X = R^n$ or let X be a smooth manifold and denote by $C^k(X)$ the space of continuous bounded functions having continuous bounded derivatives up to the kth order on the set X. Assume that G is a bounded domain in R^n with the smooth boundary $\partial G = \{x \in R^n\colon \psi(x) = 0\}$, where $\psi \in C^4(R^n)$ and $\psi(x) > 0$ for $x \in G$.

Given real-valued functions $A_i^k(x)$, $a_i(x)$, $B_l^{kp}(x)$, $c_l^p(x)$, $\alpha(x)$, and $f_l(x)$ defined on R^n, considér the first initial boundary-value problem for a system of parabolic equations in G

$$D_t u_l + (Lu)_l + \alpha u_l + f_l = 0, \qquad x \in G, \tag{1}$$

$$l = 1, 2, \ldots, m, \quad u = (u_1, \ldots, u_m),$$

where

$$(Lu)_l = \frac{1}{2} D^2_{(A,A)} u_l + D_a u_l + B_l^{kp} D_{A^k} u_p + c_l^p u_p, \tag{2}$$

$$u_l(0, x) = u_0(x), \qquad x \in G,$$

$$u_l(x) = 0, \qquad x \in \partial G. \tag{3}$$

Here,

$$D_{(A,A)} u_l = \sum_{i,j,k=1}^{n} A_i^k \frac{\partial^2 u_l}{\partial x_i \partial x_j} A_j^k, \qquad D_a u_l = \sum_{i=1}^{n} a_i \frac{\partial u_l}{\partial x_i}, \qquad D_t u_l = \frac{\partial u_l}{\partial t}.$$

In (2) and bellow, we assume the summation over all repeating indices and denote by $(\,,\,)_n$ the inner product in R^n, omitting index n if this does not lead to confusion.

Let us explain the first reduction. Given $h \in R^m$ and a set of real functions $u_p(t, x)$, $x \in R^n$, $p = 1, \ldots, m$, we denote $y = (x, h) \in R^{n+m}$ and introduce the scalar function

$$\Phi(t, y) = (h, u(t, x)).$$

It is easy to check that if $u_l(t, x)$ solve (1), then $\Phi(t, y)$ solves

$$\frac{\partial \Phi}{\partial t} + M^0 \Phi + \alpha \Phi + g = 0, \tag{4}$$

where M^0 is an elliptic operator,

$$M^0 \Phi = \frac{1}{2} Q_i^k \frac{\partial^2 \Phi}{\partial y_i \partial y_j} Q_j^k + q_i \frac{\partial \Phi}{\partial y_i}, \tag{5}$$

with the coefficients $Q(y)$ and $q(y)$, $y=(x,h)$, given by

$$Q_i^k(y) = A_i^k(x), \quad i=1,\dots,n, \qquad Q_i^k(y) = B_i^{kp} h_p, \quad i=n+1,\dots,n+m,$$

$$q_i(y) = a_i(x), \quad i=1,\dots,n, \qquad q_i(y) = c_i^p h_p, \quad i=n+1,\dots,n+m, \tag{6}$$

$$g(y) = (h,f).$$

The importance of this simple observation is connected with the fact that it gives the way of using the maximum principle and comparison theorems in the investigation of the systems of equations described above. Note that, in studying a scalar parabolic equation, one can easily check that the gradient of its solution is governed by a system of this type. The same is also true for systems in the class under consideration.

Let $\tilde{\Phi}(t,x,h,\xi) = D_\xi \Phi(t,x,h)$. Then $\tilde{\Phi}(t,z) = \tilde{\Phi}(t,x,h,\xi)$ solves the equation

$$\frac{\partial\tilde{\Phi}}{\partial t} + \frac{1}{2}\tilde{Q}_i^k \frac{\partial^2\tilde{\Phi}}{\partial z_i \partial z_j}\tilde{Q}_j^k + q_i \frac{\partial\tilde{\Phi}}{\partial z_i} + \tilde{\alpha}\tilde{\Phi} + \tilde{g} = 0, \tag{7}$$

where $z=(y,\xi)\in R^{2(n+m)}$ and

$$\tilde{Q}_i^k(z) = Q_i^k(y), \quad i=1,\dots,n+m,$$

$$\tilde{Q}_i^k(z) = D_\xi Q_i^k(x), \quad i=n+m+1,\dots,2(n+m),$$

$$\tilde{q}_i(z) = q_i(y), \quad i=1,\dots,n+m, \tag{8}$$

$$\tilde{q}_i(z) = D_\xi q_i(y), \quad i=n+m+1,\dots,2(n+m),$$

$$\tilde{g} = D_\xi g + D_\xi \alpha\Phi.$$

Below, in fact, we consider the function $\Psi(t,y,\tilde{\xi}) = D_\xi\Phi(t,y) + \xi_0\Phi(t,y) = D_{\tilde{\xi}}\Phi(t,y)$ governed by the equation

$$\frac{\partial\Psi}{\partial t} + \frac{1}{2}\Theta_i^k \frac{\partial^2\Psi}{\partial z_i \partial z_j}\Theta_j^k + \theta_i \frac{\partial\Psi}{\partial z_i} + \alpha\Psi + \tilde{g} = 0, \tag{9}$$

where

$$\Theta_i^k = \tilde{Q}_i^k, \quad i=1,\dots,2(n+m), \quad \Theta_{2(n+m)+1}^k = \pi_1^k,$$

$$\theta_i = \tilde{q}_i, \quad i=1,\dots,m+n, \tag{10}$$

$$\theta_{m+n+i} = \tilde{q}_{m+n+i} - Q_i^k\pi_1^k, \quad i=1,\dots,n, \quad \theta_{2(n+m)+1} = D_\xi\alpha.$$

Note that one can continue this consideration starting from $\Psi(t,z) = D_{\tilde{\xi}}\Phi(t,y)$ and arriving at a new function

$$D^2_{(\tilde{\xi},\tilde{\eta})}\Phi(t,y) = D^2_{\tilde{\xi}}\Phi(t,y) + D_{\tilde{\eta}}\Phi(t,y) = \mathcal{H}(t,\tilde{\xi},\tilde{\eta}).$$

Following this way, we can show that $\mathcal{H}$ should be governed by a scalar equation of the same form if one takes

$$\Theta_{2(n+m)+1+i}^k = D_\xi^2 Q_i^k + D_\eta Q_i^k, \quad i\le n,$$

$$\theta_{2(n+m)+1+i} = D_\xi^2 q_i + D_\eta q_i - 2D_\xi Q_i^k\pi_1^k - Q_i^k\pi_2^k, \quad i\le n, \tag{11}$$

$$\Theta^k_{3(n+m)+2} = \pi^k_2, \quad \theta_{3(n+m)+2} = -\|\pi_1\|^2 + D^2_\xi \alpha + D_\eta \alpha.$$

To explain the second reduction, we introduce a new vector function $v(t, x)$ given by $u(t, x) = \psi(x) v(t, x)$. It follows immediately from (1) that $v(t, x)$ gives a solution to

$$\psi D_t v_l + \frac{1}{2}[\psi D_{(A,A)} v_l + 2(D_A \psi, D_A v_l) + D_{(A,A)} \psi v_l] + D_a \psi v_l + \psi D_a v_l +$$

$$+ (B^p_l, D_A \psi) v_p + \psi c^p_l v_p + \psi \alpha v_l + f_l = 0. \tag{12}$$

It is easy to check that it has a unique bounded solution in G and, hence, we need no more boundary conditions. Actually, if both v^1 and v^2 solve (12), then $w = v^1 - v^2$ solves the system

$$\psi D_t w_l + \frac{1}{2}[\psi D_{(A,A)} w_l + 2(D_A \psi, D_A w_l) + D_{(A,A)} \psi w_l] + D_a w_l +$$

$$+ (B^p_l, D_A \psi) w_p + \psi c^p_l w_p + \psi \alpha w_l + f_l = 0,$$

which is equivalent to

$$D_t(\psi w_l) + \frac{1}{2}[D_{(A,A)}(\psi w_l)] + D_a(\psi w_l) +$$

$$+ (B^p_l, D_A(\psi w_p)) + c^p_l(\psi w_p) + \alpha(\psi w_l) = 0$$

in G. Eventually, $\psi w_l = 0$ for $x \in \partial G$ and, hence, the uniqueness of a solution of (1), (3) yields $\psi w_l = 0$ for $x \in G$. Since $\psi(x) > 0$ for $x \in G$, we get $w_l(x) = 0$ for $x \in G$.

The considerations presented above show that one may consider the parabolic system (12) by using the technique developed in [7] after some additional transformations which lead to a system of the required form.

We succeed to get the necessary form of the system following the way indicated in [9].

Introduce a new function $w_l(x, \psi(x)) = v_l(x)$ and consider it as a function defined on a surface in the space R^{n+1} given by the equation $\psi(x) = r^2$, $x \in R^{n+4}$. Meanwhile, system (12) should be treated as a system on this surface. To derive the explicit form of the system that governs functions $w_l(x, r)$ we should use the relations

$$D_{x_i} v_l = \left(D_{x_i} + \frac{1}{2\sqrt{\psi}} D_{x_i} \psi D_r\right) w_l, \tag{13}$$

$$D^2_{x_i x_j} v_l = \left(D_{x_i} + \frac{1}{2\sqrt{\psi}} D_{x_i} \psi D_r\right)\left(D_{x_j} + \frac{1}{2\sqrt{\psi}} D_j \psi D_r\right) w_l =$$

$$= D^2_{x_i x_j} w_l + \frac{1}{2r} D_{x_i} \psi D^2_{x_j r} w_l + \frac{1}{2r} D_{x_j} \psi D^2_{x_i r} w_l +$$

$$+ \frac{1}{4r^2} D_{x_i} \psi D_{x_j} \psi \left[D^2_{rr} w_l - \frac{1}{r} D_r w_l\right] + \frac{1}{2r} D^2_{x_i x_j} \psi w_l. \tag{14}$$

Let us substitute (13) and (14) into (12). As a result, we derive the system with singular coefficients

$$\psi D_t w_l + \frac{1}{2}\left[\psi D^2_{(A,A)} w_l - (D_A \psi, D_A \psi) \frac{1}{4\sqrt{\psi}} D_r w_l + \frac{\sqrt{\psi}}{2} D^2_{(A,A)} \psi D_r w_l +\right.$$

$$+ \frac{\sqrt{\psi}}{2} D_{A^k} \psi A_j^k D^2_{x_j r} w_l + \frac{1}{4} (D_A \psi, D_A \psi) D^2_{rr} + 2(D_A \psi, D_A w_l) +$$

$$+ \frac{1}{\sqrt{\psi}} (D_A \psi, D_A \psi) D_r w_l + D^2_{(A,A)} \psi w_l \Big] + \psi D_a w_l + \frac{\sqrt{\psi}}{2} D_a \psi D_r w_l +$$

$$+ \psi (B_l^p, D_A w_p) + \frac{\sqrt{\psi}}{2} (B_l^p, D_A \psi) D_r w_p + D_a \psi w_l + (B_l^p, D_A \psi) w_p +$$

$$+ \psi c_l^p w_p + \alpha \psi w_l + f_l = 0. \tag{15}$$

To overcome this obstacle note that the singular part of the system has the form

$$\frac{1}{8} \left[D^2_{rr} w_l + \frac{3}{r} D_r w_l \right] (D_A \psi, D_A \psi)$$

and the expression in square brackets coincides with the radial part of the four-dimensional Laplace operator. Hence, putting

$$\sqrt{\psi} = r = \sqrt{\sum_{\nu = n+1}^{n+4} x_\nu^2}$$

and treating the system above as a system on the manifold

$$U = \left\{ x \in R^{n+4} : \; 0 < \psi(x_1, \ldots, x_n) = \sum_{\nu = n+1}^{n+4} x_\nu^2 \right\},$$

we derive a system with good coefficients having the required form. To this end, rewrite (15) keeping in mind the relations

$$D^2_{rr} w_l + \frac{3}{r} D_r w_l = \sum_{\nu = n+1}^{n+4} D^2_{x_\nu x_\nu} w_l$$

and

$$\sum_{\nu = n+1}^{n+4} x_\nu D_{x_\nu} w_l = r D_r w_l .$$

As a result, we get the system

$$x_\nu x_\nu D_t w_l + \frac{1}{2} \Big[x_\nu A_s^k D^2_{x_s x_q} w_l x_\nu A_q^k + \frac{1}{4} D_{A^k} \psi D^2_{x_\nu x_\nu} w_l D_{A^k} \psi +$$

$$+ \frac{x_\nu}{2} D^2_{(A,A)} \psi D_{x_\nu} w_l + x_\nu D_{A^k} \psi D^2_{x_s x_\nu} w_l A_s^k + 2(D_{A^k} \psi, D_{A^k} w_l) + D^2_{(A,A)} \psi w_l \Big] +$$

$$+ \psi D_a w_l + \frac{1}{2} x_\nu D_a \psi D_{x_\nu} w_l + D_a \psi w_l + x_\nu (B_l^p, D_A w_p) x_\nu +$$

$$+ x_\nu (B_l^p, D_A \psi) D_{x_\nu} w_p + (B_l^p w_p, D_A \psi) + \psi c_l^p w_p + \alpha \psi w_l + f_l = 0,$$

which can be rewritten in the form

$$x_\nu x_\nu D_t w_l + (\overline{L}^0 w)_l + \overline{\alpha} w_l + f_l = 0,$$

$$(\overline{L}^0 w)_l = \frac{1}{2} \overline{A}_{s\nu}^k \frac{\partial^2 w_l}{\partial x_s \partial x_q} \overline{A}_{q\nu}^k + \overline{a}_s \frac{\partial w_l}{\partial x_s} + \overline{B}_{sl}^{kp} \overline{A}_{s\nu}^k \frac{\partial w_p}{\partial x_s} + \overline{c}_l^p w_p , \tag{16}$$

$$s,\ q = 1, \ldots, n+4, \quad k = 1, \ldots, n, \quad p, l = 1, \ldots, m, \quad \nu = n+1, \ldots, n+4,$$

where

$$\overline{A}^k_{i\nu} = x_\nu A^k_i, \quad \overline{A}^k_{\mu\nu} = \frac{1}{2} D_{A^k} \psi \delta_{\mu\nu}, \quad \overline{a}_i = \psi a_i + D_{A^k} \psi A^k_i,$$

$$\overline{a}_\nu = \frac{x_\nu}{2} \hat{L}\psi, \quad \overline{B}^{kp}_{l\nu} = x_\nu B^{kp}_l, \quad \overline{B}^{kp}_{li} = B^{kp}_l D_{x_i} \psi, \tag{17}$$

$$\overline{c}^p_l = \psi c^p_l + (B^p_l, D_A \psi), \quad \overline{\alpha} = \psi\alpha + \hat{L}\psi, \quad \hat{L}\psi = \frac{1}{2} D^2_{(A,A)} \psi + D_a \psi.$$

Finally, reduce the last system to the scalar equation

$$x_\nu x_\nu D_t \Phi + M^0 \Phi + \overline{\alpha}\Phi + g = 0, \quad s, q = 1, \ldots, m+n+4,$$

with respect to $\Phi(t, y) = (h, w(t, x))$, where

$$M^0 \Phi(y) = \frac{1}{2} Q^k_{s\nu} D^2_{y_s y_q} \Phi Q^k_{q\nu} + q_i D_{y_i} \Phi \tag{18}$$

and

$$Q^k_{i\nu}(y) = x_\nu A^k_i, \quad Q^k_{\mu\nu}(y) = \frac{1}{2} D_{A^k} \psi \delta_{\mu\nu},$$

$$Q^k_{n+4+li}(y) = B^{kp}_l D_{x_i} \psi h_p, \quad Q^k_{n+4+l\nu}(y) = x_\nu B^{kp}_l h_p,$$

$$Q^k_{i\nu}(y) = B^{kp}_l D_{x_i} \psi h_p, \tag{19}$$

$$q_i(y) = \psi a_i + D_{A^k} \psi A^k_i, \quad q_\mu(y) = \frac{x_\nu}{2} \hat{L}\psi,$$

$$q_{n+4+l}(y) = \psi c^p_l h_p + (B^p_l h_p, D_A \psi),$$

$i, k = 1, \ldots, n$, $\mu, \nu = n+1, \ldots, n+4$, $p, l = 1, \ldots, m$, and $\delta_{\mu\nu}$ is the Kronecker symbol.

Note that (16) should be treated either as a system on the manifold $U \in R^{n+4}$ or as a system in the linear space R^{n+4}. In a similar way, (18) should be treated as a PDE on a vector bundle $\mathcal{E}$ over U or as a PDE in R^{n+m+4}. The first approach is more bulky while the second one might be less evident. In fact, we can prove the existence of a solution of the Cauchy problem for the system only in the class of functions with support in U and, hence, we have to check that L^0 acts in this class. To explain some ideas and illustrate further calculations we would like to describe the probabilistic background of the entire approach.

Let us make one more remark. In Secs. 4 and 5, we consider a system of nonlinear elliptic equations. Evidently, all considerations presented above are valid in this case as well. The only change will be the absence of terms with time derivatives.

3. Stochastic approach. Denote by $(\Omega, \mathcal{F}, P)$ a probability space, i.e., a mesurable space with measure P such that $P(\Omega) = 1$ and by $w_{\nu k}(t)$, $k = 1, \ldots, n$, $\nu = n+1, \ldots, n+4$, a family of independent scalar Wiener processes.

Consider the system of stochastic differential equations (SDE)

$$d\xi_i = \|\zeta(t)\|^2 a_i(\xi(t))\, dt + A^k_i(\xi(t)) D_{A^k(\xi(t))} \psi(\xi(t))\, dt + \zeta_\nu(t) A^k_i(\xi(t))\, dw_{k\nu},$$

$$d\zeta_\mu = \frac{1}{2} \zeta_\mu(t) \hat{L}\psi(\xi(t))\, dt + \frac{1}{2} \delta_{\mu\nu}(t) D_{A^k(\xi(t))} \psi(\xi(t))\, dw_{k\nu}, \tag{20}$$

$$i, k = 1, \ldots, n, \quad \mu, \nu = n+1, \ldots, n+4.$$

We construct a solution $z(t) = (\xi(t), \zeta(t))$ of (20) that satisfies the conditions

$$\xi_i(0) = x_i, \quad \zeta_\nu(0) = x_\nu. \tag{21}$$

It should be noted that the coefficients in (20) are smooth functions which grow too fast. Thus, a solution of (20), (21) exists up to the blow time, which could be finite.

Nevertheless, if $z(t)$ starts at $x = z(0) \in U = \{\gamma \in R^{n+4} : \overline{\psi}(\gamma) = 0\}$, it will live for infinite time and never leaves the manifold.

Lemma 1. *Let* $z(0) = x \in U$. *Then for all* $t \geq 0$, *a solution* $z(t)$ *of* (20), (21) *does not leave* U *for all* $t \geq 0$.

Proof. Applying the Itô formula to the function $\overline{\psi}$ and process $z(t)$, we get

$$d\overline{\psi}(z(t)) = d[\psi(\xi(t)) - \|\zeta(t)\|^2] =$$

$$= [\hat{L}\psi(\xi(t))\|\zeta(t)\|^2 + D_{A^k}\psi(\xi(t))\, D_{A^k}\psi(\xi(t))]\, dt +$$

$$+ [D_{A^k}\psi(\xi(t))\,\zeta_\nu(t)\, dw_{k\nu} - \|\zeta(t)\|^2 \hat{L}\psi(\xi(t)) - D_{A^k}\psi(\xi(t))\, D_{A^k}\psi(\xi(t))]\, dt -$$

$$- \zeta_\mu(t)\, D_{A^k}\psi(\xi(t))\, dw_{k\mu} = 0. \tag{22}$$

This proves that $\psi(\xi(t)) = \|\zeta(t)\|^2$ for all t that do not exceed the blow time. Since ψ is a bounded function, it follows that $\|\zeta(t)\|^2$ is bounded for all $t \geq 0$. Now given bounded $\zeta(t)$, we should consider the equation for $\xi(t)$ and deduce from the general results for SDE that $E\|\xi(t)\|^2$ is finite for all $t \in [0, T]$ as well. Thus, the unique solution of (20), (21) is defined for all $t \geq 0$ and all the time (a.s.) lives on U.

Note that there are two ways of studying (20), (21). One of them is to regard it as a system in R^{n+4} and study its solution with initial values in U, which is proved to stay in U for all t, and the second one is to give an invariant description of the SDE that governs the process $z(t) \in U$. First, following [9], we consider (20), (21) in the entire space R^{n+4} and then describe the other approach.

Applying the standard technique of the theory of stochastic differential equations, given a smooth bounded function $w_0(x)$ defined on R^{n+4}, one can easily derive the equation to govern the function

$$w(t, x) = E w_0(z(t)) \exp\left\{\int_0^t \alpha(z(\tau))\, d\tau\right\}.$$

The corresponding equation has the form

$$D_t w + \frac{1}{2}\Big[\psi(x) A_i^k(x) D^2_{x_i x_j} w A_j^k(x) + \frac{1}{4} D_{A^k(x)}\psi(x) D^2_{x_\nu x_\nu} \tilde{w} D_{A^k(x)}\psi +$$

$$+ \frac{1}{2} x_\nu A_i^k(x) D^2_{x_i x_\nu} D_{A^k(x)}\psi(x)\Big] + \frac{1}{2} x_\nu \hat{L}\psi(x) D_{x_\nu} w +$$

$$+ \psi(x) a_k(x) D_{x_k} w + A_i^k(x) D_{A^k(x)}\psi(x) D_{x_i}\tilde{w} + \alpha(x) w = 0. \tag{23}$$

Next, we denote $\eta(t) = D_x\xi(t)$, $\beta(t) = D_x\zeta(t)$, $\gamma(t) = (\eta(t), \beta(t))$ and, by the formal differentiation of (20), (21), derive equations to govern $\eta(t)$ and $\beta(t)$

$$d\eta_i = 2(\zeta(t), \beta(t))\, a_i(\xi(t))\, dt + \|\zeta(t)\|^2 (D_x a_i(\xi(t), \eta)\, dt +$$

$$+ \frac{1}{2}\big([D_x A_i^k(\xi(t)) D_{A^k(\xi(t))}\psi(\xi(t)) + A_i^k(\xi(t)) D_{D_x A^k(\xi(t))}\psi(\xi(t)) +$$

$$+ \; A_i^k(\xi(t)) D_{A^k(\xi(t))} D_x \psi(\xi(t))], \eta(t)\big) \, dt + \beta_\nu(t) A_i^k(\xi(t)) \, dw_{\nu k} +$$

$$+ \; \zeta_\nu(t) (D_x A_i^k(\xi(t)), \eta(t)) \, dw_{\nu k},$$

$$i, k = 1, \ldots, n, \quad \mu, \nu = n+1, \ldots, n+4,$$

$$d\beta_\nu = \frac{1}{2} \beta_\nu(t) \hat{L}\psi(\xi(t)) \, dt + \frac{1}{2} \zeta_\nu(t) \big(D_x(\hat{L}\psi(\xi(t))), \eta(t)\big) \, dt +$$

$$+ \; \frac{1}{2} \big(D_x [D_{A^k(\xi(t))} \psi(\xi(t))], \eta(t)\big) \, dw_{\nu k}. \tag{24}$$

System (20), (21), (23), (24) describes the process $\tilde{\gamma}(t) = (z(t), \gamma(t)) \in R^{2(n+4)}$. It is easy to check that

$$\tilde{\Phi}(t, x, y) = E\left(\gamma(t), D_x \left[w_0(z(t)) \exp\left\{ \int_0^t \alpha(z(\tau)) \, d\tau \right\}\right]\right)$$

solves a parabolic equation similar to (22) as well as

$$\Phi(t, x, \tilde{y}) = \tilde{\Phi}(t, x, y) + y_0 w(t, x)$$

does for $x = z(0) \in U \subset R^{n+4}$, $y = \gamma(0) \in R^{n+4}$, $y_0 \in R^1$, $\tilde{y} = (y, y_0)$.

The corresponding equation is

$$D_t \Phi + \frac{1}{2} Q_i^{k\nu}(z) D^2_{z_i z_j} \Phi \, Q_j^{k\nu} + q_i D_{z_i} \Phi + \alpha \Phi = 0, \tag{25}$$

where

$$Q_i^{k\nu}(z) = x_\nu A_i^k(x_1, \ldots, x_n), \qquad Q_\mu^{k\nu}(z) = \frac{1}{2} \delta_{\nu\mu} D_{A^k} \psi(x_1, \ldots, x_n),$$

$$Q_{n+4+i}^{k\nu}(z) = D_{y_i} Q_i^{k\nu}(z), \qquad Q_{n+4+\mu}^{k\nu}(z) = D_{y_\mu} Q_\mu^{k\nu}(z),$$

$$Q_{2(n+4)+1} = \pi_1^k,$$

$$\tag{26}$$

$$q_i(z) = \psi(x) a_i(x) + \frac{1}{2} A_i^k(x) D_{A^k(x)} \psi(x),$$

$$q_{n+4+i}(z) = D_{y_i} \psi(x) + \psi(x) D_{y_i} a + \frac{1}{2} D_{y_i} [A^k(x) D_{A^k(x)} \psi(x)] - Q_i^k \pi_1^k,$$

$$q_{n+4+\nu}(z) = \frac{1}{2} y_\nu \hat{L}\psi(x), \qquad q_{2(n+4)+1} = D_y \alpha(x),$$

$$\mu, \nu = n+1, \ldots, n+4, \quad k, i = 1, \ldots, n.$$

In a similar way, one can show that system (20), (21), (23), (24) along with

$$d\alpha_i = [2\|\beta(t)\|^2 + 2(\gamma(t), \zeta(t))] a_i(\xi(t)) \, dt +$$

$$+ \; 2(\beta(t), \zeta(t))(D_x a_i(\xi(t), \eta(t)) \, dt + \|\zeta(t)\|^2 D^2_{\eta,\alpha} a_i(\xi(t)) \, dt +$$

$$+ \; \frac{1}{2}([D_x A_i^k(\xi(t)) D_{A^k(\xi(t))} \psi(\xi(t)) + A_i^k(\xi(t)) D_{D_x A^k(\xi(t))} \psi(\xi(t)) +$$

$$+ \; A_i^k(\xi(t))\, D_{A^k(\xi(t))}\, D_x\, \psi(\xi(t))],\, \alpha\big)\, dt \; +$$

$$+ \; \frac{1}{2} D_x \left[D_x A_i^k(\xi(t))\, D_{A^k(\xi(t))}\, \psi(\xi(t)) + A_i^k(\xi(t))\, D_{D_x A^k(\xi(t))}\, \psi(\xi(t)) \; + \right.$$

$$+ \; A_i^k(\xi(t))\, D_{A^k(\xi(t))}\, D_x\, \psi(\xi(t))](\eta, \eta)\, dt + \beta_\nu(t)\, A_i^k(\xi(t))\, dw_{\nu k} \; +$$

$$+ \; \xi_\nu(t)\big(D_x A_i^k(\xi(t)), \eta(t)\big)\, dw_{\nu k}, \tag{27}$$

$$d\gamma_\nu = \frac{1}{2}\gamma_\nu \hat{L}\psi(\xi(t))\, dt + \frac{1}{2}\gamma_\nu D_x[\hat{L}\psi(\xi(t))]\,\eta\, dt \; +$$

$$+ \; \frac{1}{2}[\zeta_\nu D_x(\hat{L}\psi(\xi(t)))\,\eta(t)]\,\beta(t)\, dt \; +$$

$$+ \; \frac{1}{2} D_x[\zeta_\nu D_x(\hat{L}\psi(\xi(t)))\,\eta(t)](\eta(t), \eta(t))\, dt \; +$$

$$+ \; \frac{1}{2}\big(D_x[D_{A^k(\xi(t))}\,\psi(\xi(t))], \beta(t)\big)\, dw_{\nu k} \; +$$

$$+ \; \frac{1}{2} D_x^2\, [D_{A^k(\xi(t))}\,\psi(\xi(t))](\eta(t), \eta(t))\, dw_{\nu k} \tag{28}$$

describes the process $\tilde{\gamma}(t) = (\xi(t), \zeta(t), \eta(t), \beta(t), \alpha(t), \gamma(t))$.

Here, $D^2_{(y, y^1)}\, g(x) = D^2_y\, g(x) + D_{y^1}\, g(x)$, $\tilde{z} = (x, \tilde{y}, \tilde{y}^1)$. Denote

$$Q^l_{2(n+4)+1+i}(z) = D^2_{(y, y^1)}\, Q^l_i(x), \quad Q^k_{3(n+4)+2}(z) = \pi^k_2,$$

$$q_{2(n+4)+1+i}(z) = D^2_{(y, y^1)}\, q_i(x) - 2\pi^k_1\, D_y\, Q^k_i - Q^k_i\, \pi^k_2, \tag{29}$$

$$q_{3(n+4)+2}(z) = -\sum_{k=1}^{n} |\pi^k_1|^2 + D_{(y, y^1)}\,\alpha(x).$$

Then $\Psi(t, \tilde{z}) = D^2_{\tilde{y}}\, w(t, x) + D_{\tilde{y}^1}\, w(t, x)$ solves the equation

$$D_t \Psi + \frac{1}{2} Q^l_i\, D^2_{\tilde{z}_i \tilde{z}_j} \Psi\, Q^l_j + q_i\, D_{\tilde{z}_i} \Psi + \alpha \Psi = 0, \tag{30}$$

$$i, j = 1, \ldots, 3(n+4) + 2.$$

To explain the second approach recall some notions of differential geometry. Let $l = n + 4$ and let the manifold U be given by

$$x_i = u_i, \quad i = 1, \ldots, l-1, \quad \psi(u_1, \ldots, u_n) - \sum_{\nu = n+1}^{n+3} u_\nu^2 = x_l^2.$$

Linear independent vectors $D_{u_1} x_i, \ldots, D_{u_{l-1}} x_i$ determine a tangent plane $T_x U$ at each point $x \in U$. Denote by n^i a line orthogonal to $T_x U$ at x. Let g_{ij}, $i, j = 1, \ldots, l$, be a metric tensor of the Euclidian space R^l, let $G_{\alpha\beta} = g_{ij}\, D_{u_\alpha} x_i\, D_{u_\beta} x_j$ denote a metric tensor on U, let ∇ and $\Gamma^\pi_{\alpha\beta}$ denote, respectively, the covariant derivative and Christoffel symbols, and let $\exp^U : TU \to U$ be the exponential mapping corresponding to $G_{\alpha\beta}$.

It follows from the general theory of stochastic differential equations on manifolds [7] that, given

$$\Lambda^{\gamma}_{\beta}(z(t)) = \left(\psi(\xi(t)) - \sum_{\nu=n+1}^{n+3} \zeta^2_{\nu}(t)\right)^{1/2} A^{\gamma}_{\beta}(\xi(t)), \quad \Theta^{\gamma}_{\nu\beta}(z) = \zeta_{\nu} A^{\gamma}_{\beta}(\xi),$$

the system of SDE

$$d\xi_{\beta} = \psi(\xi(t))\, a_{\beta}(\xi(t))\, dt + A^{\gamma}_{\beta}(\xi(t))\, D_{A^{\gamma}(\xi(t))}\psi(\xi(t))\, dt -$$

$$- \Lambda^{k}_{\beta}(z(t))\, dw_{4k} + \Theta^{\gamma}_{\nu\beta}(z(t))\, dw_{\gamma\nu} +$$

$$+ \frac{1}{2}[\Gamma^{\pi}_{\beta\alpha}(\Lambda^{k}_{\pi}(z(t)), \Lambda^{k}_{\alpha}(z(t))) + \Gamma^{\pi}_{\beta\alpha}(\Theta^{\gamma}_{\nu\alpha}(z(t)), \Theta^{\gamma}_{\nu\pi}(z(t)))]\, dt, \tag{31}$$

$$d\zeta_{\nu} = \frac{1}{2}\zeta_{\nu}\, \hat{L}\psi(\xi(t))\, dt + \frac{1}{2}\delta_{\mu\nu}\, D_{A^{\gamma}(\xi(t))}\psi(\xi(t))\, dw_{\mu\gamma} -$$

$$- \frac{1}{8}\Gamma^{\pi}_{\nu\alpha}(\delta_{\pi\nu} D_{A^{\gamma}(\xi(t))}\psi(\xi(t)), \delta_{\beta\nu} D_{A^{\gamma}(\xi(t))}\psi(\xi(t)))\, dt \tag{32}$$

gives a coordinate representation of SDE of the form

$$dz = \exp_{z(t)}(q(z(t))\, dt + Q(z(t))\, dw) \tag{33}$$

with respect to the stochastic process $z(t) = (\xi(t), \zeta(t)) \in U$.

By using the standard technique of the theory of stochastic differential equations on manifolds, given a smooth bounded function $w_0(t)$ defined on U, one can derive an equation to govern the function

$$\lambda(t, x) = E w_0(z(t)) \exp\left\{\int_0^t \alpha(z(\tau))\, d\tau\right\}.$$

The corresponding equation has the form

$$D_t\lambda + \frac{1}{2}\left[\nabla_{Q(x)}\nabla_{Q(x)} - \nabla_{\nabla_{Q(x)}Q(x)}\right]\lambda + \nabla_{q(x)}\lambda + \alpha(x)\lambda = 0. \tag{34}$$

Next, denote $\eta(t) = D_x\xi(t)$, $\beta(t) = D_x\zeta(t)$, and $\gamma(t) = (\eta(t), \beta(t))$ and let $\exp^T\colon T^2U \to TU$ be the exponential mapping on the manifold TU. Let ∇^{TU} denote the corresponding covariant derivative [7]. By the formal differentiation of (33), we can derive the following equation to govern $\gamma(t) \in TU$:

$$d\gamma = \exp^{T}_{\gamma(t)}[\tilde{q}(\gamma(t))\, dt + \tilde{Q}(\gamma(t))\, dw]. \tag{35}$$

Note that

$$\tilde{\Phi}(t, x, y) = E\left(\gamma(t), D_x w_0(z(t)) \exp\left\{\int_0^t \alpha(z(\tau))\, d\tau\right\}\right) +$$

$$+ E\Phi(t, x)\, D_{\gamma(t)} \exp\left\{\int_0^t \alpha(z(\tau))\, d\tau\right\}$$

solves a parabolic equation similar to (34) as well as

$$\Phi(t, x, \tilde{y}) = \tilde{\Phi}(t, x, y) + y_0 w(t, x)$$

does for $x = z(0) \in U$, $y = \gamma(0) \in TU$, $y_0 \in R^1$, $z = (x, \tilde{y}) \in TU \times R^1$.

The corresponding equation is

$$D_t \Phi + \frac{1}{2} \operatorname{Tr}\left[\nabla_Q^{TU} \nabla_Q^{TU} - \nabla_{\nabla_Q Q}^{TU}\right] \Phi + \nabla_Q^{TU} \Phi + \alpha \Phi = 0, \tag{36}$$

where $Q(z)$ and $q(z)$ are determined by

$$Q_i^k(z) = \left(\psi(x_1, \ldots, x_n) - \sum_{\nu = n+1}^{n+3} x_\nu^2\right)^{1/2} A_i^k(x_1, \ldots, x_n),$$

$$Q_i^{k\nu}(z) = x_\nu A_i^k(x_1, \ldots, x_n), \quad i, k = 1, \ldots, n,$$

$$Q_\mu^{k\nu}(z) = \frac{1}{2} \delta_{\nu\mu} D_{A^k} \psi(x_1, \ldots, x_n), \quad \mu, \nu = n+1, \ldots, n+3,$$

$$Q_{n+3+i}^k(z) = D_y Q_i^k(x), \quad Q_{n+3+\nu}^k(z) = D_y Q_i^{k\nu}(x),$$

$$Q_{n+3+\mu}^{k\nu}(z) = D_y Q_\mu^{k\nu}(x), \quad Q_{2(n+3)+1} = \pi_1^k, \tag{37}$$

$$q_i(z) = \psi(x) a_i(x) + \frac{1}{2} A_i^k(x) D_{A^k(x)} \psi(x),$$

$$q_{n+3+i}(z) = D_y \psi(x) + \psi(x) D_y a_i(x) + \frac{1}{2} D_y \left[A_i^k(x) D_{A^k(x)} \psi(x)\right] - Q_i^k \pi_1^k,$$

$$q_{n+3+\nu}(z) = \frac{1}{2}\left[x_\nu D_y [\hat{L}\psi(x) + y_\nu \hat{L}\psi(x)]\right],$$

$$q_{2(n+3)+1}(z) = D_y \alpha(x),$$

$$\mu, \nu = n+1, \ldots, n+3, \quad k, i = 1, \ldots, n.$$

In a similar way, one can describe the stochastic process $\kappa(t) \in T^2 \mathscr{C}$ having the local representation

$$\kappa(t) = (\xi(t), \zeta(t), \eta(t), \beta(t), \eta(t), \beta(t), \alpha(t), \gamma(t)) \in T^2 U$$

as a solution of the equation of the form (35) with $\exp^{TU}$ replaced by $\exp^{T^2U}$ and the coefficients Q and q given by (37) and

$$Q_{2(n+3)+1+i}^l(z) = D_{(\xi,\eta)}^2 Q_i^l(x), \quad Q_{3(n+3)+2}^k(z) = \pi_2^k,$$

$$q_{2(n+3)+1+i}(z) = D_{(\xi,\eta)}^2 q_i(x) - 2\pi_1^k D_\xi Q_i^k - Q_i^k \pi_2^k, \tag{38}$$

$$q_{3(n+3)+2}(z) = -\sum_{k=1}^n |\pi_1^k|^2 + D_{(\xi,\eta)} \alpha(x),$$

where $D_{(\xi,\eta)}^2 g(x) = D_\xi^2 g(x) + D_\eta g(x)$, $\tilde{z} = (x, \tilde{\xi}, \tilde{\eta})$.

One should mention another approach to describing a stochastic process on a surface U in R^l, which may also be useful.

Given the surface U determined by $x^i = x^i(u^1, \ldots, u^{l-1})$, $i = 1, \ldots, l$, let $e_\alpha^i = D_{u_\alpha} x_i$, $\alpha = 1, \ldots, l-1$, and let n^i be a vector orthogonal to e_α^i, i.e., $g_{ij} e_\alpha^i n^j = 0$.

Denote by $b = \{b_{\alpha\beta}\}$ the second fundamental tensor of the surface U, $b_{\alpha\beta} = g_{ij}\nabla_\alpha e^i_\beta n^j$.

Consider a system of SDE in the Stratonovich form [7]

$$dx^i = e^i_\alpha \circ dw^\alpha,$$

$$de^i_\alpha = [-\Gamma^\kappa_{\alpha\beta}(x)\, e^i_\kappa + b_{\alpha\beta} n^i] \circ dx^\beta,$$

$$dn^i = -G^{\sigma\alpha} b_{\alpha\kappa}\, e^i_\sigma \circ dx^\kappa.$$

It follows from the general results that there exists a unique solution of this system that satisfies the conditions $x(s) = x_0 \in U$, $i = 1, \ldots, l-1$, $e^i_\alpha(s) = \delta_{i\alpha}$, $n^i(s) = \delta^i_l$ and determines a diffusion process on U. Actually, one can replace the process $x(t) \in R^{l-1}$ in this system by a quasimartingale $y(t) \in R^{l-1}$ to construct the diffusion on U with required diffusion coefficients.

Our further considerations directly follow [9], though we consider equations on charts of the manifold U rather then in the entire space R^{n+4}. Given $\delta \in (0, 1)$, $\kappa = [2-\delta]^{-1}$, denote

$$m(x, y) = [\lambda(K(x)y, y) + |y_0|^2 + 1]^\kappa, \tag{39}$$

where $K(x)$ is the field of a positive definite linear operator in $T_x U$, and $(X, Y)_x = G_{\alpha\beta}(x) X^\alpha Y^\beta$ for $X, Y \in T_x U$. Let $Q(x)$, $q(x)$, $\alpha(x)$ and $Q(z)$, $q(z)$, $\alpha(z)$ be given by (26), (37).

Denote $\nabla_{\tilde{y}} w(x) = \nabla_y w(x) + y_0 w(x)$, $(x, y) \in TU$, and introduce the operator $\tilde{M}$ acting on functions $\Phi(x) = \Phi(x, \tilde{y}) = \nabla_{\tilde{y}} w(x)$ as follows:

$$\tilde{M}\Phi(z) = \frac{1}{2}\operatorname{Tr}\left[\nabla^{TU}_{Q(z)} \nabla^{TU}_{Q(z)} - \nabla^{TU}_{\nabla_{Q(z)} Q(z)}\right]\Phi(z) + \nabla^{TU}_q \Phi(z).$$

It can be checked by direct calculation that

$$\tilde{M}\Phi(z) = D_{\tilde{y}}[Lw(x)],$$

where

$$Lw(x) = \frac{1}{2}\operatorname{Tr}\left[\nabla_{Q(x)}\nabla_{Q(x)} - \nabla_{\nabla_{Q(x)} Q(x)}\right] w(x) + \nabla_{Q(x)} w(x).$$

As a result, we get the elliptic operator $\tilde{M}$ and the maximum principle and comparison theorem can be applied to studying the equation $\tilde{M}\Phi(z) + \alpha\Phi(z) = 0$.

To this end, consider $\tilde{M}[m(x, \tilde{y})]^{1/\kappa}$, put $\pi^k_1 = \lambda(K(x)y, Q^k)$, and introduce

$$\Gamma_1(y) = -\lambda\left[M(K(x)y, y) - \mu\sum_{k=1}^{l}(K(x)y, Q^k)^2 + (2-\delta)\,\alpha(K(x)y, y)\right],$$

where $M\Phi(z) = \frac{1}{2}\operatorname{Tr}\left[\nabla_{Q(z)}\nabla_{Q(z)} - \nabla_{\nabla_{Q(z)} Q(z)}\right]\Phi(z) + \nabla_{q(z)}\Phi(z)$.

The following assertion can now be verified by direct calculation:

Lemma 2. *Let*

$$\Gamma_1(y) > 0 \quad \text{and} \quad (Ky, y)^{1/2}\nabla_y \alpha \le \mu\Gamma_1(y)$$

for any $x, \tilde{y} \in U'$.

Then there exists a constant $\lambda = \lambda(\delta, \mu) \geq 1$ *such that, for* (39) *with* $\kappa = (2-\delta)^{-1}$, *the estimate* $\tilde{M} m < 0$ *holds on the set*

$$U' \cap \{(Ky, y) \geq |y_0|^2 + 1\},$$

provided $\pi_1^k = \lambda(Ky, Q^k)$ *in* $\hat{L}$.

The function $m(x, \tilde{y})$ plays a role of a barrier function for $\nabla_{\tilde{y}} w$. Note that the function

$$\mathcal{M}(x, \tilde{y}, \tilde{y}^1) = [\lambda^2 (K(x)y, y)^2 + \lambda(K^1(x)y^1, y^1) + |y_0|^4 + |y_0^1|^2 + 1]^{\kappa} \tag{40}$$

can be used as a barrier function for the second derivatives of $w(t, x)$.

In the next section, we extend the considerations presented above to the case of systems of parabolic equations. For this purpose, we need one more stochastic process to describe the so-called multiplicative operator of a Markov process $z(t)$. This process is constructed as a solution of the linear stochastic equation in R^m

$$d\mathcal{G}_r = \|\zeta(t)\|^2 c_r^p(\xi(t))\, \mathcal{G}_p(t)\, dt + (B_r^p, D_{A(\xi(t))}\psi)\, \mathcal{G}_p(t)\, dt +$$

$$+ \zeta_\nu(t) B_r^{kp}(\xi(t))\, \mathcal{G}_p(t)\, dw_{k\nu} + B_r^{kp}(\xi(t))\, D_{x_i}\psi(\xi(t))\, \mathcal{G}_p(t)\, dw_{ki}, \tag{41}$$

$$k = 1, \ldots, n, \quad p, r = 1, \ldots, m, \quad \nu = n+1, \ldots, n+4.$$

Remark. For $z(t) \in U$, it follows from the general results that there exists a unique solution of (41) and positive constants μ_1, C, C_1 such that

$$E\|\mathcal{G}(t)\|^2 \leq C_1 e^{-Ct}$$

if one imposes the additional assumption on the coefficients B and c

$$\psi(x) \sum_{k=1}^{n} \|B^k(x) h\|^2 + \|D_x \psi(x)\|^2 \sum_{k=1}^{n} \|B^k(x) h\|^2 + 2\psi(x)(c(x) h, h) +$$

$$+ 2 \sum_{k=1}^{n} (B^k(x) h, h)\, D_{A^k(x)} \psi(x) \leq -\mu_1 \|h\|^2. \tag{42}$$

A procedure similar to that described above allows us to check that the processes $\mathcal{G}' = D_x \mathcal{G}$ solve the equation

$$d\mathcal{G}_r' = 2(\zeta(t), \beta(t))\, c_r^p(\xi(t))\, \mathcal{G}_p(t)\, dt + \|\zeta(t)\|^2 D_\eta [c_r^p(\xi(t))\, \mathcal{G}_p(t)\, dt +$$

$$+ \|\zeta(t)\|^2 c_r^p(\xi(t))]\, \mathcal{G}_p'(t)\, dt + [\beta_\nu(t)\, B_r^{kp}(\xi(t))\, \mathcal{G}_p(t) +$$

$$+ \zeta_\nu(t) D_\eta [B_r^{kp}(\xi(t))]\, \mathcal{G}_p(t) + \zeta_\nu(t)\, B_r^{kp}(\xi(t))\, \mathcal{G}_p'(t)]\, dw_{k\nu} +$$

$$+ \{D_\eta [B_r^{kp}(\xi(t))\, D_{x_i}\psi(\xi(t))]\, \mathcal{G}_p(t) + B_r^{kp}(\xi(t))\, D_{x_i}\psi(\xi(t))\, \mathcal{G}_p'(t)\}\, dw_{ki}$$

and to derive the corresponding equation for $\mathcal{G}_r'' = D_x \mathcal{G}'$.

In this case, $m(z, \tilde{y})$ and $\mathcal{M}(z, \tilde{y}, \tilde{y}^1)$ for $z = (x, h) \in R^{n+m+4}$ should have the form

$$m(z, \tilde{y}) = [\lambda(K(z)y, y) + |y_0|^2 + 1]^{\kappa},$$

$$\mathcal{M}(z, \tilde{y}, \tilde{y}^1) = [\lambda^2 (K(z)y, y)^2 + \lambda(K^1(z)y^1, y^1) + |y_0|^4 + |y_0^1|^2 + 1]^{\kappa}.$$

4. *A priori* estimates for solutions of equations on a manifold. To derive *a priori* estimates for the solution $w_l(x)$ of the system

$$F_l(x, w, Dw, D^2 w_l) = 0, \quad l = 1, \ldots, m, \quad x \in U \subset R^{n+4}, \tag{43}$$

we treat it as a system in R^{n+4}, assuming F_l to be defined for $x \in R^{n+4}$ and using the equivalence discussed in Sec. 2. Recall that we assume that either F is a smooth function of its arguments or, at least, there are reasons to ensure that Dw satisfies a system of the same kind. Note that, for the system of Bellman equations

$$F_l(x, w, Dw, D^2 w_l) = \inf_{r \in \mathfrak{R}} \big[A_i^k(r, x) D^2_{x_i x_j} w_l A_j^k(r, x) + a_i(r, x) D_{x_i} w_l +$$

$$+ B_l^{kp}(r, x) A_i^k D_{x_i} w_p + c_l^p(r, x) w_p + \alpha(r, x) w_l + f_l(r, x)\big] = 0,$$

we have a situation of this type. Actually, let $r_0(x)$ be the function that corresponds to the solution of (43). Then, for each $l = 1, \ldots, m$, $s_l(y) = (L(r_0(x), y) w(y))_l + f_l(r_0(x), y)$ is a smooth function of y which attains its relative minimum value (equal to 0) at a point $x \in U$. This allows us to conclude that $D_y s_l(y) = 0$ and to derive an equation for $D_x w_l(x)$.

Let

$$[x] = \left(\sum_{\nu = n+1}^{n+4} x_\nu^2\right)^{1/2}$$

and let

$$\overline{G}(\rho) = \{x \in G \colon \operatorname{dist}(x, \partial G) \geq \rho\},$$

$$\Delta_\rho(G) = G \setminus \overline{G}(\rho) = \{x \in R^n \colon 0 < \psi(x) < \rho\},$$

$$V(\rho) = \{x \in R^{n+4} \colon (x_1, \ldots, x_n) \in \Delta_\rho(G),\ [x] < 1\},$$

$$U(\rho) = \{x \in R^{n+4} \colon \overline{\psi}(x) = 0\} \cap V(\rho).$$

Denote by $\mathscr{E}(\rho)$ a vector bundle over $U(\rho)$ with model space R^m. We treat $\mathscr{E}(\rho)$ as a subset of R^{n_1}, $n_1 = n + m + 4$. Given $\delta \in (0, 1)$, $\tilde{\xi} \in (\xi, \xi_0)$, $\tilde{\xi} \in R^{n_1+1}$, $\kappa = (2 - \delta)^{-1}$, and a positive linear operator $K(y)$, $y \in R^{n_1}$, acting in R^{n_1}, consider a function $\mathcal{N}$ defined in R^{n_1} by

$$\mathcal{N}(y, \tilde{\xi}) = [\lambda(K(y)\xi, \xi) + |\xi_0|^2 + 1]^\kappa. \tag{44}$$

Let us estimate

$$M\mathcal{N}(y, \tilde{\xi}) = \frac{1}{2} N_{sl} \frac{\partial^2 \mathcal{N}}{\partial z_s \partial z_l} + q_s \frac{\partial \mathcal{N}}{\partial z_s} + \alpha \mathcal{N}, \quad l, s = 1, \ldots, 2n_1 + 1, \tag{45}$$

where $N_{sl} = \sum_{k,\mu} Q^k_{s\mu} Q^k_{l\mu}$, q_s, $s, l = 1, \ldots, 2n_1 + 1$, $k = 1, \ldots, n$, are given by

$$Q^k_{i\nu}(z) = x_\nu A_i^k(x_1, \ldots, x_n), \quad Q^k_{\nu\mu}(z) = \frac{1}{2} \delta^{\nu\mu} D_{A^k} \psi,$$

$$Q^k_{(n+4+r)i}(z) = D_{x_i} \psi B_r^{kp} h_p, \quad Q^k_{(n+4+r)\nu}(z) = x_\nu B_r^{kp} h_p,$$

$$\mu, \nu = n+1, \ldots, n+4, \quad k, i = 1, \ldots, n, \quad r, p = 1, \ldots, m,$$

$$Q^k_{(n_1+i)\nu}(z) = D_\xi Q^k_{i\nu}(x), \quad Q^k_{(n_1+\mu)\nu}(z) = D_\xi (Q^k_{\mu\nu}(x)),$$

$$Q^k_{(n_1+n+4+r)i}(z) = D_\xi D_{x_i} \psi B^{kp}_r h_p,$$

$$Q^k_{(n_1+n+4+r)\nu}(z) = D_\xi (x_\nu B^{kp}_r h_p), \quad Q^{k\nu}_{2n_1+1} = \pi^{k\nu}_1, \tag{46}$$

$$q_i(z) = \psi(x)\, a_i(x) + \frac{1}{2} A^k_i D_{A^k} \psi(x),$$

$$q_\nu(z) = \frac{x_\nu}{2} \hat{L}\psi, \quad q_{n+4+r}(z) = \psi c^p_r h_p + (B^p_r h_p, D_A \psi),$$

$$q_{n_1+i}(z) = D_\xi \psi a_i + \psi D_\xi a_i + \frac{1}{2} D_\xi [A^k D_{A^k} \psi] - Q^k_{i\nu} \pi^{k\nu}_1,$$

$$q_{n_1+\nu}(z) = \frac{1}{2}[x_\nu D_\xi[\hat{L}\psi(x) + \xi_\nu \hat{L}\psi(x)]] - Q^{k\mu}_\nu \pi^{k\mu}_1,$$

$$q_{n_1+n+4+r}(z) = D_\xi \psi c^p_r h_p + D_\xi (B^p_r h_p, D_A \psi) - Q^{k\nu}_{n+4+r} \pi^{k\nu}_1,$$

$$q_{2n_1+1}(z) = D_\xi \alpha(x).$$

Since M is an elliptic operator, one can easily check that

$$\kappa^{-1} \mathcal{N}^{(1-\kappa)/\kappa} M \mathcal{N} \le M^0 \mathcal{N}^{1/\kappa} + \kappa^{-1} \alpha \mathcal{N}^{1/\kappa}, \tag{47}$$

where $M^0 \mathcal{N} = M\mathcal{N} - \alpha\mathcal{N}$. Choosing $\pi^{k\nu}_1 = \lambda(K(y)\xi, Q^{k\nu})$ in (46), we derive

$$M^0 \mathcal{N}^{1/\kappa} + \kappa^{-1}\alpha\mathcal{N}^{1/\kappa} = \lambda[([MK(y)]\xi, \xi) + 2(D_Q K(y)\xi, D_\xi Q) +$$
$$+ (K(y) D_\xi Q, D_\xi Q) + 2(K(y)\xi, D_\xi q)] +$$
$$+ \lambda^2 \sum_{k,\nu} [(K(y)\xi, Q^{k\nu})^2 + \alpha\kappa^{-1}[\lambda(K(y)\xi, \xi) + |\xi_0|^2 + 1] -$$
$$- 2\lambda^2 (K(y)\xi, (K(y)\xi, Q^k_\nu) Q^k_\nu) + 2(\xi_0, D_\xi \alpha)].$$

Denote by $\Gamma_1(\xi)$ the function given by

$$-\Gamma_1(\xi) = ([MK(y)](\xi, \xi) + 2(D_Q K(y)\xi, D_\xi Q) +$$
$$+ (K(y) D_\xi Q, D_\xi Q) + 2(K(y)\xi, D_\xi q) +$$
$$+ \mu \sum (K(y)\xi, Q^k_\nu)^2 + \alpha\kappa^{-1}(K(y)\xi, \xi).$$

It is easy to check that

$$M^0 \mathcal{N}^{1/\kappa} + \kappa^{-1}\alpha\mathcal{N}^{1/\kappa} = -\lambda\Gamma_1(\xi) + \alpha\kappa^{-1}[|\xi_0|^2 + 1] -$$
$$+ \lambda^2 \sum (K(y)\xi, Q^{k\nu})^2 + 2\xi_0 D_\xi \alpha + \lambda\mu \sum (K(y)\xi, Q^k_\nu)^2. \tag{48}$$

We say that condition C.1 is satisfied if

1) $\Gamma_1(\xi) > 0$,

2) $(K(y)\xi, \xi)^{1/2}[D_\xi f + D_\xi \overline{\alpha}] \le \mu \Gamma_1(\xi)$,

3) $(K(x)\xi, \xi)|f| \le \mu \Gamma_1(\xi)$, $(K(y)\xi, \xi)\overline{\alpha} \le \mu \Gamma_1(\xi)$.

Since, for $\lambda > \mu$, the right-hand side of (46) does not exceed

$$-\lambda \Gamma_1(\xi) + \alpha \kappa^{-1}(K(y)\xi, \xi) + \lambda(\mu - \lambda) \sum (K(y)\xi, Q_\nu^k)^2 + 2\xi_0 D_\xi \alpha$$

on the set

$$\tilde{T}\mathscr{E} \cap \{(K(y)\xi, \xi) > |\xi_0|^2 + 1\},$$

where

$$\tilde{T}\mathscr{E} = \{(y, \tilde{\xi}) \in R^{2n_1+1}:\ y \in \mathscr{E},\ \tilde{\xi} = (\xi, \xi_0),\ \xi \in R^{n_1},\ \xi_0 \in R^1,\ D_\xi \overline{\Psi}(y) = 0\}.$$

We get

$$M\mathscr{N} \le (4\mu - \lambda)\Gamma_1(\xi)$$

due to C.1. On the other hand, given $\lambda \ge 1$, on the set $\tilde{T}\mathscr{E}$, we have

$$\mathscr{N} \le 2\lambda^\kappa (K(y)\xi, \xi)^\kappa,$$

$$\mathscr{N}^{(1-\kappa)/\kappa} \le 2^{(1-\kappa)/\kappa}\lambda^{1-\kappa}(K(y)\xi, \xi)^{1-\kappa} \le 4\lambda^{1-\kappa}(K(y)\xi, \xi)^{1/2},$$

$$\kappa^{-1}\mathscr{N}^{(1-\kappa)/\kappa} D_\xi f \le 8\lambda^{(1-\kappa)/\kappa}(K(y)\xi, \xi)^{1/2}[|D_\xi f| +$$

$$+ (K(y)\xi, \xi)|f|] < 16\mu\lambda^{1-\kappa}\Gamma_1(\xi).$$

This yields

$$\kappa^{-1}[M\mathscr{N} + D_\xi f]\mathscr{N}^{(1-\kappa)/\kappa} \le (4\mu - \lambda + 16\mu\lambda^{1-\kappa})\Gamma_1(\xi) < 0$$

for a suitable choice of $\lambda(\mu, \delta) \ge 1$.

Thus, we have proved the following statement:

Lemma 3. *Let condition* C.1 *be satisfied for* $y = (x, h) \in \mathscr{E}$, $(y, \tilde{\xi}) \in T\mathscr{E} \times R^1$. *Then there exists a constant* $\lambda(\delta, \mu) > 0$ *such that the function*

$$\mathscr{N}(y, \tilde{\xi}) = [\lambda(K(y)\xi, \xi) + |\xi_0|^2 + 1]^\kappa$$

with $\kappa = (2 - \delta)^{-1}$ *satisfies the estimate*

$$M\mathscr{N} + D_\xi g < 0 \tag{49}$$

on the set

$$\tilde{T}\mathscr{E} \cap \{(K(y)\xi, \xi) \ge |\xi_0|^2 + 1\} \tag{50}$$

if one chooses $\pi_1^{k\nu} = \lambda(K(y)\xi, Q^{k\nu})$.

It should be noted that Lemma 3 gives a possibility to use $\mathscr{N}$ as a barrier function for the first-order derivatives of the solution of the equation

$$M^0\Phi + \alpha\Phi + g = 0, \tag{51}$$

where M^0 is given by (18), (19).

To derive estimates for the second derivatives we should introduce some additional notation. Denote $E = T\mathscr{E}$ and let $T^2\mathscr{E} = TE$ be the second tangent bundle for $\mathscr{E}$. Let $\mathscr{H} = VTE$ denote a vertical subbundle of TE,

$$\mathscr{H} = \{(x, h, \xi, b, \eta, c)\}.$$

Note that, for $(\xi, \eta) \in VTU$,

$$D_\xi \overline{\Psi}(x) = D^2{}_{(\xi,\xi)} \overline{\Psi}(x) + D_\eta \overline{\Psi}(x) = 0.$$

Let $n(x) = D_x \overline{\Psi}(x) | D_x \overline{\Psi}(x)|^{-1}$, $\tilde{D}_{x_i} = D_{x_i} - n_i(x) D_n$, and let

$$H(x) = \delta_{ij} - n_i(x) n_j(x), \qquad \hat{K} = HKH,$$

$$\Gamma_2(\xi) = \Gamma_1(\xi) - \delta\left(\hat{K} D_\xi Q^k, D_\xi Q^k\right),$$

$$\Gamma_3(\xi) = (K\xi, \xi)\left[\Gamma_2(\xi) + \left(1 - \frac{\delta}{2}\right)\alpha(K\xi, \xi)\right] - \frac{1}{2}\sum_{\nu=n+1}^{n+4}\sum_{k=1}^{n}\left(D_{Q_\nu^k} K\xi + 2KD_\xi Q_\nu^k, \xi\right)^2.$$

One can easily see that $H(x)$ is the operator of projection onto the plane tangent to U at x. Let

$$\tilde{D}_\xi = D_\xi(\xi, n(x)) D_{n(x)}, \qquad \tilde{D}_{(\xi,\eta)} = \tilde{D}_\xi \tilde{D}_\xi + \tilde{D}_\eta.$$

Note, in addition, that, for $(y, \xi, \eta) \in VTU$, we have $D_\xi = \tilde{D}_\xi$ and $\tilde{D}^2_{(\xi,\eta)} = D^2_{(\xi,\eta)}$. Consider the function

$$\mathcal{M}(y, \bar{\xi}, \bar{\eta}) = \left[\lambda^2 (K(y)\xi, \xi)^2 + \lambda\left(\hat{K}(y)\eta, \eta\right) + |\xi_0|^4 + |\eta_0|^2 + 1\right]^\kappa \tag{52}$$

defined on $\mathscr{H} \times R^2 \subset T^2\mathscr{E} \times R^2$.

We say that condition C.2 is satisfied if

$$\begin{gathered}
\Gamma_2(\xi) > 0, \quad (K\xi, \xi)^2 \le \mu\Gamma_3(\xi), \quad \alpha \le 0, \ |g| \le \mu, \\
\left(M^0 \hat{K} n, n\right)^2 + \left(K D_{Q_\nu^k} n, D_{Q_\nu^k} n\right) \le \mu, \\
\left(M^0 \hat{K}\xi, n\right)^2 \le \mu\Gamma_2(\xi), \\
\left(D_\xi n, \xi\right)^2 + \left(\hat{K}\tilde{D}^2_\xi Q_\nu^k, \tilde{D}^2_\xi Q_\nu^k\right) \le \mu\Gamma_3(\xi), \\
(K\xi, \xi)\,|\tilde{D}^2_\xi \alpha| \le \mu\Gamma_3(\xi), \quad \left(K^1 \tilde{D}^2_\xi q, \tilde{D}^2_\xi q\right) \le \mu\Gamma_3(\xi), \\
(K\xi, \xi)\left|\tilde{D}^2_\xi \alpha\right|^2 \le \mu\Gamma_3(\xi), \quad (K\xi, \xi)\left(\left|D_\xi \alpha\right|^2 + \left|D_\xi g\right|^2\right) \le \mu\Gamma_3(\xi), \\
\left(D_{Q_\nu^k} K\xi_1, \tilde{D}^2_\xi Q_\nu^k\right) \le \mu\Gamma_2^{1/2}(\xi_1)\Gamma_3^{1/2}(\xi).
\end{gathered} \tag{53}$$

Lemma 4. *Assume that condition* C.2 *is satisfied. Then there exists a constant* $\lambda(\delta, \mu) \ge 1$ *such that*

$$M\mathcal{M} + D_{(\bar{\xi}, \bar{\eta})} g < 0 \tag{54}$$

on the set

$$\mathscr{L} = \mathscr{H} \times R^2 \cap \left\{(K\xi, \xi) \ge \left(K^1\eta, \eta\right) + |\xi_0|^4 + |\eta_0|^2 + 1\right\}.$$

Proof. Compute

$$\tilde{L}^0 \mathcal{M}^{1/\kappa} = \sum_{s=1}^{4} \mathcal{I}_s,$$

where

$$\mathcal{I}_1 = -2\lambda^2\Gamma_3(\xi) - 2\lambda^2\left[\frac{2-\delta}{2}\alpha + \mu\|\pi_1\|^2 - \delta\big(\hat{K}(y)D_\xi Q_\nu^k, D_\xi Q_\nu^k\big)\right](K(y)\xi, \xi),$$

$$\mathcal{I}_2 = \lambda\Big[\big(\mathcal{A}^0 \hat{K}(y)\eta, \eta\big) + \big(\hat{K}(y)D^2_{(\xi,\eta)} Q_\nu^k, D^2_{(\xi,\eta)} Q_\nu^k\big) +$$

$$+ 2\big(D_{Q_\nu^k} \hat{K}(y)\eta, D^2_{(\xi,\eta)} Q_\nu^k\big) + 2\big(\hat{K}(y)\eta, D^2_{(\xi,\eta)} q\big)\Big],$$

$$\mathcal{I}_3 = -4\lambda^2\big(K(y)\xi, \pi_{1\nu}^k D_\xi Q_\nu^k\big) - \lambda\Big[2\big(\hat{K}(y)\eta, Q_\nu^k \pi_{2\nu}^k\big) +$$

$$+ 6|\pi_1|^2|\xi_0|^2 + 4\pi_{1\nu}^k\big(\hat{K}(y)\eta, D_\xi Q_\nu^k\big) - 4\big(D_\xi \alpha |\xi_0|^3\big)\Big],$$

$$\mathcal{I}_4 = |\pi_2|^2 + 2\eta_0\big(D^2_{(\xi,\eta)}\alpha - |\pi_1|^2\big).$$

Fix $\lambda \geq 1$ and $(y, \tilde{\xi}, \tilde{\eta}) \in \mathcal{X}$, denote

$$\xi_1 = \eta + (D_\xi n(x), \xi)n(x) = H\eta,$$

and note that, for $(x, \xi) \in TU$, we have $D_{(\xi,\eta)} = \tilde{D}_\xi^2 + D_{\xi_1}$. To illustrate the considerations necessary to obtain the required estimates, we consider one of the terms in $\mathcal{I}_2$. Using the Cauchy–Buniakowski inequality, we can prove that

$$\big(D_{Q_\nu^k} \hat{K}(y)\eta, D^2_{(\xi,\eta)} Q_\nu^k\big) = \big(D_{Q_\nu^k} \hat{K}\xi_1, D_{\xi_1} Q_\nu^k\big) + (D_\xi n, \xi)\Big\{\big(\hat{K} D_{Q_\nu^k} n, D_{\xi_1} Q_\nu^k\big) +$$

$$+ \big(\hat{K} D_{Q_\nu^k} n, \tilde{D}_\xi^2 Q_\nu^k\big)\Big\} + \big(D_{Q_\nu^k} \hat{K}(y)\xi, \tilde{D}_\xi^2 Q_\nu^k\big) \leq$$

$$\leq \big(D_{Q_\nu^k} \hat{K}(y)\xi, D_\xi Q_\nu^k\big) + \varepsilon\Big\{\big(\hat{K} D_{\xi_1} Q_\nu^k, D_{\xi_1} Q_\nu^k\big) + \Gamma_2(\xi)\Big\} + N\varepsilon^{-1}\Gamma_3(\xi).$$

Similar considerations combined with estimates from C.2 yield

$$\kappa^{-1}\mathcal{N}^{(1-\kappa)/\kappa}\big[M\mathcal{N} + D_{(\tilde{\xi},\tilde{\eta})} g\big] \leq$$

$$\leq \lambda\big(N\varepsilon^{-1} - 2\lambda\Gamma_3(\xi)\big) + \Big(2\lambda\mu - \lambda\big(2\varepsilon - 8\varepsilon^2\big)(K\xi, \xi)\sum\nolimits_{k,\nu}\big(KQ_\nu^k, X\big)^2\Big) +$$

$$+ \Big(N\lambda^{1-2\kappa} - (1-\varepsilon)\Gamma_2(\xi) + (\mu - \lambda)\sum\nolimits_{k,\nu}\big(KQ_\nu^k, \xi_1\big)^2\Big).$$

It follows from the last inequality that one can choose $\varepsilon = \varepsilon(\delta)$ and $\lambda = \lambda(\delta, \mu)$ so that its right-hand side will be negative, which completes the proof of the lemma.

Theorem 1. *Suppose that the conditions of Lemma* 3 *are satisfied and* $\Phi \in C^2_{\text{loc}}(\mathbb{R}^{n_1})$. *Assume, in addition, that there exists* $K(y)\colon R^{n_1} \to R^{n_1}$ *such that* $(K(y)\xi, \xi) > 0$ *for* $\|\xi\| \neq 0$ *on* $T\mathcal{E}$ *and*

$$\alpha \leq 0, \qquad |\Phi(y)| \leq \mu\|h\|, \qquad |D_\xi \Phi|^2 \leq C(K\xi, \xi)\|h\|^2$$

on E' *for some* $C > 0$.

Let $\Phi(y)$ *be a solution of* (51) *and assume that there exists a nonnegative function* $\psi_1 \in C^2_{\text{loc}}(R^{n+4})$ *such that* $M\psi_1 \leq 0$ *on* U. *Then*

$$\left|D_{\tilde{\xi}}\Phi\right|^2 \le \left[\tilde{\mu}^2 + N\right]\left[(K(y)\xi,\xi) + |\xi_0|^2\right] \tag{55}$$

for $(y,\tilde{\xi}) \in T\mathscr{E}\times R^1$, *where* N *depends only on* δ *and* μ.

Proof. Let $\varepsilon > 0$ and let $\mathcal{N}(x,h,\xi)$ be given by (45). For $y = (x,h) \in R^{n_1}$, we define

$$\Lambda(y,\tilde{\xi}) = D_{\tilde{\xi}}\Phi(y) - \mathcal{N}(y,\tilde{\xi}) - \varepsilon\psi_1(x). \tag{56}$$

We have to consider three possible cases:

1. $\Lambda \le 0$ on $T\mathscr{E}\times R^1$.

2. $\Lambda > 0$ at some points from $T\mathscr{E}\times R^1$ but it does not attain its upper bound on $T\mathscr{E}\times R^1$.

3. There exists a point $\left(y^0,\tilde{\xi}^0\right) \in T\mathscr{E}\times R^1$ at which Λ takes its maximal value and $\Lambda\left(y^0,\tilde{\xi}^0\right) > 0$.

First, we consider the last case. It follows from the maximum principle that the estimate $M\Lambda\left(y^0,\tilde{\xi}^0\right) - \alpha\Lambda\left(y^0,\tilde{\xi}^0\right) \le 0$ holds at $\left(y^0,\tilde{\xi}^0\right)$ and, since α is negative, we derive $M\Lambda \le 0$. Hence,

$$\varepsilon M\psi_1 + M\mathcal{N} \ge M D_{\tilde{\xi}}\varphi.$$

On the other hand,

$$M D_{\tilde{\xi}}\Phi + D_{\tilde{\xi}} g = 0$$

yields

$$\varepsilon M\psi_1 + M\mathcal{N} + D_{\tilde{\xi}} f \ge 0$$

which contradicts (49).

Thus, it follows from (49) and (50) that

$$\left(K(y^0)\xi^0,\xi^0\right) < \|\xi_0\|^2 + 1. \tag{57}$$

Let us differentiate Λ with respect to $\tilde{\xi}$ along U'. Then

$$D_{\tilde{\zeta}}\Lambda\left(y^0,\tilde{\xi}^0\right) = -\kappa\mathcal{N}^{(\kappa-1)/\kappa}\left(y^0,\tilde{\xi}^0\right)\left[2\lambda\left(K(y^0)\tilde{\xi}^0,\tilde{\zeta}^0\right) + 2\zeta_0^0\xi_0^0\right] + D_{\tilde{\xi}}\Phi = 0,$$

which implies

$$D_{\zeta}\Phi(y^0) = 2\lambda\kappa\mathcal{N}^{(\kappa-1)/\kappa}\left(y^0,\xi^0\right)\left(K(y)\xi^0,\zeta\right),$$

$$\Phi(y^0) = 2\kappa\mathcal{N}^{(\kappa-1)/\kappa}\left(y^0,\xi^0\right)$$

and

$$\Lambda\left(y^0,\xi^0\right) = 2\kappa\mathcal{N}^{(\kappa-1)/\kappa}\left(y^0,\xi^0\right)\left[\lambda\left(K(y)\xi^0,\xi^0\right) + \left|\xi_0^0\right|^2\right] -$$

$$- \mathcal{N}\left(y^0,\xi^0\right) - \varepsilon\psi_1(x^0) \le 2(\kappa-1)\mathcal{N}\left(y^0,\xi^0\right)$$

because $\mathcal{N} > 1$ and

$$\mathcal{N}^{1/\kappa}\left(y^0,\xi^0\right) = \left[\lambda\left(K(y^0)\xi^0,\xi^0\right) + \left|\xi_0^0\right|^2 + 1\right] > 1.$$

Finally, the last relations lead to

$$\mathcal{N}^{1/\kappa}\left(y^0,\xi^0\right) \le (\lambda+1)\left[\left|\xi_0^0\right|^2 + 1\right],$$

$$\mu \ge 2\kappa\xi_0^0\left[(\lambda+1)\left|\xi_0^0\right|^2+1\right]^{\kappa-1},$$

and

$$\left|\xi_0^0\right| \le N(\mu, \delta).$$

We have shown that, on $T\mathscr{E}\times R^1$, the following estimate is true:

$$\Lambda\left(y, \tilde{\xi}\right) \le \Lambda\left(y^0, \tilde{\xi}^0\right) \le 2(\kappa-1)(\lambda+1)^{\kappa}\left[\left|\xi_0^0\right|^2+1\right]^{K} \le N(\mu, \delta). \tag{58}$$

By taking $N(\mu, \delta) > 0$, one can establish that the last estimate also holds in the case 1).

To investigate the situation in the case 2), let us choose an arbitrary sequence of points $\left(y^k, \tilde{\xi}^k\right)$ in $T\mathscr{E}\times R^1$ to maximize $\Lambda\left(y, \tilde{\xi}\right)$ on this set. Note that, by the conditions of the theorem,

$$\left|D_{\tilde{\xi}}\Phi\right| \le M(K(y)\xi, \xi)^{1/2} + \mu\left|\xi^0\right|$$

and $\kappa \ge 1/2$. Hence, if $\Lambda\left(y^k, \tilde{\xi}^k\right)$ does not tend to ∞, then $\left(K(y^k)\xi^k, \xi^k\right)$, $\left|\xi_0^k\right|$, and $D_{\tilde{\xi}}\Phi(y^k)$ are bounded. Furthermore, $\Lambda\left(y^k, \tilde{\xi}^k\right) \ge 0$ and $D_{\tilde{\xi}^k}\Phi(y^k) > \varepsilon\psi_1(x^k)$ starting with a number $\tilde{k}$.

Assume that $U_\rho = \{x \in U\colon \psi_1(x) < \rho\}$ is a bounded set and denote $\overline{U}_\rho = \{x \in U\colon \psi_1(x) = \rho\}$. Then $\overline{U}_\rho$ is a compact set and, by assumption, $(K(y)\xi, \xi) \ge \delta_\rho\left\|\tilde{\xi}\right\|^2$ for $y \in \mathscr{E}_\rho = \{(x, h) \in \mathscr{E}\colon x \in \overline{U}_\rho, \|h\| = 1\}$. In addition, it follows from the boundedness of $\left(K(y)\xi^k, \xi^k\right)$ that both $\|y^k\|$ and $\|\xi^k\|$ are also bounded. We can now assume that there exists $(y^0, \xi^0) = \lim\left(y^k, \tilde{\xi}^k\right)$. Evidently, $\left(y^0, \tilde{\xi}^0\right) \notin T\mathscr{E}\times R^1$ (we consider the case 2)) and $D_{y^0}\overline{\psi}(x^0) = 0$ if $x^0 \in \{x \in R^{n+4}\colon 0 < \overline{\psi}(x) = \rho\}$.

This implies the required estimate on $T\mathscr{E}\times R^1$:

$$\Lambda\left(y, \tilde{\xi}\right) \le \lim_{k\to\infty}\Lambda\left(y^k, \tilde{\xi}^k\right) \le I(\mu_1) = \sup_{s\ge 0}\left[\mu_1 s-(s^2+1)^{\gamma}\right], \tag{59}$$

where

$$s^2 = \lambda\left(K(y^0)\xi^0, \xi^0\right) + \left|\xi_0^0\right|^2. \tag{60}$$

The investigation of three possible cases above leads to the following conclusion: At each point of the set $\mathscr{E}\times R^1$, either (58) or (59) holds, which implies that $\Lambda\left(y, \tilde{\xi}\right) \le N \vee I(\mu_1)$. Since ε can be chosen arbitrarily, we get

$$D_{\tilde{\xi}}\Phi - \mathscr{N} \le N \vee I(\mu_1)$$

on $T\mathscr{E}\times R^1$.

Let us rewrite the last inequality in the equivalent form

$$\mathscr{P}s - (s^2+1)^{\kappa} \le N \vee I(\mu_1),$$

where s is given by (60) and

$$\mathscr{P}(y, \tilde{\xi}) = s^{-1}\left[D_{\tilde{\xi}}\Phi(y)\right].$$

As a consequence, we have $I(\mathcal{P}) \leq N \vee I(\mu_1)$. To complete the proof of (55), which is equivalent to $\mathcal{P} \leq N \vee \mu_1^{1/2}$, it remains to note that, since $\alpha(s) = (s^2+1)^\kappa$ is a strictly convex function, there exists a number μ_0 such that $I(\mu)$ strictly grows in μ for $\mu \geq \mu_0$. In addition, μ_0 can be defined by

$$2\kappa s(s^2+1)^{\kappa-1} = \mu_0,$$

i.e., the line $\mu_0 s$ is the tangent line to the graph of $(s^2+1)^\kappa$.

Let us return to the estimates of second derivatives and start with some useful notation. For $(y, \hat{\xi}) \in T\mathcal{E}$, we denote

$$\tilde{D}_\xi \Phi(y) = D_\xi \Phi(y) - \left(n(x), \hat{\xi}\right) D_n \Phi(y)$$

and

$$\tilde{D}^2_{(\hat{\xi}, \eta)} = \tilde{D}_\xi \tilde{D}_\xi + \tilde{D}_\eta .$$

Theorem 2. *Suppose that* $\Phi(y)$ *solves* (51) *and the conditions of Lemma* 4 *are satisfied. Also assume that*

$$|\Phi(y)| \leq \mu, \qquad |D_\xi \Phi(y)|^2 \leq N\left(K(y)\hat{\xi}, \hat{\xi}\right),$$

and $|D_\xi \Phi(y)| \leq N\left(K(y)\hat{\xi}, \hat{\xi}\right)$ *for some constant* N *and* $\left(K(y)\hat{\xi}, \hat{\xi}\right) > 0$. *Let there exist* $\psi_1 \in C^2_{\text{loc}}(U)$ *such that* $\psi_1(x) > 0$ *and* $M\psi_1 \leq 0$. *Then, on* U'',

$$\left|D^2_\xi \Phi(t, y)\right| \leq N(\delta, \mu)\left(K(y)\hat{\xi}, \hat{\xi}\right).$$

The proof of this theorem is quite similar to the proof of the previous one. In fact, we have to consider the function

$$\mathfrak{Q}\left(y, \tilde{\hat{\xi}}, \tilde{\eta}\right) = \tilde{D}_{(\tilde{\hat{\xi}}, \tilde{\eta})} \Phi(y) - \mathcal{M}\left(y, \tilde{\hat{\xi}}, \tilde{\eta}\right) - \varepsilon \psi_1(x)$$

and investigate three cases similar to those considered in the proof of Theorem 1.

5. *A priori* estimates for solutions of a nonlinear system. In this section, we return to the problem we have started with and show what sort of results can be deduced from the estimates obtained in previous sections.

Assume that $\kappa, \rho, \delta \in (0, 1)$, $\mu \geq 1$, $n, m \geq 1$ are integers, $\psi \in C^2_{\text{loc}}(R^n)$, $G = \{x \in R^n : \psi(x) > 0\}$ is a nonempty bounded domain, $\psi \in C^2_{\text{loc}}(G)$, and $\|D_x \psi\| > 0$ on ∂G. Assume that the real-valued functions

$$A_i^k(r, x), \quad a_i(r, x), \quad B_l^{ks}(x), \quad c_l^s(r, x), \quad \alpha(r, x), \quad f_l(r, x)$$

are defined for any $x \in G$ and $r \in \mathcal{R}$, where $\mathcal{R}$ is a given set, $i, k = 1, \ldots, n$, and $s, p = 1, \ldots, m$.

Our goal is to prove *a priori* estimates for the solutions of

$$F_l(u_{lx_i x_j}, u_{x_i}, u, x) = 0 \tag{61}$$

in a bounded domain G with the boundary conditions $u_l(x) = 0$ on ∂G. Here,

$$F_l[x] = F_l(u_{lx_i x_j}, u_{x_i}, u, x) = \inf_r \left[\left(L^0(r, x)u\right)_l + \alpha(r, x)u_l + f_l(r, x)\right],$$

where

$$(L^0 u)_l = \frac{1}{2} A_i^k D_{x_i x_j} u_l A_j^k + a_i D_{x_i} u_l + B_l^{kp} A_i^k D_{x_i} u_p + c_l^p u_p.$$

As it has been done before, we change (61) for

$$\mathscr{A}[\Phi] = \mathscr{A}\left(D_y^2 \Phi, D_y \Phi, \Phi, y\right) = 0; \tag{62}$$

where

$$\Phi(y) = (h, u(x)), \quad \alpha(r, y) \equiv \alpha(r, x), \quad g(r, y) = (h, f(r, x)),$$

$$\mathscr{A}[\Phi] = (h, F[u]) = \inf_r \left[M^0(r, y)\Phi(y) + \alpha(r, y)\,\Phi(y) + g(r, y)\right],$$

and M^0 is determined by (5), (6).

Given operators L^0 and M^0, we denote $(Lu)_l = (L^0 u)_l + \alpha u_l$, $M\Phi = M^0 \Phi + \alpha \Phi$, and $\sigma_2^2(A) = \operatorname{Tr} AA^*$.

We say that condition C.5 is satisfied if the following groups of estimates are valid:

5.1. a) The functions A, a, B, c, α, f are continuous in $\overline{G}$ for any $r \in \mathfrak{R}$,

$$x \in G, \quad \xi \in R^n, \quad h \in R^m, \quad \|\xi\| = 1, \quad \|h\| = 1,$$

$$\sum_{k=1}^{n} \left\| B^k h \right\|^2 + (ch, h) \le -\delta_1,$$

$$\sigma_2^2(D_\xi Q) + \sigma_2^2(Q) + \left\| D_\xi q \right\| + \|q\| \le \mu,$$

$$\sigma_2^2\left(D_\xi^2(Q)\right) + \left\| D_\xi^2 q \right\|^2 + \left| D_\xi^2 \alpha \right| + \|g\| + \left\| D_\xi g \right\| + \left\| D_\xi^2 g \right\| \le \mu.$$

b) There exist matrix functions $K^i(y)$, $i = 1, 2$, twice continuously differentiable in $\overline{G} \times R^m$ and such that, for any $x \in \overline{G}$, $\xi \in R^n$, $h \in R^m$, $y = (x, h)$ with $\|\xi\| = 1$, $\|h\| = 1$, we have

$$\|K^i\| + \|D_\xi K^i\| + \left\| D_\xi^2 K^i \right\| \le \mu.$$

5.2. a) For any $r \in \mathfrak{R}$ and $x \in \partial G$, $\|D_x \psi\| \ge \delta$ and $L(r, x)\psi(x) \le -\delta$.

b) There exists a function $\psi_1(x)$ such that, for any $r \in \mathfrak{R}$ on G,

$$\psi_1(x) \ge \delta, \quad L(r, x)\psi_1(x) \le -\delta.$$

c) For any $r \in \mathfrak{R}$, $x \in G$, and $\xi \in R^{n+m}$,

$$(K^0 \xi, \xi) \ge \delta \|\xi\|^2,$$

$$M(r)(K^0 \xi, \xi) + 2(K^0 \xi, q_\xi) + 2\left(D_{Q^s} K^0 \xi, D_\xi Q^s\right) + (K^0 D_\xi Q^s, D_\xi Q^s) \le$$

$$\le -\delta \|\xi\|^2 + \mu \sum_{s=1}^{4n} \left(K^0 \xi, Q^s\right)^2.$$

d) For any $x \in \partial G$, $\xi \in R^n$, $\xi \perp D_x \psi$, $\|\xi\| = 1$, $\|h\| = 1$, $r \in \mathfrak{R}$,

$$(K^1 \xi, \xi) \ge \delta,$$

$$L\psi + \operatorname{Tr} PK^1 \le -\delta + \mu(PD_x \psi, D_x \psi), \quad P_{ij} = \sum_k Q_i^k Q_j^k,$$

$$(K^1\xi,\xi)M\psi + \sum_k \left(D_\xi D_{Q^k}\psi\right)^2 + 2(K^1\xi, Q^k)D_\xi D_{Q^k}\psi \le -\delta + \mu(P\psi_x, \psi_x).$$

Denote $S_1 = \{h \in R^m : \|h\| = 1,\ h_k \ge 0,\ k = 1, \ldots, m\}$ and $\overline{\Phi}(y) = \Phi(y)/\psi(y)$ and recall that $\overline{M}(\overline{\Phi}) = M\Phi$ for $\overline{M}\Phi = \overline{M}^0\Phi + \overline{\alpha}\Phi$ with $\overline{M}^0$ given by (18), (19). To derive *a priori* estimates for the solution to (61), we prove the following assertions:

Theorem 3. *Let* 5.1 *and* 5.2 *a), b), c) be satisfied and let* $\Phi \in C^2(\overline{G} \times R^m)$ *be a solution of* (62) *in* G. *Then for any* $x \in G$, $h \in S_1$, *and* $y \in R^{n+m}$, *we have*

$$|\Phi| + |D_y\Phi| \le N(\mu,\delta)\left[1 + \max_{\partial G\times S_1}\left(|\Phi| + |D_y\Phi|\right)\right]. \tag{63}$$

Proof. Consider the function

$$T_1(y) = \sup_{\xi\ne 0}\frac{D_\xi\overline{\Phi} + \xi_0\overline{\Phi}}{\left((K(y)\xi,\xi) + \xi_0^2\right)^{1/2}}. \tag{64}$$

It can be deduced from Theorem 1 that $T_1^2(y)$ is a viscosity subsolution of the equation $\mathcal{A}[\Phi] = 1$ on the set $G \times S_1 \cap \{T_1 > N\}$. Assume that the maximal value M of $T_1(y)$ on $\overline{G} \times S_1$ is attained at a point y_0. The function $\Phi(y) \equiv M$ is a supersolution of $\mathcal{A}[\Phi] = 0$ and it follows from the definition of a subsolution that y_0 does not belong to $G \times S_1 \cap \{T_1 > N\}$. Therefore, y_0 belongs either to $\{T_1 < N\}$ or to $\partial G \times S_1$. In both cases, we have

$$M \le N\left[1 + \max_{\partial G\times S_1}\left(|\Phi| + |D_\xi\Phi|\right)\right],$$

which proves the assertion of the theorem.

To derive estimates for the second mixed derivatives on the boundary we introduce some additional notation. Given $\rho > 0$, denote $G(\rho) = \{x \in G : \operatorname{dist}(x, \partial G) > \rho\}$, $\Delta_\rho G = \{x \in R^n : 0 < \psi(x) < \rho\} = G \setminus \overline{G}(\rho)$, and $\Theta_\rho(G) = \Delta_\rho G \times S_1$. Let $U(\rho) = \{x \in R^{n+4} : \overline{\psi} = 0,\ (x_1, \ldots, x_n) \notin \overline{G}(\rho)\}$ and $\Gamma = \{x \in R^{n+4} : (x_1, \ldots, x_n) \in \partial G\}$.

For our purpose, it would be convenient to slightly change the description of the set G by using a new function $\tilde{\psi}(x) = \psi(x) - \tau\psi^2(x)$. To verify that $G = \{x \in R^n : \tilde{\psi}(x) > 0\}$ and $\Delta_{\kappa_1} G = \{x \in R^n : 0 < \tilde{\psi} < \kappa_1\}$ consider an infinitely differentiable function $\chi(t)$ defined on $(-\infty, \infty)$ by

$$\chi(t) = \begin{cases} t - \tau t^2, & \text{if } t \le (8\tau)^{-1}, \\ (8t)^{-1}, & \text{if } t \ge (4\tau)^{-1}, \end{cases}$$

and $\chi'(t) \ge 0$. It is evident that $G = \{\chi(\psi) > 0\}$ and, given $\kappa_1 \le (8\tau)^{-1}$, we have $\Delta_{\kappa_1} G = \{0 < \psi < \kappa_1\} = \{0 < \chi(\psi) < \chi(\kappa_1)\}$ and, in addition, $\chi(\psi) = \tilde{\psi}$ on $\Delta_{\kappa_1} G$. Introduce the function

$$J = \psi M^0(K\xi,\xi) + D_{Q^k}\psi\left(D_{Q^k}K\xi,\xi\right) + D_\xi\psi\left[\left(D_{Q^k}K\xi, Q^k\right) + \left(KQ^k, D_\xi Q^k\right)\right] +$$
$$+ \psi\left[2\left(D_{Q^k}K\xi, D_\xi Q^k\right) + \left(KD_\xi Q^k, D_\xi Q^k\right)\right] + \sum_k \left|D_\xi D_{Q^k}\psi\right|^2 +$$

$$+\ 2\big(K\xi,\, D_\xi\big[Q^k D_{Q^k}\psi\big]\big) + 2\big(K\xi,\, D_\xi(\psi q)\big) +$$

$$+\ \frac{1}{2} D_\xi\psi D_\xi\big(\hat{L}(\psi)\big) + s^2\big[\big(KQ^k, Q^k\big) + \hat{L}(\psi)\big].$$

It can be checked that, for all $s^2 \geq |D_\xi\psi|^2/4\psi$, 5.2 d) is equivalent to

$$J < 2(1-\delta)|\tilde{\alpha}|\big((K\xi,\xi) + s^2\big) +$$

$$+\ \mu\left[\psi \sum (K\xi, Q^k)^2 + \frac{1}{2} D_\xi\psi (K\xi, Q^k) D_{Q^k}\psi + \frac{1}{4} s^2 \sum_k |D_{Q^k}\psi|^2\right].$$

A careful analysis of the right-hand side of the last inequality shows that, for such s, one can find $\delta_1 \in (0, 1)$, $\rho_0 \in (0, \kappa)$, and a positive constant $C = C(\mu, \delta)$ such that, for $x \in \Delta_{\rho_0}(G)$,

$$J \leq -\delta_1\big[s^2 + \|\xi\|^2\big] + C\big|D_\xi\psi\big|^2. \tag{65}$$

Next, consider

$$T_2(y) = \sup_{(\xi,s)\in\Theta(y)} \frac{D_\xi\overline{\Phi} + \xi_0\overline{\Phi}}{\big((K(y)\xi,\xi) + s^2 + \xi_0^2\big)^{1/2}}, \tag{66}$$

where

$$\Theta(y) = \left\{\big(\tilde{\xi}, s\big) : \tilde{\xi} \in R^{n+1},\ \big\|\tilde{\xi}\big\| \neq 0,\ s^2 \geq \frac{|D_\xi\psi(x)|^2}{4\psi(x)}\right\}.$$

In what follows, we assume that the operators $\overline{L}$ and $\overline{M}$ are defined by (16), (17) and (18), (19) with ψ replaced by $\tilde{\psi} = \psi - \tau\psi^2$, though we omit the sign $\tilde{}$.

Denote by

$$G[\Phi](y) = \sup_{r\in\mathcal{R}} (M(r, y)\Phi(y)),$$

$$\overline{\Phi}(y) = \Phi(y)\psi^{-1}(x).$$

Lemma 5. *Suppose that condition C.5 is satisfied and (63) holds. Assume that* $\Phi \in C^2\big(\overline{\Delta_{\rho_0}G} \times R^m\big)$ *is a solution of (62) in* $\Delta_{\rho_0}G \times R^m$. *Then there exist* $\rho \in (0, \rho_0)$ *and* $C_1 > 0$ *depending only on* ρ_0, δ, μ *such that, on the set* $\{T_2(y) > N_1\} \cap \Delta_\rho G \times S_1$, *the function* $T_2(y)$ *is a viscosity subsolution of the equation* $G[\psi\Phi] = 1$. *If, in addition, we assume that, for any* $y \in \Delta_{\rho_0}G \times S_1$, *equation (62) is uniformly elliptic, then the function* $T_2(y)$ *is bounded above on* $\Delta_{\rho_0}G \times S_1$.

Proof. The assertion of the lemma is a consequence of Theorem 1. To explain it we must choose $\psi_1 = \psi^{-1}$, $\psi_2 = 1$, use the equality $\overline{M}\overline{\Phi} = M\Phi$, which holds on $U(\rho)\backslash\Gamma \times R^m$, and check that C.1 is satisfied. To this end, note that $M\psi_2 = M1 = \overline{\alpha} \leq -\delta_1$ on $U(\rho) \times R^m$

Let $z = (x, y) \in U(\rho)$, $\zeta = (\xi, \eta) \perp D\tilde{\psi}(z)$. It follows from (63) that, in this case,

$$J \leq -\delta_1\big[\|\xi\|^2 + \|\eta\|^2\big] + N|D_\xi\psi|^2 - \frac{1}{2}\delta_1\|\eta\|^2.$$

Since, for $z \in U(\rho)$, we have $\psi(x) = \|y\|^2$, $2(\eta, y) = D_\xi \psi(x)$, and, as a result, $|D_\xi \psi(x)|^2 \le 4\psi(x)\|\eta\|^2$, one can verify that, for sufficiently small $\rho = \rho(\delta, \mu, \rho_0)$, we have $J \le -\delta_1$. To check the other assumptions note that if $z = (x_1, \ldots, x_{n+4}) \in U(\rho)$ and $\zeta = (\xi, \eta) \in R^n \times R^4$, then, for $w_l(z) = w_l(x_1, \ldots, x_n)$, we get

$$(\overline{D}_z \overline{\psi}, D_z w_l) = (D_x \psi, D_x w_l),$$

$$|D_z \overline{\psi}|^2 = |D_x \psi|^2 + 4 \sum_{\nu=n+1}^{n+4} x_\nu^2 = |D_x \psi|^2 + 4\psi,$$

$$\tilde{D}_\zeta w_l(z) = D_\xi w_l - \frac{(\zeta, D_z \overline{\psi})}{\|D_z \overline{\psi}\|^2} (D_x \psi, D_x w_l).$$

Let $\left(K(y)\hat{\zeta}, \hat{\zeta}\right) = \left(K^1(x)\xi, \xi\right) + \|\eta\|^2 + \|h\|^2 + \|b\|^2$ for $(y, \hat{\zeta}) = (x, h, \zeta, b) \in T\mathscr{E}$. To obtain the estimate for

$$T_3(x) = \sup_{\hat{\xi} \in T\mathscr{E}, \|\hat{\xi}\|=1, \|h\|=1} \frac{D_{\hat{\zeta}} \tilde{\Phi} + \zeta_0 \Phi}{\left((K_1 \xi, \xi) + \|\eta\|^2 + \|h\|^2 + |\xi_0|^2\right)^{1/2}}$$

we can use Lemma 5. To prove the boundedness of T_3 we must meet the nondegeneracy requirement for the diffusion operator on $U(\rho)$. This can be done by "correcting" $Q_{l\nu}^k$, $l = 1, \ldots, n+4$, in a way similar to that used in [11] to ensure that the resulting operator is nondegenerate and coincides with the original one on functions depending only on $(x_1, \ldots, x_n)$.

Theorem 4. *Suppose that condition* C.5 *is satisfied and* (63) *holds for* $\rho_0 \in (0, 1)$. *Let* $\Phi \in C^2\left(\overline{\Delta_{\rho_0} G} \times S_1\right)$ *be a solution of* (62) *in* $\Delta_{\rho_0} G$ *such that* $\Phi(y) = 0$ *if* $x \in \partial G$. *Assume that, for any* $x \in \Delta_{\rho_0} G$, *we have* $\inf_r P_{ij}(r, y)\xi_i \xi_j > 0$. *Then there exist constants* $\rho = \rho(\delta, \mu, \rho_0) \in (0, \rho_0)$, $C = C(\delta, \mu, \rho_0)$ *such that, for any* $x \in \partial G$, $h \in S_1$, *and* $\xi \in R^{n+m}$, $\|\xi\| = 1$, *we have*

$$|D_\xi \Phi| + |D^2_{\xi h} \Phi| \le C\left[1 + \max_{\partial G(\rho) \times S_1} \left(|\Phi| + |D_\xi \Phi|\right)\right]. \tag{67}$$

Proof. Let ψ, T_3, ρ be the same as in Lemma 5. Note that the function $\Phi = \psi^{-1}(x)$ is a viscosity supersolution of $G[\psi\Phi] = 1$. If follows from Lemma 5 that the function T_3 is bounded and, for any $\varepsilon > 0$, the function $T_3^2 - \varepsilon\psi^{-1}$ attains its maximal value over the set $\left(\overline{\Delta_\rho G} \setminus \partial G\right) \times S_1$.

Let us fix $\varepsilon > 0$. Denote by $y_\varepsilon \in \left(\overline{\Delta_\rho G} \setminus \partial G\right) \times S_1$ the point where $T_3^2 - \varepsilon\psi^{-1}$ attains its maximal value Q_ε. Since $Q_\varepsilon + \varepsilon\psi^{-1}$ is a supersolution of $G[\psi\Phi] = 1$ and T_3^2 is a viscosity subsolution of this equation on $\{T_3 > G_1\} \cap \Delta_\rho G$, it follows that y_ε does not belong to this set. Hence, $y_\varepsilon \in \partial G(\rho) \cup \{T_3 \le C_1\}$ and

$$T_3^2 - \varepsilon\psi^{-1} \le C_1^2 + \max_{\partial G(\rho) \times S_1} T_3^2.$$

Let $\varepsilon \to 0$ and let

$$W_1 = 1 + \max_{\Theta(\rho)} |\overline{\Phi}| + \|D_y \overline{\Phi}\|.$$

For $y \in \Delta_p G \times S_1$, we have

$$\|D_y\overline{\Phi}\| + |\overline{\Phi}| \le CT_3\big[\|\xi\| + |D_\xi\psi|\psi^{-1} + 1\big] \le CW_1\big[\|\xi\| + |D_\xi\psi|\psi^{-1} + 1\big].$$

To prove (67) we choose $\xi = \tau \perp D_x\psi$, $\hat{\xi} = (\tau, b) \in R^{n+m}$, $\|\hat{\xi}\| = 1$, to derive

$$\|D_{\hat{\tau}}\overline{\Phi}\| + |\overline{\Phi}| \le CW_1.$$

Finally, letting $x \to \partial G$, we get (67) due to

$$\lim_{x \to \partial G} \|D_y\overline{\Phi}\| + |\overline{\Phi}| = \|D_y\psi\|^{-1}\big[D_{\hat{\tau}} D_{\hat{n}} \Phi + D_y\Phi\big].$$

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