UDC 517.9

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ONE CLASS OF SOLUTIONS OF VOLTERRA EQUATION WITH REGULAR SINGULARITY

ПРО ОДИН КЛАС РОЗВ'ЯЗКІВ РІВНЯННЯ ВОЛЬТЕРРА З РЕГУЛЯРНОЮ СИНГУЛЯРНІСТЮ

The Volterra integral equation of the second genus with regular singularity is considered. Let the kernel $K(x, t)$ be a real matrix function of order $n \times n$ with continuous partial derivatives up to order $N+1$ inclusively and let $K(0, 0)$ have complex eigenvalues $\nu \pm i\mu$ $(\nu > 0)$. It is shown that if $\nu > 2\,|||K|||_C - N - 1$, then two linear independent solutions of a given equation exist.

Розглядається інтегральне рівняння Вольтерра другого роду з регулярною сингулярністю. У припущенні, що ядро $K(x, t)$ — дійсна матричнозначна функція порядку $n \times n$ з неперервними частинними похідними до порядку $N+1$ включно, і $K(0, 0)$ має комплексні власні значення $\nu \pm i\mu$ $(\nu > 0)$. Показано, що коли $\nu > 2\,|||K|||_C - N - 1$, тоді існують два лінійно незалежних розв'язки даного рівняння.

Investigations in the theory of Volterra integral equations with singularities appeared at the beginning of this century. Increasing field of its application in 1960s stimulated its further development.

The theory of linear Volterra integral equations with singularities was substantially developed in a number of papers by T. Sato, T. Takesada, L. Panov and N. Magnitsky [1–6].

In recent years, equations with real coefficients with finite smoothness were investigated at the Voronezh Forest Industry Academy. The equations were considered both in finite-dimensional spaces and in Banach spaces.

The present paper is devoted to the study of solutions of the Volterra equation with regular singularity

$$xu(x) = \int_0^x K(x, t)u(t)\,dt \quad (0 \le x \le T), \tag{1}$$

where the kernel $K(x, t)$ is a given real-valued sufficiently smooth matrix-valued function of order $n \times n$, $K(0, 0)$ has complex eigenvalues, and $u(x)$ is an unknown summable n-dimensional real-valued vector function.

The structure of a solution of equation (1) substantially depends on the algebraic properties of the matrix $K(0, 0)$. In the space $\mathbb{R}^n$, we fix the norm

$$\|u\|_{\mathbb{R}^n} = \max_{1 \le i \le n} |u_i|.$$

This norm induces the operator norm

$$|||A||| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

In the space $C(\mathbb{R}^n)$ of all vector functions $\psi(x)$ continuous on $[0, T]$ the norm, as usual, is determined by the formula

$$\|\psi\|_{C(\mathbb{R}^n)} = \max_{0 \le x \le T} \|\psi(x)\|_{\mathbb{R}^n}.$$

Finally, in the space C consisting of all $n \times n$ matrix functions $Q(x, t)$ continuous in the triangle $0 \le t \le x \le T$, the norm is denoted by

$$|||Q|||_C = \max_{0\le t\le x\le T} |||Q(x,t)|||.$$

Let the kernel $K(x,t)$ have continuous partial derivatives up to order $N+1$ inclusive, let $K(0,0)$ have an eigenvalue $\nu + i\mu$ $(\nu > 0)$ with an eigenvector $\overline{e} = e_1 + i e_2$, where ν satisfies the inequalities

$$\nu > 2|||K|||_C - N - 1. \tag{2}$$

Let us try to find a solution of equation (1) in the form

$$u(x) = x^{\nu-1}\Bigg(\sum_{i=0}^{N} a_i x^i \sin(\mu\ln x) + \sum_{i=0}^{N} b_i x^i \cos(\mu \ln x) + a_{N+1}(x)x^{N+1}\sin(\mu\ln x) + b_{N+1}(x)x^{N+1}\cos(\mu\ln x)\Bigg), \tag{3}$$

where a_i and b_i $(i = 0, 1, \ldots, N)$ are unknown vector coefficients, and vector functions $a_{N+1}(x)$, $b_{N+1}(x)$ are continuous on $[0, T]$.

Assume that the kernel $K(x,t)$ can be represented by the Taylor formula

$$K(x,t) = \sum_{\alpha+\beta=0}^{N} K^{\alpha\beta}x^{\alpha}t^{\beta} + \sum_{\alpha+\beta=N+1} \tilde{K}^{\alpha\beta}(x,t)x^{\alpha}t^{\beta} \quad (K^{00} = K(0,0)). \tag{4}$$

Substituting (3) and (4) into (1), we get on the left-hand side

$$x^{\nu}\Bigg(\sum_{i=0}^{N} a_i x^i \sin(\mu\ln x) + \sum_{i=0}^{N} b_i x^i \cos(\mu\ln x) + a_{N+1}(x)x^{N+1}\sin(\mu\ln x) + b_{N+1}(x)x^{N+1}\cos(\mu\ln x)\Bigg).$$

Using the formulas

$$\int t^{\alpha}\cos(\beta\ln t)\,dt = \frac{1}{1+(\beta/(\alpha+1))^2}\left(\frac{t^{\alpha+1}}{\alpha+1}\cos(\beta\ln t) + \frac{\beta t^{\alpha+1}}{(\alpha+1)^2}\sin(\beta\ln t)\right),$$

$$\int t^{\alpha}\sin(\beta\ln t)\,dt = \frac{1}{1+(\beta/(\alpha+1))^2}\left(\frac{t^{\alpha+1}}{\alpha+1}\sin(\beta\ln t) - \frac{\beta t^{\alpha+1}}{(\alpha+1)^2}\cos(\beta\ln t)\right),$$

we bring the right-hand side to the form

$$\left\{\sum_{\alpha+\beta=0}^{N} K^{\alpha\beta}x^{\alpha}\right\} \times \Bigg[\sum_{i=0}^{N} a_i \frac{1}{1+(\mu/(\beta+\nu+i))^2}\left(\frac{x^{\beta+\nu+i}}{\beta+\nu+i}\sin(\mu\ln x) - \frac{\mu x^{\beta+\nu+i}}{(\beta+\nu+i)^2}\cos(\mu\ln t)\right) + \sum_{i=0}^{N} b_i \frac{1}{1+(\mu/(\beta+\nu+i))^2} \times$$

$$\times\left(\frac{x^{\beta+\nu+i}}{\beta+\nu+i}\cos(\mu\ln x) + \frac{\mu x^{\beta+\nu+i}}{(\beta+\nu+i)^2}\sin(\mu\ln x)\right)\Bigg] +$$

$$+ \int_0^x K(x,t)\big(a_{N+1}(t)t^{N+\nu}\sin(\mu\ln t) + b_{N+1}(t)t^{N+\nu}\cos(\mu\ln t)\big)dt +$$

$$+\int_0^x \Big[\sum_{\alpha+\beta=N+1} \tilde{K}^{\alpha\beta}(x,t)x^{\alpha}t^{\beta}\Big]\times$$

$$\times\left(\sum_{i=0}^{N} a_i t^{\nu+i-1}\sin(\mu\ln t)+\sum_{i=0}^{N} b_i t^{\nu+i-1}\cos(\mu\ln t)\right)\Bigg]dt. \tag{5}$$

Equating the coefficients of $x^{\nu}\sin(\mu\ln x)$ and $x^{\nu}\cos(\mu\ln x)$ on both sides, we find equations for determining the coefficients a_0 and b_0:

$$\begin{cases} a_0 = K^{00}a_0\dfrac{\nu}{\nu^2+\mu^2}+K^{00}b_0\dfrac{\mu}{\nu^2+\mu^2}, \\ b_0 = -K^{00}a_0\dfrac{\mu}{\nu^2+\mu^2}+K^{00}b_0\dfrac{\nu}{\nu^2+\mu^2}. \end{cases} \tag{6}$$

It is not difficult to show that relations (6) yield

$$K^{00}(a_0+ib_0) = (\nu+i\mu)(a_0+ib_0),$$

i.e., a_0+ib_0 is the eigenvector for the matrix K^{00} corresponding to the eigenvalue $\nu+i\mu$. Thus, $a_0=e_1$, $b_0=e_2$ is a solution of the system of equations (6).

Equating the coefficients of $x^{\nu+k}\sin(\mu\ln x)$ and $x^{\nu+k}\cos(\mu\ln x)$ $(k=1,2,\ldots,N)$, we find equations for obtaining the coefficients a_k and b_k:

$$\begin{aligned} a_k = {} & \sum_{\alpha+\beta+i=k} K^{\alpha\beta}a_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{1}{\beta+\nu+i}+ \\ & +\sum_{\alpha+\beta+i=k} K^{\alpha\beta}b_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{\mu}{(\beta+\nu+i)^2}, \\ b_k = {} & -\sum_{\alpha+\beta+i=k} K^{\alpha\beta}a_i\frac{\mu}{(\beta+\nu+i)^2}\frac{1}{1+(\mu/(\beta+\nu+i))^2}+ \\ & +\sum_{\alpha+\beta+i=k} K^{\alpha\beta}b_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{1}{\beta+\nu+i}. \end{aligned} \tag{7}$$

The system of equations (7) can be rewritten as

$$\begin{aligned} a_k &= K^{00}a_k\frac{1}{1+(\mu/(\nu+k))^2}\frac{1}{\nu+k}+K^{00}b_k\frac{1}{1+(\mu/(\nu+k))^2}\frac{\mu}{(\nu+k)^2}+S_1, \\ b_k &= -K^{00}a_k\frac{\mu}{(\nu+k)^2}\frac{1}{1+(\mu/(\nu+k))^2}+K^{00}b_k\frac{1}{1+(\mu/(\nu+k))^2}\frac{1}{\nu+k}+C_1, \end{aligned} \tag{8}$$

where S_1 and C_1 have the following form:

$$\begin{aligned} S_1 = {} & \sum_{\substack{\alpha+\beta+i=k \\ \alpha+\beta\ge 1}} K^{\alpha\beta}a_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{1}{\beta+\nu+i}+ \\ & +\sum_{\substack{\alpha+\beta+i=k \\ \alpha+\beta\ge 1}} K^{\alpha\beta}b_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{\mu}{(\beta+\nu+i)^2}, \end{aligned}$$

$$C_1 = -\sum_{\substack{\alpha+\beta+i=k\\ \alpha+\beta\geq 1}} K^{\alpha\beta} a_i \frac{\mu}{(\beta+\nu+i)^2} \frac{1}{1+(\mu/(\beta+\nu+i))^2} +$$

$$+\sum_{\substack{\alpha+\beta+i=k\\ \alpha+\beta\geq 1}} K^{\alpha\beta} b_i \frac{1}{1+(\mu/(\beta+\nu+i))^2} \frac{1}{\beta+\nu+i}.$$

Assume that

A) the numbers $(\nu+k)+i\mu$ $(k=1,2,\ldots,N)$ are not eigenvalues for the matrix $K(0,0)$.

Let us show that the system of equations (8) has only one solution. If we assume the contrary, then

$$\begin{vmatrix} K^{00}\dfrac{\nu+k}{(\nu+k)^2+\mu^2}-I & K^{00}\dfrac{\mu}{(\nu+k)^2+\mu^2} \\ -K^{00}\dfrac{\nu+k}{(\nu+k)^2+\mu^2} & K^{00}\dfrac{\nu+k}{(\nu+k)^2+\mu^2}-I \end{vmatrix} = 0.$$

Then the system of equations

$$\begin{cases} \left[-K^{00}\dfrac{\nu+k}{(\nu+k)^2+\mu^2}-I\right]f_1 + K^{00}\dfrac{\mu}{(\nu+k)^2+\mu^2}f_2 = 0, \\ -K^{00}\dfrac{\mu}{(\nu+k)^2+\mu^2}f_1 + \left[K^{00}\dfrac{\nu+k}{(\nu+k)^2+\mu^2}-I\right]f_2 = 0 \end{cases} \tag{9}$$

has a nonzero solution $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Let us calculate f_1+if_2 by using relations (9):

$$f_1+if_2 = K^{00}\frac{\nu+k}{(\nu+k)^2+\mu^2}f_1 + K^{00}\frac{\mu}{(\nu+k)^2+\mu^2}f_2 -$$

$$-K^{00}\frac{i\mu}{(\nu+k)^2+\mu^2}f_1 + K^{00}\frac{i(\nu+k)}{(\nu+k)^2+\mu^2}f_2.$$

Then

$$[(\nu+k)+i\mu](f_1+f_2) = K^{00}\frac{(\nu+k)^2}{(\nu+k)^2+\mu^2}f_1 + K^{00}\frac{\mu(\nu+k)}{(\nu+k)^2+\mu^2}f -$$

$$-K^{00}\frac{i\mu(\nu+k)}{(\nu+k)^2+\mu^2}f_1 + K^{00}\frac{i(\nu+k)^2}{(\nu+k)^2+\mu^2}f_2 + K^{00}\frac{i\mu(\nu+k)}{(\nu+k)^2+\mu^2}f_1 +$$

$$+K^{00}\frac{i\mu^2}{(\nu+k)^2+\mu^2}f_2 + K^{00}\frac{\mu^2}{(\nu+k)^2+\mu^2}f_1 -$$

$$-K^{00}\frac{\mu(\nu+k)}{(\nu+k)^2+\mu^2}f_2 = K^{00}(f_1+if_2).$$

Therefore, f_1+if_2 is the eigenvector of the matrix K^{00} corresponding to the eigenvalue $(\nu+k)+i\mu$, but this contradicts assumption A). Hence, the coefficients a_k and b_k $(k=1,2,\ldots,N)$ are uniquely determined by (8).

For function (3) to be a solution of equation (1), it is necessary that vector functions $a_{N+1}(x)$, $b_{N+1}(x)$ satisfy the equation

$$a_{N+1}(x)x^{\nu+N+1}\sin(\mu\ln x) + b_{N+1}(x)x^{\nu+N+1}\cos(\mu\ln x) =$$

$$= \int_0^x K(x,t)\big(a_{N+1}(t)t^{N+\nu}\sin(\mu\ln t) + b_{N+1}(t)t^{N+\nu}\cos(\mu\ln t)\big)dt +$$

$$+ \int_0^x \Bigg[\sum_{\alpha+\beta=N+1}\tilde{K}^{\alpha\beta}(x,t)x^{\alpha}t^{\beta}\Bigg]\Bigg(\sum_{i=0}^{N}a_i t^{\nu+i-1}\sin(\mu\ln t) +$$

$$+ \sum_{i=0}^{N}b_i t^{\nu+i-1}\cos(\mu\ln t)\Bigg)dt +$$

$$+ \sum_{\alpha+\beta+i\geq N+1}K^{\alpha\beta}a_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{x^{\alpha+\beta+i+\nu}}{\beta+\nu+i}\sin(\mu\ln x) +$$

$$+ \sum_{\alpha+\beta+i\geq N+1}K^{\alpha\beta}b_i\frac{x^{\alpha+\beta+i+\nu}}{1+(\mu/(\beta+\nu+i))^2}\frac{\mu}{(\beta+\nu+i)^2}\sin(\mu\ln x) +$$

$$+ \sum_{\alpha+\beta+i\geq N+1}K^{\alpha\beta}a_i\frac{-\mu}{1+(\mu/(\beta+\nu+i))^2}\frac{x^{\alpha+\beta+i+\nu}}{(\beta+\nu+i)^2}\cos(\mu\ln x) +$$

$$+ \sum_{\alpha+\beta+i\geq N+1}K^{\alpha\beta}b_i\frac{x^{\alpha+\beta+i+\nu}}{1+(\mu/(\beta+\nu+i))^2}\frac{1}{\beta+\nu+i}\cos(\mu\ln x). \tag{10}$$

In both integrals, let us make a substitution $t = xs$. After elementary transformations, we have

$$a_{N+1}(x)x^{\nu+N+1}\sin(\mu\ln x) + b_{N+1}(x)x^{\nu+N+1}\cos(\mu\ln x) =$$

$$= \int_0^1 K(x,xs)\Big(a_{N+1}(xs)t^{\nu+N+1}s^{\nu+N}[\sin(\mu\ln x)\cos(\mu\ln s) + \sin(\mu\ln s)\cos(\mu\ln x)] +$$

$$+ b_{N+1}(xs)x^{\nu+N+1}s^{\nu+N}[\cos(\mu\ln x)\cos(\mu\ln s) - \sin(\mu\ln x)\sin(\mu\ln s)]\Big)ds +$$

$$+ \int_0^1 \Bigg[\sum_{\alpha+\beta=N+1}\tilde{K}^{\alpha\beta}(x,xs)x^{\alpha+\beta+1}s^{\beta}\Bigg]\times$$

$$\times\Bigg(\sum_{i=0}^{N}a_i x^{\nu+i-1}s^{\nu+i-1}[\sin(\mu\ln x)\cos(\mu\ln s) + \sin(\mu\ln s)\cos(\mu\ln x)] +$$

$$+ \sum_{i=0}^{N}b_i x^{\nu+i-1}s^{\nu+i-1}[\cos(\mu\ln x)\cos(\mu\ln s) - \sin(\mu\ln x)\sin(\mu\ln s)]\Bigg)ds +$$

$$+ \Bigg[\sum_{\alpha+\beta+i\geq N+1}K^{\alpha\beta}a_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{x^{\alpha+\beta+i+\nu}}{\beta+\nu+i} +$$

$$+ \sum_{\alpha+\beta+i\geq N+1}K^{\alpha\beta}b_i\frac{1}{1+(\mu/(\beta+\nu+i))^2}\frac{\mu x^{\alpha+\beta+i+\nu}}{(\beta+\nu+i)^2}\Bigg]\sin(\mu\ln x) +$$

$$+\left[\sum_{\alpha+\beta+i\geq N+1} K^{\alpha\beta} a_i \frac{-\mu}{1+(\mu/(\beta+\nu+i))^2} \frac{x^{\alpha+\beta+i+\nu}}{(\beta+\nu+i)^2} + \right.$$

$$\left. + \sum_{\alpha+\beta+i\geq N+1} K^{\alpha\beta} b_i \frac{x^{\alpha+\beta+i+\nu}}{1+(\mu/(\beta+\nu+i))^2} \frac{1}{\beta+\nu+i}\right] \cos(\mu \ln x). \tag{11}$$

Equating the coefficients of $x^{\nu+N+1}\sin(\mu \ln x)$ and $x^{\nu+N+1}\cos(\mu \ln x)$, we obtain the equations

$$a_{N+1}(x) = \int_0^1 K(x, xs) s^{N+\nu}\left[a_{N+1}(xs)\cos(\mu \ln s) - b_{N+1}(xs)\sin(\mu \ln s)\right] ds +$$

$$+ \int_0^1 \left[\sum_{\alpha+\beta=N+1} \tilde{K}^{\alpha\beta}(x, xs) s^{\beta}\right] \left(\sum_{i=0}^{N} a_i x^i s^{\nu+i-1} \cos(\mu \ln s) - \right.$$

$$\left. - \sum_{i=0}^{N} b_i x^i s^{\nu+i-1} \sin(\mu \ln s)\right) ds +$$

$$+\left[\sum_{\alpha+\beta+i\geq N+1} K^{\alpha\beta} a_i \frac{1}{1+(\mu/(\beta+\nu+i))^2} \frac{x^{\alpha+\beta+i-N-1}}{\beta+\nu+i} + \right.$$

$$\left. + \sum_{\alpha+\beta+i\geq N+1} K^{\alpha\beta} b_i \frac{1}{1+(\mu/(\beta+\nu+i))^2} \frac{\mu x^{\alpha+\beta+i-N-1}}{(\beta+\nu+i)^2}\right], \tag{12}$$

$$b_{N+1}(x) = \int_0^1 K(x, xs) s^{\nu+N}\left[a_{N+1}(xs)\sin(\mu \ln s) + b_{N+1}(xs)\cos(\mu \ln s)\right] ds +$$

$$+ \int_0^1 \left[\sum_{\alpha+\beta=N+1} \tilde{K}^{\alpha\beta}(x, xs) s^{\beta}\right] \left(\sum_{i=0}^{N} a_i x^i s^{\nu+i-1} \sin(\mu \ln s) + \right.$$

$$\left. + \sum_{i=0}^{N} b_i x^i s^{\nu+i-1} \cos(\mu \ln s)\right) ds +$$

$$+\left[\sum_{\alpha+\beta+i\geq N+1} K^{\alpha\beta} a_i \frac{-\mu}{1+(\mu/(\beta+\nu+i))^2} \frac{x^{\alpha+\beta+i-N-1}}{(\beta+\nu+i)^2} + \right.$$

$$\left. + \sum_{\alpha+\beta+i\geq N+1} K^{\alpha\beta} b_i \frac{x^{\alpha+\beta+i-N-1}}{1+(\mu/(\beta+\nu+i))^2} \frac{1}{\beta+\nu+i}\right]. \tag{13}$$

Equations (12) and (13) can be written in the operator form

$$\varphi(x) = A\varphi(x) + f(x), \tag{14}$$

where

$$\varphi(x) = \begin{pmatrix} a_{N+1}(x) \\ b_{N+1}(x) \end{pmatrix}, \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

$$f_1(x) = \int_0^1 \left[\sum_{\alpha+\beta=N+1} \tilde{K}^{\alpha\beta}(x, xs) s^{\beta}\right] \times$$

$$\times \left(\sum_{i=0}^{N} a_i x^i s^{\nu+i-1} \cos(\mu \ln s) - \sum_{i=0}^{N} b_i x^i s^{\nu+i-1} \sin(\mu \ln s) \right) ds +$$

$$+ \left[\sum_{\alpha+\beta+i \geq N+1} K^{\alpha\beta} a_i \frac{1}{1+(\mu/(\beta+\nu+i))^2} \frac{x^{\alpha+\beta+i-N-1}}{\beta+\nu+i} + \right.$$

$$\left. + \sum_{\alpha+\beta+i \geq N+1} K^{\alpha\beta} b_i \frac{1}{1+(\mu/(\beta+\nu+i))^2} \frac{\mu x^{\alpha+\beta+i-N-1}}{(\beta+\nu+i)^2} \right],$$

$$f_2(x) = \int_0^1 \left[\sum_{\alpha+\beta=N+1} \tilde{K}^{\alpha\beta}(x, xs) s^{\beta} \right] \times$$

$$\times \left(\sum_{i=0}^{N} a_i x^i s^{\nu+i-1} \sin(\mu \ln s) + \sum_{i=0}^{N} b_i x^i s^{\nu+i-1} \cos(\mu \ln s) \right) ds +$$

$$+ \left[\sum_{\alpha+\beta+i \geq N+1} K^{\alpha\beta} a_i \frac{(-\mu)}{1+(\mu/(\beta+\nu+i))^2} \frac{x^{\alpha+\beta+i-N-1}}{(\beta+\nu+i)^2} + \right.$$

$$\left. + \sum_{\alpha+\beta+i \geq N+1} K^{\alpha\beta} b_i \frac{x^{\alpha+\beta+i-N-1}}{1+(\mu/(\beta+\nu+i))^2} \frac{1}{\beta+\nu+i} \right].$$

If the vector function $\varphi(x)$ is continuous on $[0, T]$ and its values are in $\mathbb{R}^{2n}$, then the operator A acts according to the formula

$$A\varphi(x) = \int_0^1 \overline{K}(x, s) s^{N+\nu} \varphi(x, s)\, ds,$$

where

$$\overline{K}(x, s) = \begin{pmatrix} K(x, xs)\cos(\mu \ln s) & -K(x, xs)\sin(\mu \ln s) \\ K(x, xs)\sin(\mu \ln s) & K(x, xs)\cos(\mu \ln s) \end{pmatrix}.$$

Let us estimate the norm of the operator A in the space of continuous vector functions with values in $\mathbb{R}^{2n}$. We have

$$\|A\varphi\|_{C(\mathbb{R}^{2n})} = \left\| \int_0^1 \overline{K}(x, xs) s^{N+\nu} \varphi(xs)\, ds \right\|_{C(\mathbb{R}^{2n})} =$$

$$= \max_{0 \leq x \leq T} \left\| \int_0^1 \overline{K}(x, xs) s^{N+\nu} \varphi(xs)\, ds \right\|_{\mathbb{R}^{2n}} \leq$$

$$\leq \max_{0 \leq x \leq T} \int_0^1 \| \overline{K}(x, xs) s^{N+\nu} \varphi(xs) \|_{\mathbb{R}^{2n}}\, ds \leq$$

$$\leq \max_{0 \leq x \leq T} \int_0^1 ||| \overline{K}(x, xs) ||| \, s^{N+\nu} \| \varphi(xs) \|_{\mathbb{R}^{2n}}\, ds =$$

$$= \max_{0 \leq x \leq T} \int_0^1 \max_{1 \leq i \leq 2n} \sum_{j=1}^{2n} |\overline{K}_{ij}| \, s^{N+\nu} \| \varphi(xs) \|_{\mathbb{R}^{2n}}\, ds \leq$$

$$\leq \max_{0\leq x\leq T} 2\int_0^1 \max_{1\leq i\leq n} \sum_{j=1}^{n} |K_{ij}|\, s^{N+\nu}\, \|\varphi(xs)\|_{\mathbb{R}^{2n}}\, ds =$$

$$= \max_{0\leq x\leq T} 2\int_0^1 \|\!|K|\!\|\, s^{N+\nu}\, \|\varphi(xs)\|_{\mathbb{R}^{2n}}\, ds \leq$$

$$\leq 2\|\!|K|\!\|_C \frac{1}{N+\nu+1} \|\varphi\|_{C(\mathbb{R}^{2n})}.$$

Then, according to (2),

$$\|A\|_{C(\mathbb{R}^{2n})\to C(\mathbb{R}^{2n})} \leq 2\|\!|K|\!\|_C \frac{1}{N+\nu+1} < 1.$$

Hence, equation (14) can be uniquely solved by the method of successive approximations.

We can give one more solution $v(x)$ of equation (1) in the form (3) if we take the vector $\bar{f} = i\bar{e} = -e_2 + ie_1$ instead of the vector $\bar{e} = e_1 + ie_2$.

Then $u(x)$ and $v(x)$ are linearly independent. Let us show this. Let

$$C_1 u(x) + C_2 v(x) \equiv 0,$$

or

$$C_1 x^{\nu-1}\Bigg[\sum_{i=0}^{N} a_i x^i \sin(\mu \ln x) + \sum_{i=0}^{N} b_i x^i \cos(\mu \ln x) +$$

$$+ a_{N+1}(x) x^{N+1} \sin(\mu \ln x) + b_{N+1}(x) x^{N+1} \cos(\mu \ln x)\Bigg] +$$

$$+ C_2 x^{\nu-1}\Bigg[\sum_{i=0}^{N} \bar{a}_i x^i \sin(\mu \ln x) + \sum_{i=0}^{N} \bar{b}_i x^i \cos(\mu \ln x) +$$

$$+ \bar{a}_{N+1}(x) x^{N+1} \sin(\mu \ln x) + \bar{b}_{N+1}(x) x^{N+1} \cos(\mu \ln x)\Bigg] \equiv 0.$$

Therefore,

$$C_1\left(\sum_{i=0}^{N} a_i x^i + a_{N+1}(x) x^{N+1}\right) + C_2\left(\sum_{i=0}^{N} \bar{a}_i x^i + \bar{a}_{N+1}(x) x^{N+1}\right) \equiv 0,$$

$$C_1\left(\sum_{i=0}^{N} b_i x^i + b_{N+1}(x) x^{N+1}\right) + C_2\left(\sum_{i=0}^{N} \bar{b}_i x^i + \bar{b}_{N+1}(x) x^{N+1}\right) \equiv 0.$$

Thus,

$$\begin{cases} C_1 a_0 + C_2 \bar{a}_0 = 0, \\ C_1 b_0 + C_2 \bar{b}_0 = 0, \end{cases} \qquad \begin{cases} C_1 e_1 + C_2(-e_2) = 0, \\ C_1 e_2 + C_2 e_1 = 0. \end{cases}$$

Let $e_1 \neq \bar{0}$. Then

$$C_1^2 e_1 + C_2^2 e_1 = 0, \qquad (C_1^2 + C_2^2) e_1 = 0.$$

Thus, $C_1 = 0$ and $C_2 = 0$.

We arrive at the following statement:

Theorem. *Suppose that the kernel* $K(x, t)$ *has continuous partial derivatives up to order* $N+1$ *inclusive,* $K(0, 0)$ *has complex eigenvalues* $\nu \pm i\mu$ $(\nu > 0)$, *where* ν *satisfies inequalities* (2), *and condition A) is satisfied. Then equation* (1) *has two linearly independent solutions of the form* (3).

Example. Consider the equation

$$xu(x) = \int_0^x K u(t)\,dt,$$

where

$$K = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}.$$

This matrix has two complex eigenvalues $2 \pm 2i$ $(\nu = 2,\ \mu = 2)$. The eigenvector corresponding to the eigenvalue $2+2i$ is

$$\overline{e} = e_1 + ie_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Therefore, the equation has following two solutions $u(x)$ and $v(x)$:

$$u(x) = x^{\nu-1}[\, a_0 \sin(\mu \ln x) + b_0 \cos(\mu \ln x)] =$$

$$= x^{\nu-1}[\, e_1 \sin(\mu \ln x) + e_2 \cos(\mu \ln x)] =$$

$$= x\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} \sin(2 \ln x) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos(2 \ln x)\right],$$

$$\bar{f} = ie = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + i\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$v(x) = x^{\nu-1}[\, \overline{a}_0 \sin(\mu \ln x) + \overline{b}_0 \cos(\mu \ln x)] =$$

$$= x^{\nu-1}[\, (-e_2) \sin(\mu \ln x) + e_1 \cos(\mu \ln x)] =$$

$$= x\left[\begin{pmatrix} -2 \\ 0 \end{pmatrix} \sin(2 \ln x) + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cos(2 \ln x)\right].$$

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Received 10.10.96