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Cohomologies of finite abelian groups

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Abstract. We construct a simplified resolution for the trivial G-module $\mathbb{Z}$, where G is a finite abelian group, and compare it with the standard resolution. We use it to calculate cohomologies of irreducible G-lattices and their duals.

Introduction

The theory of cohomologies of groups was inspired by the works of Hurewicz on cohomologies of acyclic spaces and was founded in 1940's by Eilenberg–MacLane, Eckmann, Hopf and others. It was one of the origins of the homological algebra. It was also related to the theory of group extensions and projective representations, where cohomologies arise as factor sets. This theory is widely used in topology, number theory, algebraic geometry and other branches of mathematics. Thus it is actively studied by plenty of mathematicians. In particular, there is a lot of papers devoted to the calculation of cohomologies of concrete groups and their classes. In these investigations one often needs special sorts of resolutions, which are simpler and more convenient than the standard one. For instance, Takahashi [7] proposed a new approach to the calculation of cohomologies of finite abelian groups and gave applications of his method to the cohomologies of the trivial module and of some Galois groups.

The aim of our paper is to describe a rather simple resolution for finite abelian groups (Section 1) and to use it for calculation of cohomologies of irreducible G-lattices and their duals (Sections 4 and 5). Our approach

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is close to that of Takahashi, though it seems more explicit. We also compare our resolution with the standard one (Section 2) and prove some facts concerning duality for cohomologies of G-lattices (Section 3). The results about the second cohomologies can be useful in the study of crystallographic groups and of Chernikov groups.

1. Resolution

For a periodic element a of a group G we denote by $o(a)$ the order of a, $s_a = \sum_{i=0}^{o(a)-1} a^i$. Let $G = \prod_{i=1}^{s} G_i$ be a direct product of finite cyclic groups $G_i = \langle\, a_i \mid a_i^{o_i} = 1 \,\rangle$ of orders $o_i = o(a_i)$, $\boldsymbol{R} = \mathbb{Z}G$, $\mathbb{P} = \boldsymbol{R}[x_1, x_2, \dots, x_s]$ and $\mathbb{P}_n$ be the set of homogeneous polynomials from $\mathbb{P}$ of degree n (including 0). We define a differential $d : \mathbb{P}_n \to \mathbb{P}_{n-1}$ by the rule

$$d_n(x_1^{k_1} x_2^{k_2} \dots x_s^{k_s}) = \sum_{i=1}^{s} (-1)^{K_i} C_i x_1^{k_1} \dots x_i^{k_i - 1} \dots x_s^{k_s},$$

where $K_i = \sum_{j=1}^{i-1} k_j$ and

$$C_i = \begin{cases} a_i - 1 & \text{if } k_i \text{ is odd,} \\ s_{a_i} & \text{if } k_i > 0 \text{ is even,} \\ 0 & \text{if } k_i = 0. \end{cases}$$

When speaking of the G-module $\mathbb{Z}$, we always suppose that the elements of G act trivially.

Theorem 1.1. $\mathbb{P} = (\mathbb{P}_n, d_n)$ *is a free resolution of the G-module $\mathbb{Z}$.*

Proof. If $s = 1$, it is well-known. If $\boldsymbol{R}_i = \mathbb{Z}G_i$ and $\mathbb{P}^i$ denotes such resolution for the group G_i, then $\boldsymbol{R} = \bigotimes_{i=1}^{s} \boldsymbol{R}_i$ and $\mathbb{P}$ is the tensor product of complexes $\bigotimes_{i=1}^{s} \mathbb{P}^i$. As all groups of cycles and boundaries in the complexes $\mathbb{P}^i$ are free abelian, the claim follows from the Künneth relations [3, Theorem VI.3.1]. $\square$

2. Correspondence with standard resolution

To apply Theorem 1.1, for instance, to extensions of groups, we have to compare it with the standard resolution, which is usually used for this purpose [2, 3]. So, in what follows, $\mathbb{S}$ denotes the normalized standard resolution for $\mathbb{Z}$ as $\boldsymbol{R}$-module, $\{\,[g_1, g_2, \dots, g_n] \mid g_i \in G \setminus \{1\}\,\}$ is the usual basis of $\mathbb{S}_n$ such that the standard differential d^{s} is defined as

$$d^{\mathrm{s}}_n[g_1, g_2, \dots, g_n] = g_1[g_2, \dots, g_n] + \sum_{i=1}^{n} (-1)^i [g_1, \dots, g_i g_{i+1}, \dots, g_n] + \\ + (-1)^n [g_1, g_2, \dots, g_{n-1}],$$

setting $[g_1, g_2, \dots, g_n] = 0$ if some $g_i = 1$. Note that $\mathbb{P}_0 = \mathbb{S}_0 = \boldsymbol{R}$.

We denote $a^{\{i\}} = 1 + a + a^2 + \dots a^{i-1}$. Then $s_a = a^{\{o(a)\}}$,

$$a^{\{i+k\}} = a^{\{i\}} + a^i a^{\{k\}}, \tag{2.1}$$

in particular,

$$a^{\{m+o(a)\}} = a^{\{m\}} + a^m s_a.$$

Theorem 2.1. *There is a quasi-isomorphism* $\sigma : \mathbb{S} \to \mathbb{P}$ *such that*

$$\sigma_0 = \mathrm{id},$$

$$\sigma_1[a_1^{k_1} a_2^{k_2} \dots a_s^{k_s}] = \sum_{i=1}^{s} \Big(\prod_{j=1}^{i-1} a_j^{k_j}\Big) a_i^{\{k_i\}} x_i,$$

$$\sigma_2[a_1^{k_1} a_2^{k_2} \dots a_s^{k_s}, a_1^{l_1} a_2^{l_2} \dots a_s^{l_s}] = \sum_{i=1}^{s} \sum_{j=1}^{i} \Big(\prod_{q=1}^{i-1} a_q^{k_q} \prod_{r=1}^{j-1} a_r^{l_r} \Big) \sigma_2[a_i^{k_i}, a_j^{l_j}], \tag{2.2}$$

$$\text{where} \quad \sigma_2[a_i^k, a_j^l] = \begin{cases} [(k+l)/o_i] x_i^2 & \text{if } i = j, \\ 0 & \text{if } i < j, \\ a_j^{\{l\}} a_i^{\{k\}} x_j x_i & \text{if } i > j \end{cases}$$

Since $\mathbb{S}$ and $\mathbb{P}$ are free resolutions of $\mathbb{Z}$, σ induces isomoprhisms of cohomologies $H^n(\mathrm{Hom}_{\boldsymbol{R}}(\mathbb{S}, M)) \to H^n(\mathrm{Hom}_{\boldsymbol{R}}(\mathbb{P}, M))$. In particular, combining σ_2 with cocycles from $\mathrm{Hom}_{\boldsymbol{R}}(\mathbb{P}_2, M)$, we obtain the "usual" presentation of cocycles from $H^2(G, M)$.

Proof. Actually, we have to show that the diagram

$$\begin{array}{ccccc} \mathbb{S}_2 & \xrightarrow{d^{\mathrm{s}}_2} & \mathbb{S}_1 & \xrightarrow{d^{\mathrm{s}}_1} & \mathbb{S}_0 \\ \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} & & \| \\ \mathbb{P}_2 & \xrightarrow{d_2} & \mathbb{P}_1 & \xrightarrow{d_1} & \mathbb{P}_0. \end{array}$$

is commutative. Then the set of homomorphisms $\{\sigma_0, \sigma_1, \sigma_2\}$ extends to a quasi-isomorphism $\sigma : \mathbb{S} \to \mathbb{P}$.

Note that $gh - 1 = (g-1) + g(h-1)$ and $a^k - 1 = a^{\{k\}}(a-1)$. Therefore,

$$\begin{aligned} d_1^{\mathrm{s}}[a_1^{k_1}a_2^{k_2}\dots a_s^{k_s}] &= a_1^{k_1}a_2^{k_2}\dots a_s^{k_s} - 1 = \sum_{i=1}^{s}(\prod_{j=1}^{i-1} a_j^{k_j})(a_i^{k_i} - 1) \\ &= \sum_{i=1}^{s}(\prod_{j=1}^{i-1} a_j^{k_j})a_i^{\{k_i\}}(a_i - 1) = \sum_{i=1}^{s}(\prod_{j=1}^{i-1} a_j^{k_j})a_i^{\{k_i\}}d_1 x_i, \end{aligned}$$

hence $d_1\sigma_1 = d_1^{\mathrm{s}}$.

Set $(r)_i = \mathrm{res}(r, o_i)$, the residue of r modulo o_i. Then, for $0 \leqslant k < o_i$, $0 \leqslant l < o_i$,

$$d_2^s[a_i^k, a_i^l] = a_i^k[a_i^l] - [a_i^{k+l}] + [a_i^k],$$

thus

$$\begin{aligned} \sigma_1 d_2^{\mathrm{s}}[a_i^k, a_i^l] &= (a_i^k a_i^{\{l\}} - a_i^{\{(k+l)_i\}} + a_i^{\{l\}})x_i \\ &= (a_i^k a_i^{\{l\}} - a_i^{\{k+l\}} + a_i^{\{l\}} + [(k+l)/o_i]s_{a_i})x_i \\ &= [(k+l)/o_i]s_{a_i}x_i = d_2([(k+l)/o_i]x_i^2), \end{aligned}$$

so, if we set $\sigma_2[a_i^k, a_i^l] = [(k+l)/o_i]x_i^2$, we have

$$d_2\sigma_2[a_i^k,\, a_i^l] = \sigma_1 d_2^{\mathrm{s}}[a_i^k, a_i^l].$$

In the same way,

$$d_2^{\mathrm{s}}[a_i^k, a_j^l] = a_i^k[a_j^l] - [a_i^k a_j^l] + [a_i^k],$$

thus, if $i < j$,

$$\sigma_1 d_2^{\mathrm{s}}[a_i^k, a_j^l] = a_i^k a_j^{\{l\}}x_j - a_i^{\{k\}}x_i - a_i^k a_j^{\{l\}}x_j + a_i^{\{k\}}x_i = 0,$$

while if $i > j$

$$\begin{aligned} \sigma_1 d_2^{\mathrm{s}}[a_i^k, a_j^l] &= a_i^k a_j^{\{l\}}x_j - a_j^{\{l\}}x_j - a_j^l a_i^{\{k\}}x_i + a_i^{\{k\}}x_i \\ &= (a_i^k - 1)a_j^{\{l\}}x_j - (a_j^l - 1)a_i^{\{k\}}x_i = -d_2(a_j^{\{l\}}a_i^{\{k\}}x_j x_i). \end{aligned}$$

So, if we set

$$\sigma_2[a_i^k, a_j^l] = \begin{cases} 0 & \text{if } i < j, \\ a_j^{\{l\}}a_i^{\{k\}}x_j x_i & \text{if } i > j \end{cases}$$

we have

$$d_2\sigma_2[a_i^k,\, a_j^l] = \sigma_1 d_2^{\mathrm{s}}[a_i^k, a_j^l] \quad \text{for } i \neq j.$$

Let now σ_2 is defined by the rule (2.2). We check that $d_2\sigma_2 = \sigma_1 d_2^{\mathrm{s}}$ for $s = 3$. The general case is analogous, though a bit cumbersome. We write a, b, c instead of a_1, a_2, a_3 and x, y, z instead of x_1, x_2, x_3. Then

$$\begin{aligned}
\sigma_1 d_2^{\mathrm{s}}[a^ib^jc^r, a^kb^lc^s] &= \sigma_1(a^ib^jc^r[a^kb^lc^s] - [a^{i+k}b^{j+l}c^{r+s}] + [a^ib^jc^r]) \\
&= a^ib^jc^r(a^{\{k\}}x + a^kb^{\{l\}}y + a^kb^lc^{\{s\}}z) - a^{\{i+k\}}x - a^{i+k}b^{\{j+l\}}y \\
&\quad - a^{i+k}b^{j+l}c^{\{r+s\}}z + [(i+k)/o_a]s_ax + a^{i+k}[(j+l)/o_b]s_by \\
&\quad + a^{i+k}b^{j+l}[(r+s)/o_c]s_cx + a^{\{i\}}x + a^ib^{\{j\}}y + a^ib^jc^{\{r\}}z \\
&= (a^ib^jc^ra^{\{k\}} - a^{\{i+k\}} + a^{\{i\}} + [(i+k)/o_a]s_a)x \\
&\quad + a^i(a^kb^jc^rb^{\{l\}} - a^kb^{\{j+l\}} + b^{\{j\}} + a^k[(j+l)/o_b]s_b)y \\
&\quad + a^ib^j(a^kb^lc^rc^{\{s\}} - a^kb^lc^{\{r+s\}} + c^{\{r\}} + a^kb^l[(r+s)/o_c]s_c)z,
\end{aligned}$$

while

$$\begin{aligned}
d_2\sigma_2[a^ib^jc^r, a^kb^lc^s] &= d_2\big(-a^ia^{\{k\}}b^{\{j\}}xy - a^ib^ja^{\{k\}}c^{\{s\}}xz - a^{i+k}b^jb^{\{l\}}c^{\{r\}}yz \\
&\quad + [(i+k)/o_a]x^2 + a^{i+k}[(j+l)/o_b]y^2 + a^{i+k}b^{j+l}[(r+s)/o_c]z^2\big) \\
&= -a^i(a^k-1)b^{\{j\}}y + a^i(b^j-1)a^{\{k\}}x - a^i(a^k-1)b^jc^{\{r\}}z \\
&\quad + a^ib^j(c^r-1)a^{\{k\}}x - a^{i+k}b^j(b^l-1)c^{\{r\}}z + a^{i+k}b^j(c^r-1)b^{k\}}y, \\
&\quad + [(i+k)/o_a]s_ax + a^{i+k}[(j+l)/o_b]s_by + a^{i+k}b^{j+l}[(r+s)/o_c]s_cx \\
&= (-a^ia^{\{k\}} + a^ib^jc^ra^{\{k\}} + [(i+k)/o_a]s_a)x \\
&\quad + a^i(-a^kb^{\{j\}} + b^{\{j\}} + a^kb^jc^rb^{\{l\}} - a^kb^jb^{\{l\}} + a^k[(j+l)/o_b]s_b)y \\
&\quad + a^ib^j(c^{\{r\}} - a^kb^lc^{\{r\}} + a^kb^l[(r+s)/o_c]s_c)z.
\end{aligned}$$

Relations (2.1) immediately imply that both results are equal. □

3. Cohomologies of *G*-lattices

In this section G denotes a finite group, $\boldsymbol{R} = \mathbb{Z}G$. Recall that a G-*lattice* (or an *integral representation* of G) is a G-module M such that its abelian group is free of finite rank. They also say that M is *a lattice in the* $\mathbb{Q}G$*-module* $\tilde{M} = \mathbb{Q} \otimes_{\mathbb{Z}} M$. Two G-lattices M, N are said to be *of the same genus* if $M_p \simeq N_p$ for each prime p, where $M_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} M$ ($\mathbb{Z}_p = \{\, r/z \mid r \in \mathbb{Z},\ s \in \mathbb{Z} \setminus p\mathbb{Z}\,\}$). Then they write $M \vee N$. We also set $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, where G acts by the rule $gf(u) = f(g^{-1}u)$.

We denote by $\hat{H}^n(G,M)$ the *Tate cohomologies* of G with coefficients in M [2,3]. Let

$$\mathbb{F}: \dots \to \mathbb{F}_n \xrightarrow{d_n} \mathbb{F}_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \mathbb{F}_1 \xrightarrow{d_1} \mathbb{F}_0 \to 0$$

be a free resolution of $\mathbb{Z}$, where all modules $\mathbb{F}_n$ are finitely generated,

$$\mathbb{F}^*: 0 \to \mathbb{F}_0^* \xrightarrow{d_1^*} \mathbb{F}_1^* \xrightarrow{d_2^*} \dots \xrightarrow{d_{n-1}^*} \mathbb{F}_{n-1}^* \xrightarrow{d_n^*} \mathbb{F}_n^* \to \dots$$

be the dual complex, $d_0 : \mathbb{F}_0 \to \mathbb{F}_0^*$ be the composition of the maps $\mathbb{F}_0 \to \operatorname{coker} d_1 \simeq \mathbb{Z} \simeq \ker d_0^* \to \mathbb{F}_0^*$. Set $\mathbb{F}_{-n} = \mathbb{F}_{n-1}^*$, $d_{-n} = d_n^*$. The sequence

$$\begin{aligned}\mathbb{F}^+ : \dots \to \mathbb{F}_n \xrightarrow{d_n} \mathbb{F}_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \mathbb{F}_1 \xrightarrow{d_1} \mathbb{F}_0 \xrightarrow{d_0} \\ \xrightarrow{d_0} \mathbb{F}_{-1} \xrightarrow{d_{-1}} \mathbb{F}_{-2} \xrightarrow{d_{-2}} \dots \xrightarrow{d_{-n}} \mathbb{F}_{-n}^* \xrightarrow{d_{-n}} \mathbb{F}_{-n-1} \to \dots\end{aligned}$$

is called a *complete resolution* for the group G. Then $\hat{H}^n(G,M)$ are just the cohomologies of the complex $\operatorname{Hom}_{\boldsymbol{R}}(\mathbb{F}^+, M)$. If $\mathbb{F}_0 = \boldsymbol{R}$ and the surjection $\mathbb{F}_0 \to \mathbb{Z}$ maps g to 1, then $\mathbb{F}_{-1} \simeq \boldsymbol{R}$ and d_0 is just the *trace*, i.e. the multiplication by $\operatorname{tr}_G = \sum_{x\in G} x$. It is the case for the resolutions $\mathbb{F}$ and $\mathbb{S}$.

Proposition 3.1. *Let G be a finite group, M, N be G-lattices such that $M \vee N$. Then $\hat{H}^n(G,M) \simeq \hat{H}^n(G,N)$ for all n.*

Proof. It is known that all groups $\hat{H}^n(G,M)$ $(n > 0)$ are periodic of period $\#(G)$, hence $\hat{H}^n(G,M) \simeq \bigoplus_{p|\#(G)} \hat{H}^n(G,M)_p$. Moreover, as $\mathbb{Z}_p$ is flat over $\mathbb{Z}$, $\hat{H}^n(G,M)_p \simeq \hat{H}^n(G,M_p)$. It implies he claim. □

We denote by DM the dual G-module $DM = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{T})$, where $\mathbb{T} = \mathbb{Q}/\mathbb{Z}$.

Proposition 3.2. *Let M be a G-lattice. Then*

$$\hat{H}^{n-1}(G,DM) \simeq D\hat{H}^{-n}(G,M), \tag{3.1}$$

$$\hat{H}^{n}(G,DM) \simeq \hat{H}^{n+1}(G,M^*), \tag{3.2}$$

$$\hat{H}^{n}(G,M^*) \simeq D\hat{H}^{-n}(G,M). \tag{3.3}$$

If $M = \mathbb{Z}$, (3.3) coincides with [3, Theorem XII.6.6].

Proof. (3.1) follows from [3, Corollary XII.6.5].

Consider the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{T} \to 0$. As M is free abelian, it gives the exact sequence of G-modules

$$0 \to M^* \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}) \to DM \to 0.$$

$\hat{H}^n(G, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q})) = 0$ for all n, since the multiplication by $\#(G)$ is an automorphism of $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q})$, whence we obtain (3.2).

(3.3) follows from (3.1) and (3.2). □

We also need some information on cohomologies of direct products.

Proposition 3.3. *Let N be a normal subgroup of G, $F = G/N$ and* $\gcd(\#(N), \#(F)) = 1$. *For every G-module M and all n*

$$\hat{H}^n(G, M) \simeq \hat{H}^n(N, M)^F \oplus \hat{H}^n(F, M^N). \tag{3.4}$$

Proof. As $\#(G)$ annihilates all $H^n(G, M)$ if $n > 0$ and the same is true for N and F, in the Hochschild–Serre spectral sequence

$$H^p(F, H^q(N, M)) \Longrightarrow H^n(G, M)$$

all terms with $p > 0$ and $q > 0$ are zero. Hence, if $n > 0$,

$$\begin{aligned} \hat{H}^n(G, M) &\simeq H^0(F, \hat{H}^n(N, M)) \oplus \hat{H}^n(F, H^0(N, M)) \\ &= \hat{H}^n(N, M)^F \oplus \hat{H}^n(F, M^N). \end{aligned}$$

Suppose now that the claim holds for $\hat{H}^n$. Choose an exact sequence $0 \to L \to P \to M \to 0$, where P is a free $\mathbb{Z}G$-module. Then

$$\hat{H}^{n-1}(G, M) \simeq \hat{H}^n(G, L) \simeq \hat{H}^n(N, L)^F \oplus \hat{H}^n(F, L^N).$$

As P is also free as $\mathbb{Z}N$-module, $\hat{H}^n(N, L) \simeq \hat{H}^{n-1}(N, M)$. On the other hand, there are exact sequences

$$0 \to L^N \to P^N \to M' \to 0$$

and

$$0 \to M' \to M^N \to M^N/M' \to 0,$$

where M' is the image of the map $P^N \to M^N$. Obviously, $M' \supseteq \mathrm{tr}_N M$, thus $\#(N)(M^N/M') = 0$, whence $\hat{H}^n(F, M^N/M') = 0$. Therefore,

$$\hat{H}^{n-1}(F, M^N) \simeq \hat{H}^{n-1}(F, M') \simeq \hat{H}^n(F, L^N),$$

since P^N is a free $\mathbb{Z}F$-module. So the isomorphism (3.4) holds for $\hat{H}^{n-1}$, hence for all values of n. □

Corollary 3.4. *Let* $G = G_1 \times G_2$ *with* $\gcd(\#(G_1), \#(G_2)) = 1$, $M = M_1 \otimes_{\mathbb{Z}} M_2$, *where* M_i *is a* G_i*-lattice* $(i = 1, 2)$. *Then*

$$\hat{H}^n(G, M) \simeq \hat{H}^n(G_1, M_1) \otimes_{\mathbb{Z}} M_2^{G_2} \oplus M_1^{G_1} \otimes_{\mathbb{Z}} \hat{H}^n(G_2, M_2).$$

Proof. As M_i are free abelian, $\otimes_{\mathbb{Z}} M_i$ is an exact functor and $M^{G_i} = M_i^{G_i} \otimes_{\mathbb{Z}} M_j$ $(j \neq i)$. Hence $\hat{H}^n(G_i, M) \simeq \hat{H}^n(G_i, M_i) \otimes_{\mathbb{Z}} M_j$, where $j \neq i$. So the claim is just a reformulation of Proposition 3.3 for this special case. □

4. Cohomologies of irreducible G-lattices

A G-lattice M is called *irreducible* if there are no submodules $0 \neq N \subset M$ such that M/N is torsion free (i.e. again a G-lattice). Equivalently, $\tilde{M} = \mathbb{Q} \otimes_{\mathbb{Z}} M$ is a simple $\mathbb{Q}G$-module. If G is a finite abelian group, then any simple $\mathbb{Q}G$-module is defined by a group homomorphism $\rho : G \to \boldsymbol{K}^\times$, where $\boldsymbol{K}$ is a cyclotomic field and the image of ρ generates the ring of integers of $\boldsymbol{K}$. Therefore, any two G-lattices in $\boldsymbol{K}$ are of the same genus [4], so have the same cohomologies. In particular, if M is a G-lattice in $\boldsymbol{K}$, so is M^*, hence $M^* \vee M$ and

$$\hat{H}^n(G, M) \simeq \hat{H}^n(G, M^*) \simeq D\hat{H}^{-n}(G, M) \simeq D\hat{H}^{n-1}(G, DM). \qquad (4.1)$$

The subgroup of periodic elements of $\boldsymbol{K}$ is cyclic and generated by a primitive root of unity ζ. Hence, there is an element $a \in G$ such that $\rho(a) = \zeta$. Let $G = \prod_{i=1}^{s} C_i$, where $C_i = \langle\, a_i \mid a_i^{o_i} = 1 \,\rangle$ are cyclic groups. We can suppose that $a_1 = a$. Set $o = o_1$. Changing he generators a_i, we can make $\rho(a_i) = 1$ for $i \neq 1$. Let $G' = \langle\, a_2, a_3, \dots, a_s \,\rangle$, so $G = C_1 \times G'$. Then $M \simeq M_1 \otimes_{\mathbb{Z}} \mathbb{Z}$, where M_1 is M considered as C_1-module and $\mathbb{Z}$ is the trivial G'-module. Note that $M^G = 0$, as $\zeta v = v$ implies $v = 0$. Hence $\hat{H}^0(G, M) = 0$. Consider the trace $T = \sum_{g \in G} g = (\sum_{k=0}^{o-1} a^k)(\sum_{g \in G'} g)$. Obviously, $\sum_{k=0}^{o-1} \zeta^k = 0$, hence $TM = 0$. It implies that $\hat{H}^{-1}(G, M) = H_0(G, M) = M/(\zeta - 1)M$. If $o = p^m$ fore some m, then also $o(\zeta) = p^k$ for some k, whence $N_{\boldsymbol{K}/\mathbb{Q}}(1 - \zeta) = p$ [1] and $\hat{H}^{-1}(G, M) = H_0(G, M) \simeq \mathbb{Z}/p\mathbb{Z}$. If $o(\zeta)$ is not a degree of a prime number, then $N_{\boldsymbol{K}/\mathbb{Q}}(1 - \zeta) = 1$ and $\hat{H}^{-1}(G, M) = H_0(G, M) = 0$ (it also follows from Corollary 3.4)..

Let a finite abelian group G be a direct product $G_1 \times G_2$ and the orders of G_1 and G_2 be coprime. If $\boldsymbol{K}_i$ $(i = 1, 2)$ is a cyclotomic field arising from a simple $\mathbb{Q}G_i$-module, then $\boldsymbol{K} = \boldsymbol{K}_1 \otimes_{\mathbb{Q}} \boldsymbol{K}_2$ is again a field, hence a simple $\mathbb{Q}G$-module, and all simple $\mathbb{Q}G$-modules arise in this way. If M_i $(i = 1, 2)$ is a G_i-lattice in $\boldsymbol{K}_i$, then $M = M_1 \otimes_{\mathbb{Z}} M_2$ is a G-lattice in $\boldsymbol{K}$, unique up to genus. Corollary 3.4 shows that $\hat{H}^n(G, M) = 0$ if neither M_1 nor M_2 is trivial. If M_1 is non-trivial and M_2 is trivial, then $\hat{H}^n(G, M) \simeq \hat{H}^n(G_1, M_1)$, and if both M_1 and M_2 are trivial, then $\hat{H}^n(G, M) \simeq \hat{H}^n(G_1, \mathbb{Z}) \oplus \hat{H}^n(G_2, \mathbb{Z})$. Thus we only need to consider the case of p-groups. Note also that $\mathbb{T} = \bigoplus_p \mathbb{T}_p$ and $\mathbb{T}_p$ is the *quasicyclic p-group*, i.e. the direct limit $\varinjlim_m \mathbb{Z}/p^m\mathbb{Z}$ with respect to the natural embeddings $\mathbb{Z}/p^m Z \to \mathbb{Z}/p^{m+1}\mathbb{Z}$. Hence, if M is finitely generated, $DM \simeq \bigoplus_p DM_p$, where $D_pM = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{T}_p)$. If M is a lattice, the additive group of D_pM is a direct product of several copies of $\mathbb{T}_p$. Moreover, if G is a p-group, $\hat{H}^n(G, D_qM) = 0$ and $D_q\hat{H}^n(G, M) = 0$ for $q \neq p$, so we can always replace D by D_p in all formulae from Proposition 3.2.

So, let $G = \prod_{k=1}^s G_k$, where G_k is a cyclic group of order p^{m_k}. We calculate cohomologies of a non-trivial irreducible G-lattices. Actually, it is easier to calculate homologies.

Theorem 4.1. *Let M be a non-trivial irreducible G-lattice. Then $H_n(G, M) \simeq (\mathbb{Z}/p\mathbb{Z})^{\nu(n,s)}$, where*

$$\nu(n, s) = (-1)^n \sum_{i=0}^{n} \binom{-s}{i}. \tag{4.2}$$

Note that for fixed n the value of $\nu(n, s)$ is a polynomial of degree n with respect to s with the leading coefficient $(n!)^{-1}$. For instance,

$$\nu(0, s) = 1, \quad \nu(1, s) = s - 1,$$
$$\nu(2, s) = \frac{s^2 + s + 2}{2}, \quad \nu(3, s) = \frac{s^3 + 5s - 6}{6}.$$

Proof. We consider G as a direct product $G' \times G_s$, where $G' = \prod_{i=1}^{s-1} G_i$, and suppose that G_s acts trivially on M. Then M can be considered as the outer tensor product $M' \times_{\mathbb{Z}} \mathbb{Z}$, where $M' = M$ considered as G'-module and $\mathbb{Z}$ is considered as trivial G_s-module. Then we can use the Künneth formula [2, Corollary V.5.8]:

$$H_n(G, M) \simeq \Big(\bigoplus_{i=0}^{n} H_i(G', M') \otimes_{\mathbb{Z}} H_{n-i}(G_s, \mathbb{Z})\Big) \oplus \Big(\bigoplus_{i=0}^{n-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_i(G', M'), H_{n-i-1}(G_s, \mathbb{Z}))\Big). \tag{4.3}$$

Recall that, for a cyclic group $C = \mathbb{Z}/p^m\mathbb{Z}$,

$$H_0(C,\mathbb{Z}) = \mathbb{Z};$$
$$H_n(C,\mathbb{Z}) = \begin{cases} \mathbb{Z}/p^m\mathbb{Z} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even;} \end{cases}$$

while for a non-trivial irreducible lattice M

$$H_n(C,M) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

that is,

$$\nu(n,1) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Moreover,

$$H_0(G,M) = \mathbb{Z}/p\mathbb{Z},$$

that is,

$$\nu(0,s) = 1.$$

Thus (4.1) is valid for $n = 0$ and for $s = 1$, the minimal values of n and s. Therefore, the Künneth formula implies that $H^n(G,M) \simeq (\mathbb{Z}/p\mathbb{Z})^{\nu(n,s)}$ for some $\nu(n,s)$. Moreover, it implies that

$$\nu(n,s) = \sum_{k=0}^{n} \nu(n,s-1) = \nu(n,s-1) + \nu(n-1,s)$$

Hence we can prove (4.1) by induction, supposing that it is true for $\nu(n,s-1)$ and $\nu(n-1,s)$. Then we have

$$\begin{aligned} \nu(n,s) &= \nu(n,s-1) + \nu(n-1,s) \\ &= (-1)^n \sum_{i=0}^{n} \binom{-s+1}{i} - (-1)^n \sum_{i=0}^{n-1} \binom{-s}{i} \\ &= (-1)^n \sum_{i=0}^{n} \left(\binom{-s+1}{i} - \binom{-s}{i-1} \right) \\ &= (-1)^n \sum_{i=0}^{n} \binom{-s}{i}. \end{aligned}$$ □

Note that in this case $\hat{H}^{-1}(G,M) = H_0(G,M)$ and $\hat{H}^0(G,M) = 0$.

The formulae (4.1) and (4.2) give the following result.

Corollary 4.2. *If M is a non-trivial irreducible G-lattice, then*

$$\hat{H}^n(G,M) \simeq \hat{H}^{n-1}(G,DM) \simeq (\mathbb{Z}/p\mathbb{Z})^{\nu(|n|-1,s)}.$$

Analogous calculations give the known result for the trivial G-module $\mathbb{Z}$ (cf. [6, 7]).

Theorem 4.3. *If $n \neq 0$ and $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$, then*

$$\hat{H}^n(G,\mathbb{Z}) \simeq \bigoplus\nolimits_{k=1}^{s}(\mathbb{Z}/p^{m_k}\mathbb{Z})^{\nu(|n|-1,k)+(-1)^n}. \tag{4.4}$$

Recall that $\hat{H}^0(G,\mathbb{Z}) \simeq \mathbb{Z}/p^m\mathbb{Z}$, where $m = \sum_{k=1}^{s} m_k$.

Proof. First of all, the Künneth formula (4.3) implies that $H_n(G,\mathbb{Z})$ is a direct sum of $\mu(n,s)$ cyclic groups so that

$$\mu(n,s) = \sum_{i=1}^{n} \mu(i,s-1) + \varepsilon,$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

whence

$$\mu(n,s) = \mu(n,s-1) + \mu(n-1,s) + (-1)^{n-1}.$$

Using induction by s, we obtain that

$$\mu(n,s) = \nu(n,s) - (-1)^n,$$

hence

$$\mu(n,s) = \mu(n,s-1) + \nu(n-1,s).$$

Note that all groups $H^i(G_s,\mathbb{Z})$ are of period p^{m_s}. Therefore, by (4.3),

$$H_n(G,\mathbb{Z}) \simeq H_n(G',\mathbb{Z}) \oplus (\mathbb{Z}/p^{m_s}\mathbb{Z})^r$$

for some r. Together with the formula for $\mu(n,s)$, it gives that

$$H_n(G,\mathbb{Z}) \simeq H_n(G',\mathbb{Z}) \oplus (\mathbb{Z}/p^{m_s}\mathbb{Z})^{\nu(n,s)-(-1)^n}.$$

By induction, we obtain that

$$H_n(G,\mathbb{Z}) \simeq \bigoplus_{k=1}^{s}(\mathbb{Z}/p^{m_k}\mathbb{Z})^{\nu(n,k)-(-1)^n}.$$

In view of (4.1), it is just the formula (4.4). □

5. Explicit formulae

In this section we find explicit formulae for crossed homomoprhisms (elements of $H^1(G,M)$) and cocycles (elements of $H^2(G,M)$) for irreducible latticies and their duals (the latter are important, for instance, in study of Chernikov groups see [5]). We use the resolution defined in Section 1.

Let $G=\prod_{i=1}^{m_s} G_i$, where $G_i=\langle\, a_i \mid a_i^{p^{m_i}}=1\,\rangle$ is a cyclic group of order $o_i=p^{m_i}$. We set $s_i=s_{a_i}$. For a cochain $\mu:\mathbb{P}_n\to M$ we denote by $\partial\mu$ its coboundary, that is the composition $\mu d_{n+1}:\mathbb{P}_{n+1}\to M$. Then, if $\xi:\mathbb{P}_1\to M$, $i<j$,

$$\begin{aligned}\partial\xi(x_i^2)&=s_i\xi(x_i),\\ \partial\xi(x_ix_j)&=(a_i-1)\xi(x_j)-(a_j-1)\xi(x_i).\end{aligned}\tag{5.1}$$

Thus ξ is a cocycle if and only if

$$\begin{aligned}s_i\xi(x_i)&=0 \text{ for all } i,\\ (a_i-1)\xi(x_j)&=(a_j-1)\xi(x_i) \text{ for all } i\neq j.\end{aligned}\tag{5.2}$$

If $\gamma:\mathbb{P}_2\to M$, $i<j<k$, then

$$\begin{aligned}\partial\gamma(x_i^3)&=(a_i-1)\gamma(x^2)=0,\\ \partial\gamma(x_i^2x_j)&=s_i\gamma(x_ix_j)+(a_j-1)\gamma(x_i^2),\\ \partial\gamma(x_ix_j^2)&=(a_i-1)\gamma(x_j^2)-s_j\gamma(x_ix_j),\\ \partial\gamma(x_ix_jx_k)&=(a_i-1)\gamma(x_jx_k)-(a_j-1)\gamma(x_ix_k)+(a_k-1)\gamma(x_ix_j).\end{aligned}$$

Thus γ is a cocycle if and only if

$$\begin{gathered}(a_i-1)\gamma(x_i^2)=0 \;\text{ for all } i,\\ s_i\gamma(x_ix_j)=-(a_j-1)\gamma(x_i^2),\quad s_j\gamma(x_jx_i)=(a_i-1)\gamma(x_j^2),\\ (a_j-1)\gamma(x_ix_k)=(a_i-1)\gamma(x_jx_k)+(a_k-1)\gamma(x_ix_j).\end{gathered}\tag{5.3}$$

Finally, if we identify an element $u\in M$ with the homomorphism $\mathbb{P}_0\to M$ which maps a to au, then $\partial u(x_i)=(a_i-1)u$.

First suppose that $M=\mathbb{Z}$. Then the element s_i acts on M as p^{m_i} and the formulae (5.2) show that $H^1(G,\mathbb{Z})=0$. As a_i-1 acts as 0, the formulae (5.3) mean that γ is a cocycle if and only if $\gamma(x_ix_j)=0$. The formulae (5.1) imply that, adding a coboundary, we can reduce $\gamma(x_i^2)$ modulo p^{m_i}. Therefore, $H^2(G,\mathbb{Z})\simeq\bigoplus_{i=1}^{m_s}\mathbb{Z}/p^{m_i}\mathbb{Z}$ and generators of this group can be chosen as the cohomology classes of the cocycles $\gamma_k : \mathbb{P}_2 \to \mathbb{Z}$ such that $\gamma_k(x_i x_j) = 0$ for all i, j and $\gamma_k(x_i^2) = \delta_{ik}$.

For the dual module $D_p\mathbb{Z} = \mathbb{T}_p$, the formulae (5.2) mean that ξ is a cocylce if and only if $p^{m_i}\xi(x_i) = 0$. Hence $H^1(G, \mathbb{T}_p) \simeq \bigoplus_{i=1}^{m_s} \mathbb{T}_{m_i}$, where $\mathbb{T}_{m_i} = \{\, u \in \mathbb{T}_p \mid p^{m_i}u = 0 \,\}$ (it is a cyclic group of order p^{m_i}). As $\mathbb{T}_p$ is divisible, the formulae (5.1) imply that, adding a coboundary to a 2-dimensional cocycle γ, one can always make $\gamma(x_i^2) = 0$. Then the formulae (5.3) mean that $p^{m_{ij}}\gamma_{x_i x_j} = 0$, where $m_{ij} = \min\{m_i, m_j\}$. Hence $H^2(G, \mathbb{T}_p) \simeq \bigoplus_{i<j} \mathbb{T}_{m_{ij}} \simeq \bigoplus_{i<j} \mathbb{Z}/p^{m_{ij}}\mathbb{Z}$, and generators of this group are the classes of cocycles γ_{kl} $(1 \leqslant k < l \leqslant s)$ such that $\gamma_{kl}(x_i^2) = 0$ for all i, while $\gamma_{kl}(x_i x_j) = \delta_{ki}\delta_{lj}u_{kl}$, where u_{kl} is a fixed element of $\mathbb{T}_p$ of order $p^{m_{kl}}$.

Let now M be a lattice in a cyclotomic field $\boldsymbol{K}$ of order p^m such that a_1 acts as the multiplication by the primitive root ζ of unity of order p^m and all a_i $(i > 1)$ act trivially. As we can choose any lattice in the same genus, we can suppose that $M = \mathbb{Z}[\zeta]$. Therefore, the formulae (5.2) show that ξ is a cocycle if and only if $\xi(x_i) = 0$ for $i > 1$. As $\zeta - 1$ is a prime element in $\mathbb{Z}[\zeta]$ with the norm p [1], $M/(\zeta - 1)M \simeq \mathbb{Z}/p\mathbb{Z}$. Hence, adding a coboundary ∂u to ξ, one can make $\xi(x_1) = \lambda$, where $\lambda \in \mathbb{Z}$ is defined modulo p. Thus $H^1(G, M) \simeq \mathbb{Z}/p\mathbb{Z}$. The formulae (5.3) show that γ is a cocycle if and only if $\gamma(x_1^2) = 0$, $\gamma(x_i x_j) = 0$ if $1 < i < j$ and $p^{m_i}\gamma(x_1 x_i) = (\zeta - 1)\gamma(x_i^2)$. The formulae (5.1) imply that, adding a coboundary, one can make $\gamma(x_1 x_i) = \lambda_i$, where $\lambda_i \in \mathbb{Z}$ is defined modulo p. Then $\gamma(x_i^2)$ is uniquely defined. Thus $H^2(G, M) \simeq (\mathbb{Z}/p\mathbb{Z})^{s-1}$. The generators of this group are the classes of cocycles γ_k $(1 < k \leqslant s)$ such that $\gamma_k(x_1^2) = \gamma_(x_i x_j) = 0$ for all $1 < i < j$, $\gamma_k(x_1 x_i) = \delta_{ik}$, $\gamma_k(x_i)^2 = 0$ if $i \neq k$ and $(1 - \zeta)\gamma_k(x_k) = p^{m_k}$.

Consider the dual module D_pM. As the multiplication by $\zeta - 1$ is injective on M, it is surjective on D_pM. On the other hand, the subgroup $\{\, u \in D_pM \mid (\zeta - 1)u = 0 \,\}$ is dual to $M/(\zeta - 1)M$, so it is generated by one element u_0 of period p. Thus, adding a couboundary ∂u to a 1-cocycle ξ, one can make $\xi(x_1) = 0$. Then $(\zeta - 1)\xi(x_i) = 0$ if $i > 1$, whence $\xi(x_i) = \lambda_i u_0$, where $\lambda_i \in \mathbb{Z}/p\mathbb{Z}$. Hence $H^1(G, D_pM) \simeq \boldsymbol{P}_1^{s-1} \simeq (\mathbb{Z}/p\mathbb{Z})^{s-1}$. In the same way, adding a coboundary to a 2-cocycle γ, we can make $\gamma(x_1 x_i) = 0$ for $i > 1$. Then the conditions (5.3) give $(\zeta - 1)\gamma(x_i^2) = 0$ for all i, whence $\gamma(x_i^2) = \lambda_i u_0$ $(\lambda_i \in \mathbb{Z}/p\mathbb{Z})$, and $(\zeta - 1)\gamma(x_i x_j) = 0$ for $1 < i < j$, whence $\gamma(x_i x_j) = \lambda_{ij} u_0$ $(\lambda_{ij} \in \mathbb{Z}/p\mathbb{Z})$. Therefore $H^2(G, D_pM) \simeq \mathbb{T}_1^{(s^2-s+2)/2} \simeq (\mathbb{Z}/p\mathbb{Z})^{(s^2-s+2)/2}$. Th generators of this group are cocycles γ_k $(1 \leqslant k \leqslant s)$

and γ_{kl} $(1 < k < l \leqslant s)$ such that $\gamma_k(x_1x_i) = \gamma_{kl}(x_1x_i)$ for $i > 1$, $\gamma_k(x_i^2) = \delta_{ik}u_0$, $\gamma_k(x_ix_j) = 0$ for $i \neq j$, $\gamma_{kl}(x_i^2 = 0$ for all i and $\gamma_{kl}(x_ix_j) = \delta_{ik}\delta_{jl}u_0$.

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