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SURVEY ARTICLE

A survey article on some subgroup embeddings and local properties for soluble PST-groups

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Abstract. Let G be a group and p a prime number. G is said to be a Y_p-group if whenever K is a p-subgroup of G then every subgroup of K is an S-permutable subgroup in $N_G(K)$. The group G is a soluble PST-group if and only if G is a Y_p-group for all primes p.

One of our purposes here is to define a number of local properties related to Y_p which lead to several new characterizations of soluble PST-groups. Another purpose is to define several embedding subgroup properties which yield some new classes of soluble PST-groups. Such properties include weakly S-permutable subgroup, weakly semipermutable subgroup, and weakly seminormal subgroup.

1. Introduction and statement of results

All groups considered in this survey are finite.

A subgroup H of a group G is said to permute with a subgroup K of G if HK is a subgroup of G. H is said to be permutable (S-permutable) if it permutes with all the subgroups (Sylow subgroups, respectively) of G. Examples of permutable subgroups include the normal subgroups of G. However, if G is a modular, non-Dedekind p-group, p a prime, we see permutability is quite different from normality. For instance, letting C_n denote the cyclic group of order n, we see that C_2 is permutable but not

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normal in the group $C_8 \rtimes C_2$ where the generator for C_2 maps a generator of C_8 to its fifth power. Kegel [12] proved that an S-permutable subgroup is always subnormal. In particular, a permutable subgroup of a group is subnormal.

A group G is called a PST-group (PT-group) if S-permutability (permutability, respectively) is a transitive relation. By Kegel's result, a group G is a PST-group (PT-group) if every subnormal subgroup of G is S-permutable (permutable, respectively) in G.

A number of research papers have been written on these groups. See, for example, [1–10, 16].

Another class of groups is the so called T-groups. A group G is a T-group if normality in G is transitive, that is, if $H \trianglelefteq K \trianglelefteq G$ then $H \trianglelefteq G$. There are several nice characterizations of T-groups in [15].

Soluble PST-, PT- and T-groups were characterized by Agrawal [1], Zacher [18] and Gaschütz [11] respectively.

Theorem 1. 1) *A soluble group G is a PST-group if and only if the nilpotent residual L of G is an abelian Hall subgroup of G on which G acts by conjugation as a group of power automorphisms.*

2) *A soluble PST-group G is a PT-group (T-group) if and only if G/L is a modular (Dedekind, respectively) group.*

Note that if G is a soluble T-, PT- or PST-group then every subgroup and every quotient of G inherits the same properties.

We mention that in [6, Chapter 2] many of the beautiful results on these classes of groups are presented.

Subgroup embedding properties closely related to permutability and S-permutability are semipermutability and S-semipermutability. A subgroup X of a group G is said to be *semipermutable* (*S-semipermutable*) in G provided that it permutes with every subgroup (Sylow subgroup, respectively) K of G such that $\gcd(|X|, |K|) = 1$. A semipermutable subgroup of a group need not be subnormal. For example, a 2-Sylow subgroup of the non-abelian group of order 6 is semipermutable but not subnormal.

Note that a subnormal semipermutable (S-semipermutable) subgroup X of a group G must be normalized by every subgroup (Sylow subgroup, respectively) P of G such that $\gcd(|X|, |P|) = 1$. This observation was the basis for Beidleman and Ragland [10] to introduce the following subgroup embedding properties.

A subgroup X of a group G is said to be *seminormal* (*S-seminormal*)[1] in G if it is normalized by every subgroup (Sylow subgroup, respectively) K of G such that $\gcd(|X|, |K|) = 1$.

By [10, Theorem 1.2], a subgroup of a group is seminormal if and only if it is S-seminormal. Furthermore, seminormal subgroups are not necessarily subnormal; it is enough to consider a non-subnormal subgroup H of a group G such that $\pi(H) = \pi(G)$. To see some of the properties of these subgroups see Examples 1, 2 and 3 in Section 2. However, a p-subgroup of a group G, p a prime, which is also seminormal is subnormal ([10, Theorem 1.3]).

Semipermutable, S-semipermutable and seminormal subgroups have been investigated in [10, 17, 19, 20].

The following result gives some embedding properties on subnormal subgroups of a group which yields several characterizations of soluble PST-groups.

Theorem 2 ([10, Theorem 1.5]). *Let G be a soluble group. Then the following statements are pairwise equivalent:*

1) *G is a PST-group;*
2) *all the subnormal subgroups of G are seminormal in G;*
3) *all the subnormal subgroups of G are semipermutable in G;*
4) *all the subnormal subgroups of G are S-semipermutable in G.*

The following is a beautiful result of H. Wielandt which seems to have inspired the authors of [5] to introduce the concept of weakly S-permutable subgroups of a subgroup H of a group G.

Theorem 3 ([13, Theorem 7.3.3]). *Let H be a subgroup of a group G. Then the following statements are equivalent:*

1) *H is subnormal in G;*
2) *H is subnormal in $\langle H, H^g \rangle$ for all $g \in G$;*
3) *H is subnormal in $\langle H, g \rangle$ for all $g \in G$.*

This embedding property led to several new characterizations of soluble PST-groups which are presented in the following theorem from [5].

Theorem 4 ([5]). *Let G be a group. The following statements are pairwise equivalent:*

1) *G is a soluble PST-group.*
2) *Every subgroup of G is weakly S-permutable in G.*

[1] Note that the term *seminormal* has several different meanings in the literature.

3) *For every prime number p, every p-subgroup of G is weakly S-permutable in G.*

Theorems 3 and 4 motivate the following definition.

Definition 1. Let H be a subgroup of a group G.

1) H is said to be weakly S-permutable in G if whenever $g \in G$ and H is S-permutable in $\langle H, H^g \rangle$, then H is S-permutable in $\langle H, g \rangle$.
2) H is said to be weakly semipermutable in G if whenever $g \in G$ and H is semipermutable in $\langle H, H^g \rangle$, then H is semipermutable in $\langle H, g \rangle$.
3) H is said to be weakly S-semipermutable in G if whenever $g \in G$ and H is S-semipermutable in $\langle H, H^g \rangle$, then H is S-semipermutable in $\langle H, g \rangle$.
4) H is said to be weakly seminormal in G if whenever $g \in G$ and H is seminormal in $\langle H, H^g \rangle$, then H is seminormal in $\langle H, g \rangle$.

The next theorem relates the concept of S-permutable subgroups of a group G with weakly S-permutable subgroups of G.

Theorem 5 ([5])**.** *A subgroup H of a group G is S-permutable in G if and only if H is S-permutable in $\langle H, g \rangle$ for every $g \in G$.*

Theorem 5 and its proof are used to establish Theorem 6 in [7].

Theorem 6 ([7])**.** *Let H be a subnormal subgroup of a group G. Then*

1) *H is S-semipermutable in G if and only if H is S-semipermutable in $\langle H, g \rangle$ for every $g \in G$.*
2) *H is seminormal in G if and only if H is seminormal in $\langle H, g \rangle$ for every $g \in G$.*

A class of groups G is a PST-group if and only if Sylow permutability is a transitive relation in G.

We next define several local properties which provide a number of new local characterizations of soluble PST-groups.

Definition 2. Let G be a group and p be a prime. Then

1) G is a Y_p-group if for every p-subgroup K of G every subgroup of K is S-permutable in $N_G(K)$.
2) G is a $\hat{Y}_p$-group if for every p-subgroup K of G every subgroup of K is semipermutable in $N_G(K)$.
3) G is a $\tilde{Y}_p$-group if for every p-subgroup K of G every subgroup of K is S-semipermutable in $N_G(K)$.

4) G is a $\tilde{\tilde{Y}}_p$-group if for every p-subgroup K of G every subgroup of K is seminormal in $N_G(K)$.
5) G is a $\underline{Y}_p$-group if for every p-subgroup K of G every subgroup of K is weakly S-permutable in $N_G(K)$.
6) G is a $\underline{\tilde{Y}}_p$-group if for every p-subgroup K of G every subgroup of K is weakly S-semipermutable in $N_G(K)$.
7) G is a $\underline{\tilde{\tilde{Y}}}_p$-group if for every p-subgroup K of G every subgroup of K is weakly seminormal in $N_G(K)$.

The following result is a very nice local characterization of soluble PST-groups.

Theorem 7 ([6, Theorem 2.2.9] and [4, Theorem 4]). *A group G is a soluble PST-group if and only if it satisfies Y_p for all primes p.*

Our next result shows how some of the classes in Definition 2 are related to the class Y_p.

Theorem 8 ([10, Theorem 1.8]). *Let p be a prime and G a group. Then $Y_p = \hat{Y}_p = \tilde{Y}_p = \tilde{\tilde{Y}}_p$.*

Using Theorems 7 and 8 we note that the next result shows all of the classes $\underline{Y}_p$, $\underline{\tilde{Y}}_p$ and $\underline{\tilde{\tilde{Y}}}_p$ are just Y_p.

Theorem 9 ([7]). *Let p be a prime and G a group. Then*
1) *$G \in Y_p$ if and only if $G \in \underline{Y}_p$.*
2) *$G \in \tilde{Y}_p$ if and only if $G \in \underline{\tilde{Y}}_p$.*
3) *$G \in \tilde{\tilde{Y}}_p$ if and only if $G \in \underline{\tilde{\tilde{Y}}}_p$.*

From Theorems 8 and 9 we obtain several results that yield new local characterizations of soluble PST-groups.

Corollary 1. *Let p be a prime. Then*

$$Y_p = \underline{Y}_p = \hat{Y}_p = \tilde{Y}_p = \underline{\tilde{Y}}_p = \tilde{\tilde{Y}}_p = \underline{\tilde{\tilde{Y}}}_p.$$

Using Theorem 9 and Corollary 1 we obtain one of the main results of this survey paper.

Theorem 10 ([7]). *Let G be a group and p a prime. Then the following statements are pairwise equivalent:*
1) *G is a soluble PST-group.*
2) *G is a Y_p-group for all primes p.*

3) *G is a $\underline{Y}_p$-group for all primes p.*
4) *G is a $\hat{Y}_p$-group for all primes p.*
5) *G is a $\tilde{Y}_p$-group for all primes p.*
6) *G is a $\underline{\tilde{Y}}_p$-group for all primes p.*
7) *G is a $\tilde{\tilde{Y}}_p$-group for all primes p.*
8) *G is a $\underline{\tilde{\tilde{Y}}}_p$-group for all primes p.*

2. Examples

Example 1. Let S_4, A_4 and K_4 denote, respectively, the symmetric group of order 4, the alternating group of order 4, and the Klein 4-group. Let $G = S_4$ and let $H = \langle (123) \rangle$. The H is S-semipermutable in G but it is not semipermutable in G since it does not permute with an element of order 2 in K_4, the Sylow 2-subgroup of A_4.

An S-permutable subgroup of a group is subnormal. That this is not the case with S-semipermutable subgroups can be seen in the subgroup H in S_4. Notice that H is not seminormal in S_4.

Example 2. Let $D_{10} = \langle x, y \mid x^5 = y^2 = 1, x^y = x^{-1} \rangle$, the dihedral group of order 10, and $C_{15} = \langle t, s \mid t^5 = s^3 = 1, ts = st \rangle$, the cyclic group of order 15. Let $G = D_{10} \times C_{15}$ and let $K = \langle y \rangle \times \langle t \rangle$. Since $\langle s \rangle$ centralizes K it follows that K is seminormal in G. Note that K is not subnormal in G.

Example 3. Let $H = \langle x \rangle \rtimes \langle y \rangle$ be a semidirect product of a cyclic group, $\langle x \rangle$, of order 11 by a cyclic group, $\langle y \rangle$, of order 5. Let $G = H \times S_4$. Set $K = \langle x \rangle \times S_3$ where S_3 is a copy of the symmetric group on three elements in S_4. Then K is a seminormal subgroup of G which is not subnormal.

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