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Finite automaton actions of free products of groups

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Abstract. It is shown that for groups G and H that act faithfully by finite state automorphisms on regular rooted trees their free product $G * H$ admits a faithful action by finite state automorphisms on some regular rooted tree.

The class of groups that act faithfully on regular rooted trees by finite state automorphisms (equivalently, groups defined by finite initial automata over finite alphabets) constitute a remarkable family among residually finite groups. It is rich of many interesting and important groups with a solid influence on different branches of mathematics. One can easily show that this class is closed under finite direct products. The Kaloujnine-Krasner theorem implies that it is closed under finite extensions as well. The purpose of this short note is to show that this class is closed under finite free products. Thus, we positively solve the following problem from Kourovka Notebook (see [1, Problem 16.85]).

Suppose that groups G, H act faithfully on a regular rooted tree by finite state automorphisms. Can their free product $G * H$ act faithfully on a regular rooted tree by finite state automorphisms?

In fact, to obtain the affirmative solution it is sufficient to prove the following

Theorem 1. *The free product* $\mathrm{FAut}\, T_n * \mathrm{FAut}\, T_n$ *is isomorphic to a subgroup of* $\mathrm{FAut}\, T_{3n}$, $n \geqslant 2$.

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All notions and notations we use are standard for groups defined by finite automata and can be found in [2] and [3]. In particular, by $\mathrm{FAut}\, T_m, m \geqslant 2$ we denote the group of all finite state automorphisms of m-regular rooted tree.

The idea of the main construction used in the proof is influenced by the paper [4].

Proof of Theorem 1. We start with three disjoint alphabets X, Y and Z, each of cardinality n. We assume that their elements are enumerated, i.e. let

$$\mathsf{X} = \{x_0, \dots, x_{n-1}\}, \quad \mathsf{Y} = \{y_0, \dots, y_{n-1}\} \quad \text{and} \quad \mathsf{Z} = \{z_0, \dots, z_{n-1}\}.$$

Consider two isomorphic copies G and H of the group $\mathrm{FAut}\, T_n$ as groups whose elements are defined by finite initial automata over alphabets X and Y correspondingly. The elements of the group $\mathrm{FAut}\, T_{3n}$ will be defined by finite initial automata over the alphabet $\mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z}$. We will construct isomorphic embeddings Ψ_1 and Ψ_2 of the groups G and H correspondingly into the group $\mathrm{FAut}\, T_{3n}$ and show that the subgroup generated by their images splits into the free product $\Psi_1(G) * \Psi_2(H)$. The proof is divided into three parts.

Step 1. Construction of Ψ_1, Ψ_2.

We describe the map $\Psi_1 : G \to \mathrm{FAut}\, T_{3n}$. The map Ψ_2 is defined analogously replacing X by Y and vice versa in the constructions below.

For each element $g \in G$ fix a finite initial automaton

$$A_g = (Q, \mathsf{X}, \lambda, \mu, q_g)$$

such that g is defined by A_g. Here Q denotes the set of inner states of A_g, λ and μ its transition and output functions respectively, q_g the initial state.

The definition of Ψ_1 consists of two stages.

At the first stage we define another automaton B_g by adding to the set Q of states of A_g two states s_g, d_g. The functions λ, μ are extended by the equalities

$$\lambda(s_g, x_0) = q_g, \quad \lambda(s_g, x) = d_g,\ x \in \mathsf{X} \setminus \{x_0\}, \quad \lambda(d_g, x) = d_g,\ x \in \mathsf{X},$$
$$\mu(s_g, x) = \mu(d_g, x) = x,\ x \in \mathsf{X}.$$

Then

$$B_g = (Q \cup \{s_g, d_g\}, \mathsf{X}, \lambda, \mu, s_g).$$

Denote by $\psi_1(g)$ the element of the group G defined by the automaton B_g. Then the rule

$$g \mapsto \psi_1(g)$$

defines an isomorphic embedding of G into G. Indeed, the definition of the automaton B_g implies that for arbitrary $x \in \mathsf{X}, w \in \mathsf{X}^*$ the following equality holds

$$(xw)^{\psi_1(g)} = \begin{cases} xw^g, & \text{if } x = x_0, \\ xw & \text{otherwise.} \end{cases}$$

Hence, ψ_1 is injective and preserves multiplication.

Note, that the map ψ_1 is nothing but the identification of G with the first term in the direct product

$$G^{(0)} \times \ldots \times G^{(n-1)}, \quad G^{(i)} \simeq G, 0 \leqslant i \leqslant n-1.$$

This direct product is a natural subgroup of G as soon as we look at the action of G on X^*.

Let us proceed to the second stage of the definition. We construct a finite initial automaton C_g over the alphabet $\mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z}$. Its set of inner states is $Q \cup \{s_g, d_g\} \cup Q \times \mathsf{Z}$.

The transition function λ is extended by the following rules:

$$\begin{gathered} \lambda(d_g, y) = d_g, \quad y \in \mathsf{Y}, \qquad \lambda(d_g, z) = s_g, \quad z \in \mathsf{Z}, \\ \lambda(q, y) = s_g, \quad q \in Q \cup \{s_g\} \cup Q \times \mathsf{Z},\ y \in \mathsf{Y}, \\ \lambda(s_g, z) = s_g, \quad z \in \mathsf{Z}, \qquad \lambda(q, z) = (q, z), \quad q \in Q,\ z \in \mathsf{Z}, \\ \lambda((q, z), t) = s_g, \quad q \in Q,\ z \in \mathsf{Z},\ t \in \mathsf{Y} \cup \mathsf{Z}, \\ \lambda((q, z), x) = \lambda(s_g, x), \quad q \in Q,\ z \in \mathsf{Z},\ x \in \mathsf{X}. \end{gathered}$$

The output function μ is extended on new states by the rule:

$$\mu((q, z), x) = \mu(s_g, x), \quad q \in Q,\ z \in \mathsf{Z},\ x \in \mathsf{X}.$$

Let us define the action of the output function μ in new states on letters from Y. For arbitrary $q \in Q, i, j \in \{0, \ldots, n-1\}$ the value $\mu(q, x_{(j-i) \bmod n})$ is a well-defined letter $x \in \mathsf{X}$. Then $x = x_k$ for some $k \in \{0, \ldots, n-1\}$ and we define

$$\mu((q, z_i), y_j) = y_{(k+i) \bmod n}.$$

In other words, the permutation π_{Y} on the alphabet Y defined in the state (q, z_i) "mimics" the permutation π_{X} on X defined in the state q. More precisely, if we denote by σ the one-to-one correspondence between X and Y given by the rule $x_j \mapsto y_{(j+k) \bmod n}, 0 \leqslant j \leqslant n-1$, then $\pi_{\mathsf{Y}} = \sigma(\pi_{\mathsf{X}}(\sigma^{-1}))$.

None of the rest of the states change letters from Y. Finally, none of the states change letters from Z.

Then the automaton

$$C_g = (Q \cup \{s_g, d_g\} \cup Q \times \mathsf{Z}, \mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z}, \lambda, \mu, s_g)$$

is well-defined.

Denote by $\Psi_1(g)$ the element of the group $\mathrm{FAut}\, T_{3n}$ defined by the automaton C_g.

Step 2. Ψ_1, Ψ_2 *are isomorphic embeddings.*

As above, we consider Ψ_1 only. The proof for Ψ_2 is completely analogous.

Let $g \in G$, $w \in (\mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z})^*$. We examine the word $w^{\Psi_1(g)}$ to show that Ψ_1 preserve multiplication in G. The main idea is to split w into sub-words such that under the action of $\Psi_1(g)$ each image coincides with the one obtained by the action of g. Under the action of $\Psi_1(g)$ in w letters from $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ are transformed into the letters of the same alphabet. The letters from Z are preserved under the action of $\Psi_1(g)$.

For $w \in \mathsf{X}^*$ the definition of the automaton C_g directly implies the equality

$$w^{\Psi_1(g)} = w^{\psi_1(g)}.$$

In particular, the mapping Ψ_1 is injective.

Let the first letter of w belongs to X and differs from x_0. If w contain no letters from Z then w is a fixed point under $\Psi_1(g)$. In other case $w = xw_1zw_2$ for some $w_1 \in (\mathsf{X} \cup \mathsf{Y})^*$, $z \in \mathsf{Z}$, $w_2 \in (\mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z})^*$. Then

$$w^{\Psi_1(g)} = (xw_1zw_2)^{\Psi_1(g)} = xw_1zw_2^{\Psi_1(g)}.$$

Assume that $w \in (\mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z})^* \setminus \mathsf{X}^*$ and the first letter of w either belongs to $\mathsf{Y} \cup \mathsf{Z}$ or equals x_0. Then w can be uniquely written in the form

$$w = w_1u_1w_2u_2,$$

where $w_1, w_2 \in \mathsf{X}^*$, $u_1 \in (\mathsf{Y} \cup \mathsf{Z})^+$, $u_2 \in (\mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z})^*$ and the word w_1 either empty or its first letter equals x_0. In the former case from the definition of C_g we obtain the following equality:

$$(w_1u_1w_2u_2)^{\Psi_1(g)} = w_1u_1(w_2u_2)^{\Psi_1(g)}.$$

In the latter case if the first letter of u_1 belongs to Y or the length of u_1 is greater than 1 and the first and the second letters of u_1 belong to Z the definition of C_g implies

$$(w_1u_1w_2u_2)^{\Psi_1(g)} = w_1^{\Psi_1(g)}u_1(w_2u_2)^{\Psi_1(g)}.$$

It is left to consider two cases.

Case I. Let $w_1 = x_0w_3$ for some $w_3 \in \mathsf{X}^*$, $u_1 = z$ for some $z \in \mathsf{Z}$ and the word w_2 is non-empty. In this case $w = x_0w_3zxw_4u_2$ for some $x \in \mathsf{X}, w_4 \in \mathsf{X}^*$ and we obtain

$$(x_0w_3zxw_4u_2)^{\Psi_1(g)} = x_0w_3^g z(xw_4u_2)^{\Psi_1(g)}.$$

Case II. Let $w_1 = x_0w_3$ for some $w_3 \in \mathsf{X}^*$ and $u_1 = zyu_3$ for some $z \in \mathsf{Z}$, $y \in \mathsf{Y}$, $u_3 \in (\mathsf{Y}\cup\mathsf{Z})^*$. Then $y = y_j, z = z_i$ for some $i, j \in \{0, \ldots, n-1\}$, $w = x_0w_3z_iy_ju_3w_2u_2$ and we obtain

$$(x_0w_3z_iy_ju_3w_2u_2)^{\Psi_1(g)} = (x_0w_3z_iy_j)^{\Psi_1(g)}u_3(w_2u_2)^{\Psi_1(g)}.$$

Further we obtain

$$(x_0w_3z_iy_j)^{\Psi_1(g)} = (x_0w_3)^{\psi_1(g)}z_iy_{(i+k) \bmod n},$$

where number k is uniquely determined by the equality

$$(x_0w_3x_{(j-i) \bmod n})^{\Psi_1(g)} = x_0w_3^g x_k.$$

The last equality means that $\Psi_1(g)$ acts on words of the form $x_0w_3z_iy_j$ as g acts on words $w_3x_{(j-i) \bmod n}$, i.e. corresponding permutation groups are isomorphic.

Step 3. $\langle\Psi_1(G), \Psi_2(H)\rangle$ *splits as* $\Psi_1(G) * \Psi_2(H)$.

It is required to prove that for arbitrary positive integer m and non-identity elements $g_1, \ldots, g_m \in G$, $h_1, \ldots, h_m \in H$ the product

$$\Psi_1(g_1)\Psi_2(h_1)\ldots\Psi_1(g_m)\Psi_2(h_m)$$

defines a non-trivial permutation on the set $(\mathsf{X}\cup\mathsf{Y}\cup\mathsf{Z})^*$. Denote by $a_1, b_1, \ldots, a_m, b_m$, the elements

$$\Psi_1(g_1), \quad \Psi_1(g_1)\Psi_2(h_1), \quad \ldots, \quad \Psi_1(g_1)\Psi_2(h_1)\ldots\Psi_1(g_m)\Psi_2(h_m)$$

correspondingly. We will find words

$$u_1, \ldots, u_m, u_{m+1} \in \mathsf{X}^* \quad \text{and} \quad v_1, \ldots, v_m \in \mathsf{Y}^*$$

and letters

$$t_1, \ldots, t_m, s_1, \ldots, s_m \in \mathsf{Z}$$

such that the word w of the form

$$w = u_1 t_1 v_1 s_1 \ldots u_m t_m v_m s_m u_{m+1}$$

is not a fixed point under the action of the element b_m.

For any non-trivial element $g \in G$ there exist a non-empty word from X^* that is not a fixed point under the action of g. In each non-fixed word of the shortest possible length the last letter does not coincide with the last letter of its image under g. Note, that all other letters are preserved under the action of g. Hence, there exist a word $w(g) \in \mathsf{X}^*$ and numbers $i(g), k(g) \in \{0, \ldots, n-1\}$ such that $i(g) \neq k(g)$ and

$$(w(g) x_{i(g)})^g = w(g) x_{k(g)}.$$

Denote $j(g) = (i(g) - k(g)) \bmod n$. Then $j(g) \neq 0$. In the same way for any non-trivial element $h \in H$ one can choose a word $w(h) \in \mathsf{Y}^*$ and numbers $i(h), k(h) \in \{0, \ldots, n-1\}$ such that $i(h) \neq k(h)$ and

$$(w(h) y_{i(h)})^h = w(h) y_{k(h)}.$$

Denote $j(h) = (i(h) - k(h)) \bmod n$. Then $j(h) \neq 0$.

Now define

$$u_1 = x_0 w(g_1), \quad u_r = x_{j(h_{r-1})} w(g_r),\ 2 \leqslant r \leqslant m, \quad u_{m+1} = x_{j(h_m)},$$

$$v_r = y_{j(g_r)} w(h_r),\ 1 \leqslant r \leqslant m, \quad \text{and} \quad t_r = z_{k(g_r)}, s_r = z_{k(h_r)},\ 1 \leqslant r \leqslant m.$$

Hence, we put

$$w = x_0 w(g_1) z_{k(g_1)} y_{j(g_1)} w(h_1) z_{k(h_1)} x_{j(h_1)} \cdots \\ \cdots x_{j(h_{m-1})} w(g_m) z_{k(g_m)} y_{j(g_m)} w(h_m) z_{k(h_m)} x_{j(h_m)}.$$

The definition of Ψ_1 and Ψ_2 implies that under the action of mappings $a_1, b_1, \ldots, b_{m-1}, a_m, b_m$ in the word w letters

$$y_{j(g_1)}, x_{j(h_1)}, \ldots, x_{j(h_{m-1})}, y_{j(g_m)}, x_{j(h_m)}$$

one by one become

$$y_0, x_0, \ldots, x_0, y_0, x_0.$$

More precisely, one can verify by induction the following $2m$ equalities

$$\begin{aligned}
w^{a_1} &= (u_1)^{a_1} t_1 y_0 w(h_1) s_1 u_2 \dots u_m t_m v_m s_m u_{m+1},\\
w^{b_1} &= (u_1 t_1 v_1)^{b_1} s_1 x_0 w(g_2) \dots u_m t_m v_m s_m u_{m+1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
w^{a_m} &= (u_1 t_1 v_1 s_1 \dots u_m)^{a_m} t_m y_0 w(g_m) s_m u_{m+1},\\
w^{b_m} &= (u_1 t_1 v_1 s_1 \dots u_m t_m v_m)^{b_m} s_m x_0.
\end{aligned}$$

In particular, $w^{b_m} \neq w$. The proof is complete. □

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