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R. M. Cherniha (Inst. Math. Ukrainian Acad.Sci., Kiev),
H. Wilhelmsson (Inst. for Electromagnetic Field Theory, Chalmers Univ., Göteborg)

SYMMETRY AND EXACT SOLUTION OF HEAT-MASS TRANSFER EQUATIONS IN THERMONUCLEAR PLASMA

СИМЕТРІЯ І ТОЧНІ РОЗВ'ЯЗКИ РІВНЯНЬ ТЕПЛОМАСОПЕРЕНОСУ В ТЕРМОЯДЕРНІЙ ПЛАЗМІ

For the nonlinear system of partial differential equation, which describe the evolution of temperature and density in TOKAMAK plasmas, multiparameter families of exact solutions are constructed. The solutions are constructed by the Lie-method reduction of initial systems of equations to system of ordinary differential equations. Examples of non-Lie ansätze and exact solutions are also presented.

Побудовано багатопараметричні сім'ї точних розв'язків для нелінійних систем рівнянь з частинними похідними, якими описується еволюція температури та густини плазми у термоядерній плазмі. Розв'язки побудовані шляхом ліївської редукції розглядуваних нелінійних систем до систем звичайних диференціальних рівнянь. Наведено також приклади неліївських анзаців та точних розв'язків.

1. Introduction. In the papers [1, 2], a system of nonlinear equation is introduced to describe temperature and density evolution in thermonuclear plasma. This system of equations has the form

$$\begin{aligned} U_t &= x^{1-n}(x^{n-1}A_1(U)U_x)_x + x^{1-n}(x^{n-1}B_1(U,V)V_x)_x + C_1(U,V), \\ V_t &= x^{1-n}(x^{n-1}A_2(U)V_x)_x + x^{1-n}(x^{n-1}B_2(U,V)U_x)_x + C_2(U,V), \end{aligned} \tag{1}$$

where $U = U(t,x)$, $V = V(t,x)$, $x^2 = x_1^2 + x_2^2 + \ldots + x_n^2$, lower indices designate differentiation with respect to t and x. Nonlinearities A_k, B_k, C_k, $k = 1, 2$, in real situations can be given by functions

$$\begin{aligned} A_1(U) &= a_1 U^{\alpha_1}, \quad B_1(U,V) = b_1 U^{\alpha_1+1}V^{-1}, \quad C_1(U,V) = \sum_{i=1}^{n_i} c_{1i} U^{\delta_{1i}} V^{\kappa_{1i}}, \\ A_2(U) &= a_2 U^{\alpha_2}, \quad B_2(U,V) = b_2 U^{\alpha_2-1}V, \quad C_2(U,V) = \sum_{j=1}^{n_j} c_{2j} U^{\delta_{2j}} V^{\kappa_{2j}}, \end{aligned} \tag{2}$$

where $a_k, b_k, \alpha_k, c_{1i}, c_{2j}, \delta_{1i}, \delta_{2j}, \kappa_{1i}, \kappa_{2j} \in \mathbb{R}$, $n_i, n_j \in \mathbb{N}$, $k = 1, 2$.

In the case of one space variable $(n = 1)$ taking into account (2), system of equations (1) has the form

$$\begin{aligned} U_t &= a_1(U^{\alpha_1}U_x)_x + b_1((U^{\alpha_1+1}/V)V_x)_x + \sum_{i=1}^{n_i} c_{1i} U^{\delta_{1i}} V^{\kappa_{1i}}, \\ V_t &= a_2(U^{\alpha_2}V_x)_x + b_2(U^{\alpha_2-1}VU_x)_x + \sum_{j=1}^{n_j} c_{2j} U^{\delta_{2j}} V^{\kappa_{2j}}. \end{aligned} \tag{3}$$

There is no complete symmetry analysis of the nonlinear systems (1) and (3), and classes of exact solutions for these systems have not been constructed. Let us note that some classes of nonlinear systems of two parabolic-type equations that are invariant under the Galilei algebra and its extensions are constructed in [3–5].

In the current paper, we carry out symmetry analysis for system of equations (3), reduce it to systems of ordinary differential equations, and construct classes of exact solutions. Due to fact that obtained exact solutions of system of equations (3) contain several arbitrary constants, we can satisfy boundary conditions [1], which are characteristic for thermonuclear plasma, by choosing appropriate constants.

Below, we assume that system of equations (3) is a second-order system, and $a_1 b_1 \neq 0$ or $a_2 b_2 \neq 0$ and, thus, the nondiagonal elements do not vanish, and there are nonvanishing elements among c_{1i}, c_{2j}.

2. Symmetry Properties of system of equations (3). It is evident that system of equations (3) admits operators of shift by coordinates t and x, namely,

$$P_x = \frac{\partial}{\partial x} \equiv \partial_x, \quad P_t = \frac{\partial}{\partial t} \equiv \partial_t. \tag{4}$$

To construct other symmetry operators for system of equations (3), we take into account that dilation operators

$$D = kx\partial_x + lt\partial_t + m_1 U\partial_U + m_2 V\partial_V \tag{5}$$

are characteristic for power nonlinearities, where

$$k,\ l,\ m_1,\ m_2 \in \mathbb{R}, \quad \partial_U = \frac{\partial}{\partial U}, \quad \partial_V = \frac{\partial}{\partial V}.$$

Depending on relations between the coefficients k, l, m_1, m_2, where nonlinearities (2) are taken into account, we obtain three types of D (5).

Note that the case $k = l = 0$ is not considered below because it has the corresponding operator D, and the reduction by such an operator $D = m_1 U\partial_U + m_2 V\partial_V$ transforms system of equations (3) to an overdetermined system of partial differential equations.

***Proposition* 1.** *system of equations* (3) *is invariant with respect to the operator*

$$D = x\partial_x + 2t\partial_t + m_1 U\partial_U + m_2 V\partial_V, \tag{6}$$

iff it has the form

$$\begin{aligned} U_t &= a_1 U_{xx} + b_1 (UV^{-1} V_x)_x + \sum_{i=1}^{n_i} c_{1i} U^{\delta_{1i}} V^{\kappa_{1i}}, \\ V_t &= a_2 V_{xx} + b_2 (U^{-1} V U_x)_x + \sum_{j=1}^{n_j} c_{2j} U^{\delta_{2j}} V^{\kappa_{2j}}, \end{aligned} \tag{7}$$

where the powers δ_{1i}, κ_{1i}, δ_{2j}, κ_{2j} *for all* i, j *satisfy the system of algebraic equations*

$$\begin{aligned} m_1(1 - \delta_{1i}) &= m_2 \kappa_{1i} + 2, \\ m_1 \delta_{2j} &= m_2(1 - \kappa_{2j}) - 2; \end{aligned} \tag{8}$$

or the form

$$\begin{aligned} U_t &= a_1 (U^{\alpha_1} U_x)_x + b_1 (U^{\alpha_1 + 1} V^{-1} V_x)_x + V^{\kappa_1} \sum_{i=1}^{n_i} c_{1i} U^{\delta_{1i}}, \\ V_t &= a_2 (U^{\alpha_2} V_x)_x + b_2 (U^{\alpha_2 - 1} V U_x)_x + V^{1+\kappa_1} \sum_{j=1}^{n_j} c_{2j} U^{\delta_{2j}}, \end{aligned} \tag{9}$$

when $m_1 = 0$, $m_2 = -2/\kappa_1$, $\kappa_1 \neq 0$.

The proof of the above statement can be carried out by using transformations generated by operator (6)

$$x' = ex, \quad x' = e^2 t, \quad U' = e^{m_1} U, \quad V' = e^{m_2} V, \tag{10a}$$

where e is the group parameter.

Indeed, it follows from (10a) that

$$\begin{gathered} U'_{t'} = U_t e^{m_1 - 2}, \quad U'_{x'} = U_x e^{m_1 - 1}, \quad U'_{x'x'} = U_{xx} e^{m_1 - 2}, \\ V'_{t'} = V_t e^{m_2 - 2}, \quad V'_{x'} = V_x e^{m_2 - 1}, \quad V'_{x'x'} = V_{xx} e^{m_2 - 2}. \end{gathered} \tag{10b}$$

Therefore, by substituting (10) for functions U and V and their derivatives in system of equations (3) written with primes, we get

$$\begin{gathered} e^{m_1 - 2} U_t = e^{m_1 - 2 + \alpha_1 m_1} a_1 (U^{\alpha_1} U_x)_x + e^{m_1 - 2 + \alpha_1 m_1} b_1 (U^{\alpha_1 + 1} V^{-1} V_x)_x + \\ + \sum_i c_{1i} e^{m_1 \delta_{1i} + m_2 \kappa_{1i}} U^{\delta_{1i}} V^{\kappa_{1i}}, \\ e^{m_2 - 2} V_t = e^{m_2 - 2 + \alpha_2 m_1} a_2 (U^{\alpha_2} V_x)_x + e^{m_2 - 2 + \alpha_2 m_1} b_2 (U^{\alpha_2 - 1} V U_x)_x + \\ + \sum_j c_{2j} e^{m_1 \delta_{2j} + m_2 \kappa_{2j}} U^{\delta_{2j}} V^{\kappa_{2j}}. \end{gathered} \tag{3'}$$

It is obvious that system of equations (3′) coincides with system of equations (3) iff powers of the group parameter e are the same in each equation, namely,

$$0 = \alpha_1 m_1 = m_1 \delta_{1i} + m_2 \kappa_{1i} - m_2 + 2,$$

$$0 = \alpha_2 m_1 = m_1 \delta_{2j} + m_2 \kappa_{2j} - m_2 + 2.$$

It is easy to notice that the above system of algebraic equations with $\alpha_1 = \alpha_2 = 0$ can be reduced to (8) and, thus, system of equations (3) takes the form (7). In the other case, we get $m_1 = 0$ and

$$m_2 = -\frac{2}{\kappa_1}, \quad \kappa_i = \kappa_{1i}, \quad i = 1, 2, \dots, n_i,$$

$$\kappa_{2j} = 1 + \kappa_1, \quad j = 1, 2, \dots, n_j.$$

Thus, we get system of equations (9), which was to be proved.

For the case considered separately in [1], namely,

$$\delta_{2j} = \kappa_{2j} = 0, \quad \delta_{11} = 0.5, \quad \delta_{12} = 0, \quad \kappa_{11} = 0, \quad \kappa_{12} = 1$$

we get a system of equations

$$\begin{gathered} U_t = a_1 U_{xx} + b_1 ((U/V) V_x)_x + c_{11} U^{1/2} + c_{12} V, \\ V_t = a_2 V_{xx} + b_2 (U^{-1} V U_x)_x + c_2, \end{gathered} \tag{11}$$

which is invariant under the action of the operator

$$D = x\partial_x + 2t\partial_t + 4U\partial_U + 2V\partial_V.$$

Proposition 2. *system of equations* (3) *is invariant under the action of the operator*

$$D = x\partial_x + m_1 U\partial_U + m_2 V\partial_V, \tag{12}$$

iff it has the form

$$U_t = a_1(U^\alpha U_x)_x + b_1((U^{\alpha+1}/V)V_x)_x + \sum_{i=1}^{n_i} c_{1i} U^{\delta_{1i}} V^{\kappa_{1i}},$$
$$V_t = a_2(U^\alpha V_x)_x + b_2(U^{\alpha-1} V U_x)_x + \sum_{j=1}^{n_j} c_{2j} U^{\delta_{2j}} V^{\kappa_{2j}}, \tag{13}$$

where $\alpha = 2/m_1$, $m_1 \neq 0$, *and the powers* δ_{1i}, κ_{1i}, δ_{2j}, κ_{2j} *satisfy the system of algebraic equations*

$$(1-\delta_{1i})m_1 - \kappa_{1i}m_2 = 0, \quad -\delta_{2j}m_1 + (1-\kappa_{2j})m_2 = 0.$$

The proof of this statement is similar to that of Proposition 1.

For the case $\delta_{2j} = \kappa_{2j} = 0$, we get the system of equations

$$U_t = a_1(U^\alpha U_x)_x + b_1((U^{\alpha+1}/V)V_x)_x + U\sum_{i=1}^{n_i} c_{1i} V^{\kappa_{1i}},$$
$$V_t = a_2(U^\alpha V_x)_x + b_2(U^{\alpha-1} V U_x)_x + c_2, \tag{14}$$

which is invariant under the action of

$$D = x\partial_x + m_1 U x \partial_U, \quad m_1 = 2/\alpha, \quad \alpha \neq 0. \tag{15}$$

***Proposition* 3.** *System of equations* (3) *is invariant under the action of the operator*

$$D = kx\partial_x + t\partial_t + m_1 U\partial_U + m_2 V\partial_V, \quad k \neq 1/2,$$

iff it has the form (13), *where* $\alpha = (2k-1)/m_1$, $\alpha m_1 \neq 0$, *and the powers* δ_{1i}, κ_{1i}, δ_{2j}, κ_{2j} *satisfy the system of algebraic equations*

$$(1-\delta_{1i})m_1 - \kappa_{1i}m_2 = 1, \quad -\delta_{2j}m_1 + (1-\kappa_{2j})m_2 = 1.$$

The proof of this statement is similar to that of Proposition 1.

For the particular case

$$\delta_{2j} = \kappa_{2j} = 0, \quad \kappa_{11} = \delta_{12} = 0, \quad \kappa_{12} = 1, \quad \delta_{11} = 0.5,$$

we get the system of equations

$$U_t = a_1(U^\alpha U_x)_x + b_1((U^{\alpha+1}/V)V_x)_x + c_{11}U^{1/2} + c_{12}V,$$
$$V_t = a_2(U^\alpha V_x)_x + b_2(U^{\alpha-1} V U_x)_x + c_2, \tag{17}$$

which is a generalization of system of equations (11), invariant under the action of the operator

$$D = (\alpha + 0.5)x\partial_x + t\partial_t + 2U\partial_U + V\partial_V. \tag{18}$$

The following system is also worth attention:

$$U_t = a_1(U^\alpha U_x)_x + b_1((U^{\alpha+1}/V)V_x)_x + c_1 U/V,$$
$$V_t = a_2(U^\alpha V_x)_x + b_2(U^{\alpha-1} V U_x)_x + c_2, \tag{19}$$

since it is invariant under the action of two dilation operators

$$D = t\partial_t - \frac{1}{2}U\partial_U + V\partial_V,$$
$$D = x\partial_x + \frac{2}{\alpha}U\partial_U, \quad \alpha \neq 0 \tag{20}$$

(it satisfies the conditions of Propositions 2 and 3).

It is known that, among nonlinear heat equations of the form

$$U_t = (A(U)U_x)_x, \quad A \neq \text{const},$$

there is only one equation that is invariant under the action of the operator of conformal transformation [6]

$$K = x^2\partial_x - 3xU\partial_U. \tag{21}$$

The nonlinearity generating this operator has the form $A = aU^{-4/3}$, $a \in \mathbb{R}$.

Consider a natural generalization of the operator (21) for the case of two functions

$$K = x^2\partial_x + m_1 xU\partial_U + m_2 xV\partial_V. \tag{22}$$

***Proposition* 4.** *system of equations* (3) *is invariant under the action of operator* (22) *iff it has the form*

$$U_t = a_1(U^{-2}U_x)_x - a_1(U^{-1}V^{-1}V_x)_x + U\sum_{i=1}^{n_i} c_{1i}(U/V)^{\kappa_{1i}},$$
$$V_t = V\sum_{j=1}^{n_j} c_{2j}(U/V)^{\kappa_{2j}} \tag{23}$$

and $m_1 = m_2 = -2$.

This statement can be proved by application of transformations generated by operator (22)

$$t' = t, \quad x' = x/(1-ex), \quad U' = U(1-ex)^{-m_1}, \quad V' = V(1-ex)^{-m_2} \tag{24a}$$

to system of equations (3), where e is group parameter.

Indeed, since

$$U'_{t'} = E^{-m_1}U_t, \quad U'_{x'} = E^{2-m_1}\left(U_x + \frac{em_1}{E}U\right), \quad E \equiv 1 - ex,$$
$$U'_{x'x'} = E^{4-m_1}\left(U_{xx} + \frac{2(m_1-1)e}{E}U_x + \frac{m_1e^2(m_1-1)}{E^2}U\right),$$
$$V'_{t'} = E^{-m_2}V_t, \quad V'_{x'} = E^{2-m_2}\left(V_x + \frac{em_2}{E}V\right),$$
$$V'_{x'x'} = E^{4-m_2}\left(V_{xx} + \frac{2(m_2-1)e}{E}V_x + \frac{m_2e^2(m_2-1)}{E^2}V\right), \tag{24b}$$

by substituting expressions (24b) for functions U and V and their derivatives in system of equations (3) written with primes, for the first equation of the system, we obtain the expression

$$U_t = a_1 U^{\alpha_1} E^{4-m_1\alpha_1}\left(U_{xx} + \frac{2(m_1-1)}{E}U_x + \frac{m_1e^2(m_1-1)e}{E^2}U\right) +$$

$$+ b_1 U^{1+\alpha_1} V^{-1} E^{4-m_1\alpha_1-m_1+m_2}\left(V_{xx} + \frac{2e(m_2-1)}{E}V_x + \frac{m_2 e^2(m_2-1)}{E^2}V\right) +$$

$$+ a_1\alpha_1 E^{4-m_1\alpha_1} U^{\alpha_1-1}\left(U_x^2 + \frac{2em_1}{E}UU_x + \frac{e^2 m_1^2}{E^2}U^2\right) -$$

$$- b_1 E^{4-m_1\alpha_1} U^{\alpha_1+1} V^{-2}\left(V_x^2 + \frac{2em_2}{E}VV_x + \frac{e^2 m_2^2}{E^2}V^2\right) +$$

$$+ b_1(\alpha_1+1) E^{4-m_1\alpha_1} U^{\alpha_1} V^{-1}\left(U_x V_x + \frac{e^2 m_1 m_2}{E^2}UV + \frac{em_2}{E}VU_x + \frac{em_1}{E}UV_x\right) +$$

$$+ \sum_i^{n_i} c_{1i} U^{\delta_{1i}} V^{\kappa_{1i}} E^{m_1-m_1\delta_{1i}-m_2\kappa_{1i}}. \tag{25}$$

Expression (25) can be reduced to the first equation of system of equations (3) only if the sums of coefficients with U^{α_1+1} and $U^{\alpha_1}V_x$ are both equal to zero. Then we obtain the conditions

$$a_1 + b_1 = 0, \quad m_1 = m_2, \quad \alpha_1 = \frac{2}{m_1} - 1.$$

The rest of expression (25) can be reduced to the first equation of system (3) only if

$$4 - m_1\alpha_1 = 0, \quad m_1\delta_{1i} + m_2\kappa_{1i} = m_1.$$

Thus,

$$\alpha_1 = -2, \quad m_1 = m_2 = -2, \quad \delta_{1i} + \kappa_{1i} = 1. \tag{26a}$$

Similarly, having applied transformations (24a) to the second equation in system of equations (3), we get

$$a_2 = b_2 = 0, \quad \delta_{2i} + \kappa_{2j} = 1. \tag{26b}$$

Thus, system of equations (3) with conditions (26) takes for form of system of equations (23), which was to be proved.

Unlike the finite transformations generated by the dilation operator D (5), transformations (24a) generate a nontrivial formula for the multiplication of solutions: Indeed, if $U_0(t,x)$ and $V_0(t,x)$ is a solution, then

$$U = U_0(t, x/(1-ex))(1-ex)^2,$$

$$V = V_0(t, x/(1-ex))(1-ex)^2$$

is a new solution of system of equations (23).

***Note* 1.** The above statement can be generalized to the case of evolution systems of equations of the form

$$\begin{aligned} U_t &= (A_1(U)U_x)_x + (B_1(U,V)V_x)_x + C_1(U,V), \\ V_t &= (A_2(U)V_x)_x + (B_2(U,V)U_x)_x + C_2(U,V). \end{aligned} \tag{27}$$

System of equations (27) is invariant under the action of operator (22) iff it has the form

$$\begin{aligned} U_t &= a_1(U^{-4/3}U_x)_x + b_1(V^{-4/3}V_x)_x + Uf(U/V), \\ V_t &= a_2(U^{-4/3}V_x)_x + b_2(U^{-4/3}U_x)_x + Ug(U/V); \end{aligned} \tag{28}$$

or

$$U_t = a_1 [U^{-2} U_x - V_x / (UV)]_x + U f(U/V),$$
$$V_t = a_2 [V^{-2} V_x - U_x / (UV)]_x + U g(U/V), \tag{29}$$

where f and g are arbitrary differentiable functions. For system of equations (28), we have $m_1 = m_2 = -3$ in operator (22) and, for system of equations (29), we have $m_1 = m_2 = -2$.

3. Anzätze and Reduction of system of equations (3) to Ordinary Differential Equations. Since system of equations (3) is invariant under the action of the translation operators (4), evidently, we obtain an ansatz for solutions of the traveling wave-type

$$U = Y(\omega), \quad V = Z(\omega), \quad \omega = x - vt, \quad v \in \mathbb{R}. \tag{30}$$

Substitution of (30) into (3) gives a nonlinear system of ordinary differential equations

$$-Y_\omega = a_1 (Y^{\alpha_1} Y_\omega)_\omega + b_1 (Y^{\alpha_1 + 1} Z^{-1} Z_\omega)_\omega + \sum_{i=1}^{n_i} c_{1i} Y^{\delta_{1i}} Z^{\kappa_{1i}},$$
$$-Z_\omega = a_2 (Y^{\alpha_2} Z_\omega)_\omega + b_2 (Y^{\alpha_2 - 1} Z Y_\omega)_\omega + \sum_{j=1}^{n_j} c_{2j} Y^{\delta_{2j}} Z^{\kappa_{2j}} \tag{31}$$

(the index ω designates differentiation with respect to this variable). The invariance of system of equations of the form (3) under the action of the dilation operators (5) allows us to construct anzätze for self-similar solutions. Let us illustrate this statement by a special system of equations of the form (3), namely,

$$U_t = a_1 (U^\alpha U_x)_x + b_1 (U^{\alpha+1} V^{-1} V_x)_x + c_1 U \sum_{i=1}^{n_i} V^{\kappa_{1i}},$$
$$V_t = a_2 (U^\alpha V_x)_x + b_2 (U^{\alpha-1} V U_x)_x + c_2. \tag{32}$$

According to Proposition 1, system of equations (32) admits operator (6) for $m_2 = 2$ and an arbitrary constant m_1 only if it has the form

$$U_t = a_1 U_{xx} + b_1 ((U/V) V_x)_x + c_1 U/V,$$
$$V_t = a_2 V_{xx} + b_2 ((V/U) U_x)_x + c_2. \tag{33}$$

Operator D (6) for system of equations (33) generates an ansatz, which, if we take into account operators (4) can easily be generalized to

$$U = (t - t_0)^{m_1/2} Y(\omega), \quad V = (t - t_0) Z(\omega), \quad \omega = (x - x_0)(t - t_0)^{-1/2}. \tag{34}$$

Substituting ansatz (34) into system of equations (33), we obtain a nonlinear system of ordinary differential equations

$$\frac{m_1}{2} Y - \frac{1}{2} \omega Y_\omega = a_1 Y_{\omega\omega} + b_1 (Y Z^{-1} Z_\omega)_\omega + c_1 Y Z^{-1},$$
$$Z - \frac{1}{2} \omega Z_\omega = a_2 Z_{\omega\omega} + b_2 (Z Y^{-1} Y_\omega)_\omega + c_2, \tag{34'}$$

4. Certain Classes of Exact Solutions of System of Equations (3). Exact solutions of the traveling wave-type are obtained by means of solutions of the nonlinear system (31). For some particular cases, it was done in [7, 8]. We succeeded to obtain families of exact solutions with several arbitrary constants for the reduced systems of ordinary differential equations (35), (38), and (41).

Assume that the condition $b_1 = -a_1 \neq 0$, $b_2 = -a_2 \neq 0$ is satisfied for the coefficients of the original system of equations (3). Then we transform system of equations (35), (38) with $\kappa_{1i} = 0$, and (41) by changing Y, Z to new variables W, Z according to the following formula:

$$Y = WZ, \qquad Y_\omega = W_\omega Z + WZ_\omega. \tag{42}$$

After transformation (42), the system of ordinary differential equations (35) is reduced to

$$\begin{aligned} -\frac{1}{2}\omega(WZ)_\omega &= a_1(WZ)_\omega + c_1 W, \\ Z - \frac{1}{2}\omega Z_\omega &= -a_2(W_\omega Z W^{-1})_\omega + c_2. \end{aligned} \tag{43}$$

If we assume now that $W = e_0\omega^\gamma$, e_0, $\gamma \in \mathbb{R}$, then (43) is reduced to an overdetermined system

$$\left(a_1\gamma + \frac{1}{2}\omega\right)Z_\omega + \left(\frac{\gamma}{2} + a_1\gamma(\gamma-1)\omega^{-1}\right)Z = -c_1\omega,$$

$$\left(a_2\gamma - \frac{1}{2}\omega^2\right)Z_\omega + \left(\omega - a_2\gamma\omega^{-1}\right)Z = c_2\omega,$$

which is compatible and has the solution

$$Z = \frac{\omega^2 R(\omega)}{2}, \quad R(\omega) = \frac{c_1\omega^2 - c_2\omega + 8c_1a_2}{-a_1\omega^2 + a_2\omega - 8a_1a_2}$$

if $\gamma = -2$, and

$$\begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix} = 0.$$

Thus, taking (42) into account, we obtain the function $Y = e_0 R(\omega)$, $e_0 \in \mathbb{R}$. Thus, we obtain the three-parameter set of solutions for system of equations (33) (see (34) for $m_1 = 0$)

$$U = e_0 R(t,x), \qquad V = \frac{1}{2}(x - x_0)^2 R(t,x), \tag{44}$$

where

$$R(t,x) = \frac{c_1(x-x_0)^2 - c_2(x-x_0)\sqrt{t-t_0} + 8c_1a_2(t-t_0)}{-a_1(x-x_0)^2 + a_2(x-x_0)\sqrt{t-t_0} - 8a_1a_2(t-t_0)},$$

$$\begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix} = 0, \quad e_0, x_0, t_0 \in \mathbb{R}.$$

Substitution (42) reduces system of equations (38) with $\kappa_{1i} = 0$, $b_k = -a_k$, $k = 1, 2$, to a nonlinear system of ordinary differential equations

$$\left(\frac{2\gamma_0}{\alpha} - c_1\right)WZ - \gamma_0\omega(WZ)_\omega = a_1(W^\alpha Z^{\alpha+1} W_\omega)_\omega,$$

$$\gamma_0 \omega Z_\omega = a_2 (W^{\alpha-1} Z^{\alpha+1} W_\omega)_\omega - c_2,$$

which, in turn, for $\gamma_0(1+2/\alpha) - c_1 = 0$, by integration of the first equation over ω, is reduced to the form

$$e - \gamma_0 \omega W Z = a_1 W^\alpha Z^{\alpha+1} W_\omega, \quad e \in \mathbb{R},$$
$$\gamma_0 \omega Z_\omega = a_2 (W^{\alpha-1} Z^{\alpha+1} W_\omega)_\omega - c_2. \tag{45}$$

The nonlinear systems of ordinary differential equations (45) with $e = 0$ can be completely integrated, and we can obtain the solutions:

(i) $$Z = e_0 \omega^\gamma + c_0, \quad \gamma = -\frac{a_2 + a_1}{a_1}, \quad c_0 = \frac{a_2 c_2}{\gamma_0 (a_1 + a_2)},$$
$$W = \left[e_1 - \frac{\alpha \gamma_0}{a_1} \int \omega \left(e_0 \omega^\gamma + c_0 \right)^{-\alpha} d\omega \right]^{1/\alpha} \tag{46}$$

in the case $a_1 - a_2$,

(ii) $$Z = e_0 + \frac{c_2}{\gamma_0} \ln \omega,$$
$$W = \left[e_1 - \frac{\alpha \gamma_0}{a_1} \int \omega \left(e_0 + \frac{c_2}{\gamma_0} \ln \omega \right)^{-\alpha} d\omega \right]^{1/\alpha} \tag{47}$$

in the case $a_1 = -a_2$.

In formulas (46), (47) and below, the coefficients e_0 and e_1 are arbitrary constants, $\alpha \neq 0$.

Thus, for the nonlinear system

$$U_t = a_1 (U^\alpha U_x)_x - a_1 ((U^{\alpha+1}/V) V_x)_x + c_1 U,$$
$$V_t = a_2 (U^\alpha V_x)_x - a_2 (U^{\alpha-1} V U_x)_x + c_2, \quad \alpha \neq 0, \tag{48}$$

taking correlations (37) and (42) into account, we obtain the following four-parameter sets of solutions $(e_0, e_1, x_0, t_0 \in \mathbb{R})$:

(i) $$U = \exp \frac{2\gamma_0}{\alpha}(t - t_0) \left[\frac{e_0 (x - x_0)^\gamma}{\exp \gamma_0 \gamma (t - t_0)} + c_0 \right] W(\omega), \quad \omega = \frac{x - x_0}{\exp \gamma_0 (t - t_0)},$$
$$V = \frac{e_0 (x - x_0)^\gamma}{\exp \gamma_0 \gamma (t - t_0)} + c_0, \quad \gamma_0 = \frac{\alpha c_1}{\alpha + 2} \tag{49}$$

(γ, c_0, and W are defined in (46)) in the case $a_1 \neq a_2$,

(ii) $$U = \exp \frac{2\gamma_0}{\alpha}(t - t_0) \left[e_0 + \frac{c_2}{\gamma_0} \ln \frac{x - x_0}{\exp \gamma_0 (t - t_0)} \right] W(\omega),$$
$$V = e_0 + \frac{c_2}{\gamma_0} \ln \frac{x - x_0}{\exp \gamma_0 (t - t_0)} \tag{50}$$

(W is defined in (47)) in the case $a_1 \neq -a_2$.

Finally, substitution (42) reduces system of equations (41) to

$$\lambda_0(WZ) - \lambda\omega(WZ)_\omega = a_1(W^\alpha Z^{\alpha+1}W_\omega)_\omega + c_1 W,$$

$$Z - \lambda\omega Z_\omega = -a_2(W^{\alpha-1}Z^{\alpha+1}W_\omega)_\omega + c_2,$$

which, for $\lambda = (\alpha+2)^{-1}$, $\alpha \neq -2$, and $c_1 = 0$, after the integration of the first equation over ω, takes the form

$$e - \lambda\omega WZ = a_1 W^\alpha Z^{\alpha+1}W_\omega, \quad e \in \mathbb{R},$$

$$Z - \lambda\omega Z_\omega = -a_2(W^{\alpha-1}Z^{\alpha+1}W_\omega)_\omega + c_2. \tag{51}$$

It is easy to see that the system of ordinary differential equations (51) for $e=0$ coincides in structure with system (45), whence we get its solution, namely,

$$Z = e_0\omega^\gamma + c_0, \quad \gamma = \frac{(\alpha+2)a_1 - a_2}{a_1+a_2}, \quad c_0 = \frac{a_1c_2(\alpha+2)}{a_2 - a_1(\alpha+2)},$$

$$W = \left[e_1 - \frac{\alpha}{(\alpha+2)a_1}\int \omega\left(e_0\omega^\gamma + c_0\right)^{-\alpha}d\omega\right]^{1/\alpha}, \quad a_1 + a_2 \neq 0, \quad a_2 \neq a_1(\alpha+2). \tag{52}$$

In the case $a_1 + a_2 = 0$, we obtain $Z = \text{const}$, which is not interesting from the physical point of view. In the case $a_2 - a_1(\alpha+1) = 0$, solution (51) for $e=0$ has the form

$$Z = e_0 - c_0\ln\omega, \quad c_0 = \frac{\alpha+2}{\alpha+3}, \quad \alpha \neq -3,$$

$$W = \left[e_1 - \frac{\alpha}{(\alpha+2)a_1}\int \omega(e_0 - c_0\ln\omega)^{-\alpha}d\omega\right]^{1/\alpha}. \tag{53}$$

Note that, $\alpha \neq -3$, we again obtain $Z = \text{const}$. Thus, for the nonlinear system of equations

$$U_t = a_1(U^\alpha U_x)_x - a_1((U^{\alpha+1}/V)V_x)_x,$$

$$V_t = a_2(U^\alpha V_x)_x - a_2(U^{\alpha-1}VU_x)_x + c_2, \tag{54}$$

taking relations (40), (42), (52), and (53) into account, we obtain the following four-parameter families of solutions:

(i)

$$U = (t-t_0)^{-1/(\alpha+2)}\left[e_0(x-x_0)^\gamma(t-t_0)^{-\gamma/(\alpha+2)} + c_0\right]W(\omega), \quad \omega = \frac{x-x_0}{(t-t_0)^{1/(\alpha+2)}},$$

$$V = (t-t_0)\left[e_0(x-x_0)^\gamma(t-t_0)^{-\gamma/(\alpha+2)} + c_0\right]$$

(γ, c_0, and $W(\omega)$ are defined in (52)) in the case $a_2 \neq a_1(\alpha+2)$,

(ii)

$$U = (t-t_0)^{-1/(\alpha+2)}\left[e_0 - c_0\ln\frac{(x-x_0)}{(t-t_0)^{1/(\alpha+2)}}\right]W(\omega),$$

$$V = (t-t_0)\left[e_0 - c_0\ln\frac{(x-x_0)}{(t-t_0)^{1/(\alpha+2)}}\right]$$

(c_0 and $W(\omega)$ are defined in (53)) in the case $a_2 = a_1(\alpha+2)$, $\alpha \neq -2$.

5. Non-Lie ansätze and exact solutions of a nonlinear system of the form (3). Nowadays, the construction of non-Lie ansätze and exact solutions of nonlinear evolution equations is a very vital problem. Here, we apply an approach to the construction of such ansätze and exact solutions that is based on the consideration of a given nonlinear system together with additional conditions in the form of high-order ordinary differential equations. This approach was used in [9] for obtaining solutions of some nonlinear evolution equations, which describe real processes in physics and chemistry.

Consider a particular case of the system of equations (3), namely,

$$
\begin{aligned}
U_t &= (UU_x)_x + c_{11}UV + c_1, \\
V_t &= a(UV_x)_x - a(VU_x)_x + c_2,
\end{aligned}
\tag{55}
$$

where a, c_{11}, c_1, and c_2 are arbitrary constants. For $c_{11}c_1c_2 \neq 0$, system (55) is invariant only under the Lie algebra (4), which generates solutions of the form (30).

Taking as additional condition the non-coupled system of ordinary differential equations

$$
\begin{aligned}
\alpha_1(t)\frac{dU}{dx} + \alpha_2(t)\frac{d^2U}{dx^2} + \frac{d^3U}{dx^3} &= 0, \\
\alpha_1(t)\frac{dV}{dx} + \alpha_2(t)\frac{d^2V}{dx^2} + \frac{d^3V}{dx^3} &= 0,
\end{aligned}
\tag{56}
$$

where $\alpha_1(t)$, $\alpha_2(t)$ are arbitrary continuous functions and the variable t is regarded as a parameter, we can easily find its general solution

$$
\begin{aligned}
U &= \varphi_0(t) + \varphi_1(t)\exp(\gamma_1(t)x) + \varphi_2(t)\exp(\gamma_2(t)x), \\
V &= \Psi_0(t) + \Psi_1(t)\exp(\gamma_1(t)x) + \Psi_2(t)\exp(\gamma_2(t)x),
\end{aligned}
\tag{57}
$$

where

$$
\gamma_{1,2}(t) = \frac{1}{2}\left(\pm(\alpha_2^2 - 4\alpha_1)^{1/2} - \alpha_2\right) \neq 0
$$

and $\gamma_1 \neq \gamma_2$.

Consider relations (57) as an ansatz for our system (55). It is important to note that this ansatz contains six functions φ_i and Ψ_j. This enables us to reduce system (55) to a quasilinear system of ordinary differential equations of the first order for the unknown functions φ_i and Ψ_j. Thus, we find after cumbersome calculations a family of non-Lie solution of the form

$$
\begin{aligned}
U &= \varphi_0(t) + \exp\left[A(t) + a\gamma^2\int\varphi_0(t)\,dt\right]\left(d_1\exp\gamma x + d_2\exp(-\gamma x)\right), \\
V &= d_0 + c_2t - \frac{\gamma^2}{2c_{11}}\exp\left[A(t) + a\gamma^2\int\varphi_0(t)\,dt\right]\left(d_1\exp\gamma x + d_2\exp(-\gamma x)\right),
\end{aligned}
\tag{58}
$$

where γ, d_0, d_1, d_2 are arbitrary constants, $A(t) = c_{11}t(d_0 + c_2t/2)$, and $\varphi_0(t)$ is an arbitrary solution of the equation

$$
\frac{d\varphi_0}{dt} - c_{11}(c_2 + d_0)\varphi_0 = c_1 - \gamma^2 d_1 d_2\exp\left[2A(t) + 2a\gamma^2\int\varphi_0(t)\,dt\right].
$$

Indeed, solution (58) is just a non-Lie solution because it is not of the form (30).

Note that in the case $\gamma = i\gamma_0$, $i^2 = -1$, $\gamma_0 \in \mathbb{R}$, any complex solution of the form

(58) generates two real solutions that contain the periodic functions cos and sin. Such solutions can describe a process of evolution of the temperature and density in Tokamak plasmas [1].

In the case $c_{11} = 0$, the system of equations (55) has no solutions of the form (58), but, in this case, we can use the non-Lie ansatz

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2,$$
$$V = \Psi_0(t) + \Psi_1(t)x + \Psi_2(t)x^2, \qquad (59)$$

which is generated by the additional condition (56) if $\alpha_1 = \alpha_2 = 0$. Substituting ansatz (59) into the system of equations (55) for $c_{11} = 0$, one can find a family of exact solutions of the form

$$U = \varphi_0(t) + d_1(t_0 - t)^{-1}x + \frac{1}{6}(t_0 - t)^{-1}x^2,$$
$$V = \Psi_0(t) + \left[6d_1d_2 + d_0|t_0 - t|^{a/3}\right]x + d_2x^2,$$

where the functions $\varphi_0(t)$ and $\Psi_0(t)$ are an arbitrary solution of the linear system of ordinary differential equations

$$\frac{d\varphi_0}{dt} = \frac{1}{3}(t_0 - t)^{-1}\varphi_0 + d_1^2(t_0 - t)^{-2} + c_1,$$
$$\frac{d\Psi_0}{dt} = \frac{a}{3}(t_0 - t)^{-1}\Psi_0 + 2ad_2\varphi_0 + c_2.$$

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