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On associative algebras satisfying the identity $x^5 = 0$

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Abstract. We study Kuzmin's conjecture on the index of nilpotency for the variety $\mathcal{N}il_5$ of associative nil-algebras of degree 5. Due to Vaughan-Lee [11] the problem is reduced to that for k-generator $\mathcal{N}il_5$-superalgebras, where $k \leq 5$. We confirm Kuzmin's conjecture for 2-generator superalgebras proving that they are nilpotent of degree 15.

1. Introduction

For a positive integer n let $\mathcal{N}il_n$ be the variety of associative algebras over a field F of characteristic zero defined by the identity $x^n = 0$. The classical Dubnov-Ivanov-Nagata-Higman theorem [3], [5] states that every algebra from $\mathcal{N}il_n$ is nilpotent of degree $2^n - 1$. Razmyslov [8] improved the nilpotency degree in this theorem to n^2. Kuzmin [7] showed that the degree cannot be less than $f(n) = n(n+1)/2$ and conjectured that the last number gives the exact estimate of nilpotency degree for the variety $\mathcal{N}il_n$.

It is easy to see that Kuzmin's conjecture is true for $n = 2$, and Higman's results implied that it is also true for $n = 3$. It was natural to try to use computer for checking the conjecture for other small values of n. Vaughan-Lee in [10] did this for $n = 4$, confirming Kuzmin's conjecture in this case. We consider the next value $n = 5$.

In principle the calculations required are quite straightforward. Let $A = \mathcal{N}il_n[a_1, \ldots, a_{f(n)}]$ be the free algebra of the variety $\mathcal{N}il_n$ with free generators $a_1, a_2, \ldots, a_{f(n)}$. We need to show that $a_1 a_2 \cdots a_{f(n)} = 0$.

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In characteristic zero the identity $x^n = 0$ is equivalent to the multilinear identity

$$s_n(x_1, x_2, \ldots, x_n) \stackrel{df}{=} \sum_{\sigma \in \mathrm{Sym}(n)} x_{1\sigma} x_{2\sigma} \cdots x_{n\sigma} = 0.$$

If we let $F\langle T\rangle$ be the (absolutely) free associative algebra over F with the set of free generators $T = \{t_1, t_2, \ldots, t_{f(n)}\}$, then $A \cong F\langle T\rangle/I$, where I is the ideal of $F\langle T\rangle$ generated by

$$\{s_n(u_1, u_2, \ldots, u_n) |\ u_1, u_2, \ldots, u_n \in F\langle T\rangle\}.$$

So we need to show that the product $t_1 t_2 \cdots t_{f(n)}$ is a linear combination of terms of the form

$$v s_n(u_1, u_2, \ldots, u_n) w,$$

where $u_1, u_2, \ldots, u_n, v, w$ are products of the free generators of $F\langle T\rangle$ (with v and w possibly empty). In fact, we may suppose that v and w are empty because it is well known that the IDEAL of an algebra generated by all the n-th powers ot its elements coincides with the VECTOR SPACE spanned by the elements $s_n(u_1, u_2, \ldots, u_n)$, where $u_1, u_2, \ldots, u_n$ are monomials of positive degree. We may also assume that the term $u_1 u_2 \cdots u_n$ is multilinear in $t_1, t_2, \ldots, t_{f(n)}$. There are only finitely many expressions of the form $s_n(u_1, u_2, \ldots, u_n)$ satisfying this condition. And so the problem reduces to a finite calculation in the $f(n)!$-dimensional space spanned by

$$\{t_{1\sigma} t_{2\sigma} \cdots t_{f(n)\sigma} |\ \sigma \in \mathrm{Sym}(f(n))\}.$$

We see that the dimensions are too big already for $n = 4, 5$.

Vaughan-Lee applied to the problem the representation theory of symmetric groups and the superalgebra technique, and reduced for $n = 4$ the original calculation in 10!-dimensional space to 8 smaller calculations in $\frac{10!}{4!3!2!}$ and $\frac{10!}{4!3!3!}$-dimensional spaces. Let us briefly explain the main idea of this reduction.

2. The superalgebra method

The dimension is certainly smaller when not all t_i are different. So, it seems natural to try to reduce the number of different variables in this problem.

It is well known that reduction of this kind exists for symmetric multilinear functions: every such a function on n variables may be obtained by a *linearization* or *polarization* of a function of degree n on one variable.

Now, assume that we have a skew-symmetric multilinear function $\Phi : B^n \to B$ defined on an associative algebra B over a field F of characteristic zero. Take the Grassmann algebra G over F generated by Grassmann variables $e_1, e_2, \ldots$; that is the unital associative algebra over F subject to the relations $e_i e_j = -e_j e_i,\ i, j = 1, 2, \ldots$. Form the tensor product $G \otimes B$, and extend the function Φ to it by setting

$$\Phi(g_1 \otimes x_1, \ldots, g_n \otimes x_n) = g_1 \cdots g_n \otimes \Phi(x_1, \ldots, x_n).$$

Then Φ becomes a symmetric function on the variables $y_i = e_i \otimes x_i$; moreover,

$$n! e_1 \cdots e_n \otimes \Phi(x_1, \ldots, x_n) = \Phi(z, \ldots, z),$$

where $z = e_1 \otimes x_1 + \cdots + e_n \otimes x_n$. It is clear, for example, that $\Phi(x_1, \ldots, x_n) = 0$ if and only if $\Phi(z, \ldots, z) = 0$. So, in a skew-symmetric case we also can reduce the number of variables, only the new variables lie not in B but in $G \otimes B$.

The problem is that in general $G \otimes B$ does not belong to the same variety as B; for instance, if $B = F$ then $G \otimes F = G$ is already not commutative. Nevertheless, $G \otimes B$ satisfies certain *graded identities* related with those of B.

The Grassmann algebra G has a basis over F consisting of 1 together with all the possible products $e_i e_j \cdots e_k$ with $1 \leq i < j < \ldots < k$. We can write $G = G_0 + G_1$, where G_0 is spanned by the products of even length, and G_1 is spanned by the products of odd length. Then $G_i G_j \subseteq G_{i+j \ (mod\ 2)}$. So G is a $\mathbb{Z}_2$-graded algebra, or a *superalgebra*. We call G_0 the even part of G, and G_1 the odd part. If $g \in G_0$ and $h \in G$ then $gh = hg$. But if $g, h \in G_1$ then $gh = -hg$.

The $\mathbb{Z}_2$-grading of G is inherited by $G \otimes B = G_0 \otimes B + G_1 \otimes B$. If $B = \mathcal{V}\langle X \rangle$ is a free algebra of a certain variety $\mathcal{V}$ of algebras, then the $\mathbb{Z}_2$-graded identities of the superalgebra $G \otimes \mathcal{V}\langle X \rangle$ define a variety $\widetilde{\mathcal{V}}$ of so called $\mathcal{V}$-*superalgebras*. For instance, the variety $\widetilde{Com}$ of commutative superalgebras is defined by the graded identities of G

$$\begin{aligned} ab &= ba, \\ ax &= xa, \\ xy &= -yx, \end{aligned}$$

where the elements a, b are even and x, y are odd. The variety $\widetilde{\mathcal{N}il_3}$ of

$\mathcal{N}il_3$-superalgebras is defined by

$$\begin{aligned} abc + bca + cab + acb + bac + cba &= 0, \\ abx + bxa + xab + axb + bax + xba &= 0, \\ axy + xya - yax - ayx + xay - yxa &= 0, \\ xyz + yzx + zxy - xzy - yxz - zyx &= 0, \end{aligned}$$

where a, b, c are even and x, y, z are odd. Similarly, the varieties $\widetilde{\mathcal{N}il}_4$ and $\widetilde{\mathcal{N}il}_5$ are defined by five and six graded identities, respectively. In general, the variety $\widetilde{\mathcal{N}il}_n$ is defined by $n+1$ graded identities which may be united in the following superidentity

$$\widetilde{s}_n(x_1, x_2, \ldots, x_n) \stackrel{df}{=} \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}_{\mathrm{odd}}(\sigma) x_{1\sigma} x_{2\sigma} \ldots x_{n\sigma} = 0,$$

where $\mathrm{sign}_{\mathrm{odd}}(\sigma)$ is the sign of the permutation afforded by σ on the odd x_i.

Now the idea is the following: at the first step we substitute our multilinear identity $t_1 t_2 \cdots t_{f(n)} = 0$ by the equivalent set of multilinear identities $\Phi_i = 0$, where in every Φ_i the $n!$ variables are divided in $m_i < n$ groups, and Φ_i is symmetric or skew-symmetric on the variables in each group. And on the second step we reduce the number of the variables in each of these identities, substituting instead of every group of symmetric variables an even element, and instead of skew-symmetric, an odd one. The point is that these new elements are not from the free algebra of a variety, but from the free superalgebra.

It was Kemer [6] who first applied superalgebras to the investigation of varieties of associative algebras, in his solution of the famous Specht problem. Then this method was extended in the papers by Zel'manov [12] and Zel'manov-Shestakov [13] to investigation of nilpotence and solvability problems in non-associative algebras. Finally, Vaughan-Lee [10] applied superalgebras to reduce the number of variables for his computer calculations in $\mathcal{N}il_4$.

The two steps mentioned above were formalized by Vaughan-Lee [11] in two theorems below. First we establish some notations.

If C is an algebra generated by $c_1, c_2, \ldots, c_m$, and if w is a product of these generators, then we define the multiweight of w to be $\underline{w} = (w_1, w_2, \ldots, w_m)$ if there are w_i occurrences of the generator c_i in w, for $i = 1, 2, \ldots, m$. The subspace of C spanned by all products of multiweight $\underline{w}$ is called the multiweight $\underline{w}$ component of C.

Let $\mathcal{V}$ be a variety of ungraded algebras over a field F of characteristic zero determined by a set of multilinear identities. Let $\mathcal{V}\langle X\rangle$ be the free algebra of rank m of $\mathcal{V}$ with the set of free generators $X = \{x_1, x_2, \ldots, x_m\}$, and let M be the space of all the multilinear elements of degree m in $\mathcal{V}\langle X\rangle$. We turn M into an $F\,\mathrm{Sym}(m)$ module, letting permutations in $\mathrm{Sym}(m)$ permute $x_1, x_2, \ldots, x_m$. If S is a non-empty subset of $\{1, 2, \ldots, m\}$ then we let

$$\varphi_S^+ = \sum_{\sigma\in\mathrm{Sym}(S)} \sigma \qquad \text{and} \qquad \varphi_S^- = \sum_{\sigma\in\mathrm{Sym}(S)} \mathrm{sign}(\sigma)\cdot\sigma.$$

Theorem 2.1. *Let d be the sum of the dimensions of the irreducible representations of $F\,\mathrm{Sym}(m)$. Then $M = \sum_{i=1}^d M_i$, where each M_i has the form $M\cdot\varphi_{S_1}^{\varepsilon_1}\cdot\varphi_{S_2}^{\varepsilon_2}\cdot\cdots\cdot\varphi_{S_k}^{\varepsilon_k}$, for some partition of $\{1, 2, \ldots, m\}$ into disjoint non-empty subsets $S_1, S_2, \ldots, S_k$ with $\frac{1}{2}k(k+1) \le m$, and for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k = \pm$.*

Theorem 2.2. *Let $S_1, S_2, \ldots, S_k$ be a partition of $\{1, 2, \ldots, m\}$ into disjoint non-empty subsets. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k = \pm$, and let $m_i = |S_i|$ for $i = 1, 2, \ldots, k$. Then*

$$\dim(M\cdot\varphi_{S_1}^{\varepsilon_1}\cdot\varphi_{S_2}^{\varepsilon_2}\cdot\cdots\cdot\varphi_{S_k}^{\varepsilon_k}) = \dim N,$$

where N is the multiweight $(m_1, m_2, \ldots, m_k)$ component of the free superalgebra Z of rank k in the variety $\widetilde{\mathcal{V}}$, where for $i = 1, 2, \ldots, k$ the i-th generator of Z is even, if $\varepsilon_i = +$ and is odd if $\varepsilon_i = -$.

In fact, Theorem 2.1 is just a reformulation of the well known application of the representation theory of $\mathrm{Sym}(m)$ to the study of identities and reduces a given identity to its irreducible components corresponding to all the possible Young tableaux. Theorem 2.2 admits to associate with each Young tableau a certain superalgebra on $k < \sqrt{2m}$ homogeneous generators and to do calculations in this superalgebra. The number of generators needed corresponds to the number of horizontal (even variables) and vertical (odd variables) strips needed to subdivide Young diagrams for $\mathrm{Sym}(m)$.

3. The nilpotency of 2-generator superalgebras from $\widetilde{\mathcal{N}il}_5$

Following Vaughan-Lee, it is easy to see that in order to confirm Kuzmin's conjecture for $n = 5$, it suffices to prove that any superalgebra from $\widetilde{\mathcal{N}il}_5$ on $k \le 5$ homogeneous generators is nilpotent of degree $f(5) = 15$. However this is a huge computation, even on a supercomputer.

For example, if we consider 2-generator superalgebra, then in the case of multiweight $(7, 8)$ component we need to show that every of $\frac{15!}{7!8!} = 6435$

words is equal to zero. For the multiweight $(5,5,5)$ component of 3-generator superalgebra we need already to consider $\frac{15!}{5!5!5!} = 756756$ words.

Using the GAP computer package, we prove that every 2-generator superalgebra of $\widetilde{\mathcal{N}il}_5$ is nilpotent of degree 15.

The main difference of our algorithm and that of Vaughan-Lee is the following one. In the algorithm of Vaughan-Lee, like in the algorithms of the programs 'Albert' [4] and 'Malcev' [1], a base of a relatively free nilpotent algebra or superalgebra is constructed, while we DO NOT construct a base of a relatively free nilpotent 2-generator superalgebra, but only show that every word of length 15 in it equals zero. Observe that neither 'Albert' nor 'Malcev' programs could construct a base for 2-generator free superalgebras in $\widetilde{\mathcal{N}il}_5$ even in the simplest case of two even generators.

Let $\widetilde{\mathcal{N}}_5 = \widetilde{\mathcal{N}il}_5[a,b]$ be the free superalgebra on homogeneous generators a and b. We have to consider separately the cases when a, b are both even, both odd and when one of them is even and another is odd. In each case we consider all the homogeneous components $\widetilde{\mathcal{N}}_5(m,n)$, of degree m on a and of degree n on b, where $m+n=15$. For each set of words $u_1,\ldots,u_5 \in \widetilde{\mathcal{N}}_5$ such that their total degree on a is m and on b is n, we have an equality $\widetilde{s}_5(u_1,\ldots,u_5)=0$ relating some of $\frac{15!}{m!n!}$ words of multiweight (m,n) on a, b.

In fact, we can omit from the very beginning the words with zero subwords. For example, if both a and b are odd, then the words a^{10}, b^{10}, $(ab)^5$, $(ba)^5$ are zero and we have to consider

6365	words of multiweight	$(8,7)$,
4970	–"–	$(9,6)$,
2990	–"–	$(10,5)$,
1340	–"–	$(11,4)$,
415	–"–	$(12,3)$,
75	–"–	$(13,2)$,
5	–"–	$(14,1)$.

If a is even then the multiweight $(13,2)$ and $(14,1)$ components are trivially zero. Note that when the generators a and b have different parity, the dimensions of the multiweight (m,n) and (n,m) components are not necessary the same.

Thus, we get a linear homogeneous system on $k \le \frac{15!}{m!n!}$ unknowns, i.e. words of multiweight (m,n) on a, b. Our purpose is to calculate the rank of the matrix T of this system. If the rank $r(T) < k$, then there exists a non-zero word on a, b of the given multiweight, and $\widetilde{\mathcal{N}}_5^{15} \neq 0$; and if $r(T) = k$, then all the words of length 15 and multiweight (m,n)

on a, b are equal 0.

We do not write down all equations at once, since the parameters of the most of the obtained systems are too big for the GAP. Instead, we start with some reasonable number of them, reduce the system by GAP, and then add more equations step by step. Because of the symmetry of the identity $\widetilde{s}_5(u_1, \ldots, u_5) = 0$, we can always suppose that $deg(u_1) \leq \cdots \leq deg(u_5)$. In any case we stopped our calculations when the rank of the system reached the number of the unknowns.

Observe that the matrix T consists of integers. Clearly, for every prime $p > 0$, its rank $r_p(T)$ over the field $\mathbb{Z}(p)$ is less or equal than $r(T)$. Hence, in order to prove that $r(T) = k$ it suffices to prove that $r_p(T) = k$ for some $p > 0$. We do the calculations for $p = 13$ and prove that in all the cases the rank of the system is equal to the number of variables. Thus, we have the following result.

Theorem 3.1. *Every 2-generator superalgebra from the variety $\widetilde{\mathcal{N}il}_5$ is nilpotent of degree 15.*

Corollary 3.2. *Let $g(x_1, \ldots, x_n, y_1, \ldots, y_m)$, $n + m = 15$, be a multilinear polynomial that is symmetric or skew-symmetric on each of the groups of the variables, x-s and y-s. Then $g = 0$ identically in $\mathcal{N}il_5$.*

Proof. Assume, for the definiteness, that g is symmetric on x-s and skew-symmetric on y-s. Let $A = \mathcal{N}il_5[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the free algebra of the variety $\mathcal{N}il_5$. Consider the unital commutative associative algebra $H = alg\langle \alpha_1, \alpha_2, \ldots \mid \alpha_i^2 = 0\rangle$ and the Grassmann algebra $G = alg\langle e_1, e_2, \ldots \mid e_i e_j = -e_j e_i\rangle$. Furthermore, let $B = (H \otimes G) \otimes A$. It is easy to see that with respect to the $\mathbb{Z}_2$-grading inherited from G, $B = B_0 + B_1$ is a $\mathcal{N}il_5$-superalgebra. Let $a = \alpha_1 \otimes x_1 + \cdots + \alpha_n \otimes x_n \in B_0$, $b = e_1 \otimes y_1 + \cdots + e_m \otimes y_m \in B_1$, then by the Theorem 3.1, $(alg\langle a, b\rangle)^{15} = 0$. In particular, we have

$$0 = g(\underbrace{a, \ldots, a}_{n}, \underbrace{b, \ldots, b}_{m}) =$$
$$= n!m!(\alpha_1 \cdots \alpha_n \otimes e_1 \cdots e_m) \otimes g(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

Since $\operatorname{char}(F) = 0$, it follows that $g = 0$ in A. $\square$

In fact we were looking for a possible counter-example to the conjecture. The arguments of Kuzmin and the previous cases give certain evidence that such a counter-example, if it existed, might already appear in case of two generators. The proof of the corollary above can be easily modified for this imaginary non-nilpotency situation. Below we give a more easy superalgebra–algebra passage for this case.

Proposition 3.3. *Assume that for some natural n and m, a 2-generator $\widetilde{\mathcal{N}il}_n$-superalgebra $B = B_0 + B_1$ is not nilpotent of degree m, that is, $B^m \neq 0$. Then $x_1 x_2 \cdots x_m \neq 0$ in $\mathcal{N}il_n$.*

Proof. Let $B = alg\langle a, b\rangle$, where a and b are homogeneous. We may assume that at least one of a, b is odd since otherwise the proposition is evident. Since $B^m \neq 0$, there exists a multilinear monomial $u(y_1, \ldots, y_k, z_1, \ldots, z_l)$ of degree $m = k + l$ such that

$$u(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{l}) \neq 0.$$

Consider the algebra $A = G_0 \otimes B_0 + G_1 \otimes B_1$ where $G = G_0 + G_1$ is the Grassmann algebra; then $A \in \mathcal{N}il_n$ (see [9], [11]). Let a be even and b be odd; then $1 \otimes a, e_1 \otimes b, \ldots, e_l \otimes b \in A$, and we have

$$u(\underbrace{1 \otimes a, \ldots, 1 \otimes a}_{k}, e_1 \otimes b, \ldots, e_l \otimes b) = \pm e_1 \cdots e_l \otimes u(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{l}) \neq 0.$$

Hence $A^m \neq 0$. If both generators a and b are odd then $e_1 \otimes a, \ldots, e_k \otimes a$, $e_{k+1} \otimes b, \ldots, e_{k+l} \otimes b \in A$, and as above we have $A^m \neq 0$. □

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