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Dg algebras with enough idempotents, their dg modules and their derived categories

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Abstract. We develop the theory dg algebras with enough idempotents and their dg modules and show their equivalence with that of small dg categories and their dg modules. We introduce the concept of dg adjunction and show that the classical covariant tensor-Hom and contravariant Hom-Hom adjunctions of modules over associative unital algebras are extended as dg adjunctions between categories of dg bimodules. The corresponding adjunctions of the associated triangulated functors are studied, and we investigate when they are one-sided parts of bifunctors which are triangulated on both variables. We finally show that, for a dg algebra with enough idempotents, the perfect left and right derived categories are dual to each other.

Introduction

All throughout this paper, we fix a commutative ground ring K with unit and the term 'category' will mean 'K-linear category', unless otherwise specified, and all functors will be K-linear.

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Small differential graded (dg) categories and their dg modules have played a fundamental role in Mathematics for a long time. In the 70's and 80's of last century, they were the major tool to study matrix problems related to representation theory of algebras (see [13], [6], [3], [4],...), which, among other things, led to Drozd's proof of the tame-wild dichotomy theorem (see [3] and [4], Theorem 2). In modern times, their main importance comes from a fundamental result of Keller (see [10, Theorem 4.3]) which states that any compactly generated algebraic triangulated category is equivalent to the derived category of a small dg category. That importance grew even bigger when Tabuada (see [21]) showed that the category *Dgcat* of small dg categories admits a model structure on which the weak equivalences are the so-called quasi-equivalences and Toën (see [22]) studied in depth the associated homotopy category Ho(*Dgcat*), showing in particular that it had an internal Hom and deriving several applications of this fact to Homotopy Theory and Algebraic Geometry.

By definition, a small dg category is a small category with a grading and a differential satisfying certain conditions (see the details in next section). But from the time of Gabriel's thesis (see [5]) one knows that small categories may be viewed as algebras with enough idempotents (or rings with several objects in the spirit of [14]), and vice versa. Furthermore, if A is such a small category then the category $[A^{\mathrm{op}}, \mathrm{Mod}-K]$ of contravariant functors is equivalent to $\mathrm{Mod}-A$, the category of unitary right A-modules, when A is viewed as an algebra with enough idempotents. It is natural to expect that the mentioned one-to-one correspondence extends to the dg setting. That requires the development of a theory of dg algebras with enough idempotents and their dg modules. This development is, in some sense, a demand of a part of the mathematical community. Indeed, apart from the unavoidable technicalities concerning the use of signs, the language of small dg categories and their dg modules is very technical and elusive for many people and, although the terminology is sometimes similar, concepts as dg modules or dg bimodules over small dg categories are intuitively far from the traditional concepts of module or bimodule over an associative algebra. This leads some mathematicians to avoid the topic and others to present results about small dg categories and their derived categories only in terms of dg algebras (equivalently, dg categories with just one object). This demand is the main motivation for these notes. They were initially thought as an appendix to a joint paper with Alexander Zimmermann (see [20]), where we needed to use some adjunctions between categories of dg bimodules, which we could not find explicit in the literature of small dg categories and which became excessively unintuitive in that language (see [18]). As the notes grew longer than expected, we decided to offer them as a separated paper. Since a thorough development of the topic is out of question, we have concentrated on the basic aspects, with emphasis on those needed for [20], leaving aside other important features of the theory.

Our goal in the paper is to develop the basics of the theory of dg algebras with enough idempotents and their dg modules, to show its equivalence with the theory of small dg categories and their dg modules, and to revisit dg functors between categories of dg modules and their derived versions. In particular, we construct explicitly the correspondents in the new setting of the covariant tensor-Hom and the contravariant Hom-Hom adjunctions of (bi)module categories over algebras with unit, together with their derived versions. Since the notes are specially aimed at making the dg world more accessible to people that work with algebras and modules in the classical way, even at the cost of an excessive length, we have taken care in checking essentially all the details in proofs. This care has been special on what concerns signs in equations, which are most elusive for beginners and very important in the dg context, but whose associated calculations are rarely found explicit in the literature.

The organization of the paper goes as follows. In Section 1 we recall the definitions of dg category (not necessarily small) and dg functor. In Section 2 we define what a dg algebra with enough idempotents is and give its category $\mathrm{Dg}-A$ of right dg modules, proving in Section 3 that there is a one-to-one correspondence between small dg categories and dg algebras with enough idempotents, and showing a dg equivalence between $\mathrm{Dg}-A$ and the category $\mathcal{C}_{\mathrm{dg}}A$ of dg modules over the associated small dg category (see Theorem 3.1). In Sections 4 and 5 we define left dg modules and dg bimodules, respectively, and show that their corresponding categories can be realized as categories of right dg modules. In Section 6 we introduce the homotopy and derived category of a dg algebra with enough idempotents, and state the corresponding version of the mentioned Keller's theorem (see Corollary 6.11). In Section 7 we study derived functors of dg functors between categories of dg modules over algebras with enough idempotents and study when they appear as 'one-sided part' of a bifunctor which is triangulated on both variables. In our approach, a fundamental role is played by the concept of dg adjunction (see Definition 7.7). In section 8 we define the correspondents of the classical tensor and Hom bifunctors. Concretely, we show that if A, B and C are dg algebras with enough idempotents, then there are canonical dg functors $\overline{\mathrm{HOM}}_A(?,?)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}} \otimes (B-\mathrm{Dg}-A) \longrightarrow B-\mathrm{Dg}-C$

and $?\otimes_B?\colon (C-\mathrm{Dg}-B)\otimes(B-\mathrm{Dg}-A)\longrightarrow C-\mathrm{Dg}-A$, where $\overline{\mathrm{HOM}}_A(M,X) := B\,\mathrm{HOM}_A(M,X)C$ is the 'unitarization' of the non-unitary dg $B-C$-bimodule $\mathrm{HOM}_A(M,X)$. In Section 9, we show that if X is a dg $B-A$-bimodule, then we have dg adjunctions $(?\otimes_B X\colon C-\mathrm{Dg}-B\longrightarrow C-\mathrm{Dg}-A, \overline{\mathrm{HOM}}_A(X,?)\colon C-\mathrm{Dg}-A\longrightarrow C-\mathrm{Dg}-B)$ and $(\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(?,X)^o\colon B-\mathrm{Dg}-C\longrightarrow (C-\mathrm{Dg}-A)^{\mathrm{op}}, \overline{\mathrm{HOM}}_A(?,X)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}}\longrightarrow B-\mathrm{Dg}-C)$, both of which give rise to adjunctions between the corresponding derived functors (see Theorems 9.1 and 9.5). In the final Section 10 we use the last contravariant adjunction to prove that, for any dg algebra with enough idempotents A and taking $X=A$, the adjunction $(\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(?,A), \overline{\mathrm{HOM}}_A(?,A))$ gives rise to quasi-inverse dualities $\mathrm{per}(A^{\mathrm{op}}) \overset{\cong^o}{\longleftrightarrow} \mathrm{per}(A)$ between the left and right perfect derived categories.

The paper tries to be as self-contained as possible, but some classical concepts are used without being explicitly introduced. For the general theory of modules over algebras, the reader is referred to [1] and [25], and specifically for modules over nonunital rings and algebras, we refer to [24]. All right (resp. left) modules M over an algebra A will be assumed to be unitary. That is, we will assume that $MA=M$ (resp. $AM=M$). The corresponding category is denoted by $\mathrm{Mod}-A$ (resp. $A-\mathrm{Mod}$). When a non-unitary module eventually appears it will be explicitly mentioned. On what concerns graded algebras (or rings) and graded modules, the reader is referred to [15] for the basic concepts. Although this reference deals with graded unital rings, only a minimal adaptation is needed when passing to graded nonunital algebras. Finally, we freely use some terminology about triangulated categories. Basic references for this are [16] and [9, Chapter 10+ss], but, for a given triangulated category, we denote by $?[1]$ the shift or suspension functor, that was denoted by Σ or T in these references. Given a triangulated category $\mathcal{D}$, a subcategory $\mathcal{T}$ is a *thick subcategory* when it is closed under extensions, shifts and direct summands. When $\mathcal{S}$ is a class of objects of $\mathcal{D}$, we shall denote by $\mathrm{thick}_{\mathcal{D}}(\mathcal{S})$ the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{S}$. Recall (see [16, Definition 2.1.1]) that a functor $F\colon\mathcal{D}\longrightarrow\mathcal{D}'$ between triangulated categories is a *triangulated functor* when there is a natural isomorphism $\phi_F\colon F\circ(?[1])\overset{\cong}{\longrightarrow}(?[1])\circ F$ such that, for each triangle $X\overset{u}{\longrightarrow}Y\overset{v}{\longrightarrow}Z\overset{w}{\longrightarrow}X[1]$ in $\mathcal{D}$, the sequence $F(X)\overset{F(u)}{\longrightarrow}F(Y)\overset{F(v)}{\longrightarrow}F(Z)\overset{\phi_{F,X}\circ F(w)}{\longrightarrow}F(X)[1]$ is a triangle in $\mathcal{D}'$. If $F,G\colon\mathcal{D}\longrightarrow\mathcal{D}'$ are triangulated functors, then a *natural transformation of triangulated functors* $\tau\colon F\longrightarrow G$ is a natural transformation such that, for each triangle in $\mathcal{D}$ as above, one has $\phi_{G,X}\circ\tau_{X[1]}=\tau_X[1]\circ\phi_{F,X}$. It is well-known, and will be frequently used through these notes, that if $\tau\colon F \longrightarrow G$ is a natural transformation as above, then the class of objects $X \in \mathrm{Ob}(\mathcal{D})$ such that τ_X is an isomorphism is a thick subcategory of $\mathcal{D}$.

1. Dg categories and dg functors

In this section we collect some basic notions, mainly taken from [10] and [12], which will be used all throughout these notes. Recall that a differential graded (dg) K-module is a graded K-module $V = \bigoplus_{n\in\mathbb{Z}} V^n$, together with a graded K-linear map $d\colon V \longrightarrow V$ of degree $+1$ such that $d \circ d = 0$. The category which will be indistinctly denoted by $\mathrm{Dg} - K$ or $\mathcal{C}_{\mathrm{dg}}K$ has as objects the dg K-modules. Moreover each space of morphisms $\mathrm{HOM}_K(V,W)$ has a structure of dg K-module given by the following data:

i) The grading is $\mathrm{HOM}_K(V,W) = \bigoplus_{n\in\mathbb{Z}} \mathrm{HOM}_K^n(V,W)$, where $\mathrm{HOM}_K^n(V,W)$ consists of the graded K-linear maps $\alpha\colon V \longrightarrow W$ of degree n, i.e., such that $\alpha(V^k) \subseteq W^{k+n}$, for all $k \in \mathbb{Z}$.

ii) The differential $d\colon \mathrm{HOM}_K(V,W) \longrightarrow \mathrm{HOM}_K(V,W)$, which is a graded K-linear map of degree $+1$ such that $d \circ d = 0$, is defined by the rule $d(\alpha) = d_W \circ \alpha - (-1)^{|\alpha|}\alpha \circ d_V$, where $|?|$ denotes the degree, whenever α is a homogeneous element of $\mathrm{HOM}_K(V,W)$.

For any dg K-module V and for any $n \in \mathbb{Z}$, one puts $d^n := d_{|V^n}\colon V^n \longrightarrow V^{n+1}$, and defines $Z^n(V) := \mathrm{Ker}(d^n)$, $B^n(V) := \mathrm{Im}(d^{n-1})$ and $H^n(V) := Z^n(V)/B^n(K)$, which are called respectively the *(K)-module of n-cycles*, the *module of n-boundaries* and the *n-homology module* of V, respectively. We say that V is *acyclic* when $H^n(V) = 0$, for all $n \in \mathbb{Z}$.

Note that if V and W are dg K-modules, the tensor product $V \otimes W := V \otimes_K W$ also becomes an object of $\mathrm{Dg} - K$, where the grading is given by $(V \otimes W)^n = \oplus_{i+j=n} V^i \otimes W^j$ and the differential $d\colon V \otimes W \longrightarrow V \otimes W$ by the rule

$$d_{V\otimes W}(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d_W(w),$$

for all homogeneous elements $v \in V$ and $w \in W$. All throughout these notes, we use the unadorned symbol $\otimes$ to denote $\otimes_K$. Given a dg K-module V, one has an associated dg K-module $V[1]$, where the grading is given by $V[1]^n = V^{n+1}$, for each $n \in \mathbb{Z}$, and where $d_{V[1]} = -d_V[1]$. That is, $d_{V[1]}(v) = -d_V(v)$, for each homogeneous element $v \in V$.

The category $\mathrm{Dg} - K$ is the prototype of a *differential graded (=*dg*) category*. This is any category $\mathcal{A}$ such that, for each pair (A,B) of its objects, the K-module of morphisms, denoted indistinctly by $\mathcal{A}(A,B)$ or $\mathrm{Hom}_{\mathcal{A}}(A,B)$, has a structure of differential graded K-module so that the composition map $\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\longrightarrow\mathcal{A}(A,C)$ ($g\otimes f\rightsquigarrow g\circ f$) is a morphism of degree zero of the underlying graded K-modules which commutes with the differentials. This means that $d(g\circ f)=d(g)\circ f+(-1)^{|g|}g\circ d(f)$ whenever $g\in\mathcal{A}(B,C)$ and $f\in\mathcal{A}(A,B)$ are homogeneous morphisms. If $\mathcal{A}$ is a dg category, then the *opposite* dg *category* $\mathcal{A}^{\mathrm{op}}$ has the same class of objects as $\mathcal{A}$ and the differential on morphisms $d\colon\mathcal{A}^{\mathrm{op}}(A,B)=\mathcal{A}(B,A)\longrightarrow\mathcal{A}(B,A)=\mathcal{A}^{\mathrm{op}}(A,B)$ is the same as in $\mathcal{A}$, but the composition of morphisms is given as $\beta^o\circ\alpha^o=(-1)^{|\alpha||\beta|}(\alpha\circ\beta)^o$, where we use the superscript ^{o} to emphasize that a morphism is viewed as one in $\mathcal{A}^{\mathrm{op}}$.

If $\mathcal{A}$ and $\mathcal{B}$ are dg categories, then the *tensor product* dg *category* $\mathcal{A}\otimes\mathcal{B}$ has $\mathrm{Ob}(\mathcal{A})\times\mathrm{Ob}(\mathcal{B})$ as its class of objects and, for all pairs $(A,B),(A',B')\in\mathrm{Ob}(\mathcal{A})\times\mathrm{Ob}(\mathcal{B})$, we define $\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}[(A,B),(A',B')]=\mathcal{A}(A,A')\otimes\mathcal{B}(B,B')$, with its canonical structure of dg K-module. The composition of homogeneous morphisms in $\mathcal{A}\otimes\mathcal{B}$ is given by the rule

$$(\alpha_1\otimes\beta_1)\circ(\alpha_2\otimes\beta_2)=(-1)^{|\alpha_2||\beta_1|}(\alpha_1\circ\alpha_2)\otimes(\beta_1\circ\beta_2).$$

When $\mathcal{A}$ and $\mathcal{B}$ are dg categories, a dg *functor* $F\colon\mathcal{A}\longrightarrow\mathcal{B}$ is a graded functor (i.e. $F(\mathcal{A}^n(A,A'))\subseteq\mathcal{B}^n(F(A),F(A'))$, for all $n\in\mathbb{Z}$ and $A,A'\in\mathrm{Ob}(\mathcal{A})$) such that $F(d_{\mathcal{A}}(\alpha))=d_{\mathcal{B}}(F(\alpha))$, for each homogeneous morphism α in $\mathcal{A}$. We will frequently use the following criterion for dg functors from a tensor product dg category.

Lemma 1.1. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be* dg *categories and let $F\colon\mathcal{A}\otimes\mathcal{B}\longrightarrow\mathcal{C}$ be an assignment on objects $(A,B)\rightsquigarrow F(A,B)$ and an assignment on homogeneous morphisms $\alpha\otimes\beta\rightsquigarrow F(\alpha\otimes\beta)$ such that $|F(\alpha\otimes\beta)|=|\alpha|+|\beta|$. The following assertions are equivalent:*

1) *The given assignments define a* dg *functor $F\colon\mathcal{A}\otimes\mathcal{B}\longrightarrow\mathcal{C}$.*
2) *The following conditions hold:*
   (a) *For any fixed object $A\in\mathcal{A}$, the assignments $B\rightsquigarrow F(A,B)$ on objects and $\beta\rightsquigarrow F(1_A\otimes\beta)$ on morphisms define a* dg *functor $\mathcal{B}\longrightarrow\mathcal{C}$.*
   (b) *For any fixed object $B\in\mathcal{B}$, the assignments $A\rightsquigarrow F(A,B)$ on objects and $\alpha\rightsquigarrow F(\alpha\otimes 1_B)$ on morphisms define a* dg *functor $\mathcal{A}\longrightarrow\mathcal{C}$.*

(c) *For all homogeneous morphisms* $\alpha\colon A \longrightarrow A'$ *and* $\beta\colon B \longrightarrow B'$, *in* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively, there is the equality*

$$\begin{aligned}(-1)^{|\alpha||\beta|}&F(1_{A'} \otimes \beta) \circ F(\alpha \otimes 1_B) \\ &= F(\alpha \otimes \beta) = F(\alpha \otimes 1_{B'}) \circ F(1_A \circ \beta).\end{aligned}$$

Proof. 1) $\Longrightarrow$ 2) Since F is a dg functor it commutes with the differentials, so that $d_{\mathcal{C}}(F(\alpha \otimes \beta)) = F(d_{\mathcal{A}\otimes\mathcal{B}}(\alpha \otimes \beta))$, for all homogeneous morphisms $\alpha\colon A \longrightarrow A'$ in $\mathcal{A}$ and $\beta\colon B \longrightarrow B'$ in $\mathcal{B}$. That is, we have an equality

$$d_{\mathcal{C}}(F(\alpha \otimes \beta)) = F(d_{\mathcal{A}}(\alpha) \otimes \beta) + (-1)^{|\alpha|}F(\alpha \otimes d_{\mathcal{B}}(\beta)). \qquad (*)$$

On the other hand, by the definition of composition of morphisms in $\mathcal{A} \otimes \mathcal{B}$, we have an equality

$$(\alpha \otimes 1_{B'}) \circ (1_A \circ \beta) = \alpha \otimes \beta = (-1)^{|\alpha||\beta|}(1_{A'} \otimes \beta) \circ (\alpha \otimes 1_A).$$

Applying F to all members of this equality and using the functoriality of F, we get condition 2.c.

We next check condition 2.a, condition 2.b following by an analogous argument. The fact that, for fixed $A \in \mathcal{A}$, the assignments $B \rightsquigarrow F(A, B)$ and $\beta \rightsquigarrow F(1_A \otimes \beta)$ define a K-linear graded functor $F_A\colon \mathcal{B} \longrightarrow \mathcal{C}$ follows directly from the functoriality of F. (The corresponding construction fixing an object B of $\mathcal{B}$ is denoted F^B). We just need to check the dg condition of F_A. That is, we need to prove that if $B, B' \in \mathcal{B}$ are any two objects, then the following square is commutative

$$\begin{tikzcd} \mathcal{B}(B,B') \arrow[r, "d_{\mathcal{B}}"] \arrow[d, "F_A"] & \mathcal{B}(B,B') \arrow[d, "F_A"] & \\ \mathcal{C}(F_A(B),F_A(B')) \arrow[r, "d_{\mathcal{C}}"] & \mathcal{C}(F_A(B),F_A(B')) \arrow[r, equal] & \mathcal{C}(F(A,B),F(A,B')). \end{tikzcd}$$

Indeed we have $(F_A \circ d_{\mathcal{B}})(\beta) = F(1_A \otimes d_{\mathcal{B}}(\beta))$ while

$$\begin{aligned}(d_{\mathcal{C}} \circ F_A)(\beta) &= d_{\mathcal{C}}(F(1_A \otimes \beta)) \\ &= F(d_{\mathcal{A}}(1_A) \otimes \beta) + (-1)^{|1_A|}F(1_A \otimes d_{\mathcal{B}}(\beta)) = F(1_A \otimes d_{\mathcal{B}}(\beta)),\end{aligned}$$

due to the equality $(*)$ above and the fact that $d_{\mathcal{A}}(1_A) = 0$.

2) $\Longrightarrow$ 1) Let $\alpha_1\colon A_1 \longrightarrow A_2$ and $\alpha_2\colon A_2 \longrightarrow A_3$ be homogeneous morphisms in $\mathcal{A}$ and let $\beta_1\colon B_1 \longrightarrow B_2$ and $\beta_2\colon B_2 \longrightarrow B_3$ be homogeneous

morphisms in $\mathcal{B}$. Due to condition 2.c, we have

$$\begin{aligned}
&F[(\alpha_2 \otimes \beta_2) \circ (\alpha_1 \otimes \beta_1)] = \\
&\quad = (-1)^{|\alpha_1||\beta_2|} F((\alpha_2\alpha_1) \otimes (\beta_2\beta_1)) \\
&\quad = (-1)^{|\alpha_1||\beta_2|} F((\alpha_2\alpha_1) \otimes 1_{B_3}) \circ F(1_{A_1} \circ (\beta_2\beta_1))) \\
&\quad = (-1)^{|\alpha_1||\beta_2|} F^{B_3}(\alpha_2\alpha_1) \circ F_{A_1}(\beta_2\beta_1) \\
&\quad = (-1)^{|\alpha_1||\beta_2|} F^{B_3}(\alpha_2) \circ F^{B_3}(\alpha_1) \circ F_{A_1}(\beta_2) \circ F_{A_1}(\beta_1) \\
&\quad = (-1)^{|\alpha_1||\beta_2|} F(\alpha_2 \otimes 1_{B_3}) \circ F(\alpha_1 \otimes 1_{B_3}) \circ F(1_{A_1} \otimes \beta_2) \circ F(1_{A_1} \otimes \beta_1).
\end{aligned}$$

and

$$F(\alpha_2\otimes\beta_2)\circ F(\alpha_1\otimes\beta_1)=F(\alpha_2\otimes 1_{B_3})\circ F(1_{A_2}\otimes\beta_2)\circ F(\alpha_1\otimes 1_{B_2})\circ F(1_{A_1}\otimes\beta_1).$$

We then get that

$$F[(\alpha_2 \otimes \beta_2) \circ (\alpha_1 \otimes \beta_1)] = F(\alpha_2 \otimes \beta_2) \circ F(\alpha_1 \otimes \beta_1)$$

because, by condition 2.c, we have

$$F(1_{A_2} \otimes \beta_2) \circ F(\alpha_1 \otimes 1_{B_2}) = (-1)^{|\alpha_1||\beta_2|} F(\alpha_1 \otimes 1_{B_3}) \circ F(1_{A_1} \otimes \beta_2).$$

Moreover, we have $F(1_A \otimes 1_B) = F_A(1_B) = 1_{F_A(B)} = 1_{F(A,B)}$, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, due to the functoriality of $F_A \colon \mathcal{B} \longrightarrow \mathcal{C}$. Therefore F is a (clearly graded) K-linear functor $\mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$.

It remains to check that F is a dg functor, which amounts to prove the equality (*) above for all α and β as there. Indeed, using condition 2.c, we have

$$\begin{aligned}
d_{\mathcal{C}}(F(\alpha \otimes \beta)) &= d_{\mathcal{C}}(F(\alpha \otimes 1_{B'}) \circ F(1_A \otimes \beta)) \\
&= d_{\mathcal{C}}(F(\alpha \otimes 1_{B'})) \circ F(1_A \otimes \beta) \\
&\quad + (-1)^{|F(\alpha\otimes 1_{B'})|} F(\alpha \otimes 1_{B'}) \circ d_{\mathcal{C}}(F(1_A \otimes \beta)) \\
&= (d_{\mathcal{C}} \circ F^{B'})(\alpha) \circ F(1_A \otimes \beta) + (-1)^{|\alpha|} F(\alpha \otimes 1_{B'}) \circ (d_{\mathcal{C}} \circ F_A)(\beta).
\end{aligned}$$

But the fact that F_A and $F^{B'}$ are dg functors implies that $d_{\mathcal{C}} \circ F^{B'} = F^{B'} \circ d_{\mathcal{A}}$ and $d_{\mathcal{C}} \circ F_A = F_A \circ d_{\mathcal{B}}$. Then, using condition 2.c again, we have

$$\begin{aligned}
&d_{\mathcal{C}}(F(\alpha \otimes \beta)) \\
&\quad = (F^{B'} \circ d_{\mathcal{A}})(\alpha) \circ F(1_A \otimes \beta) + (-1)^{|\alpha|} F(\alpha \otimes 1_{B'}) \circ (F_A \circ d_{\mathcal{B}})(\beta) \\
&\quad = F(d_{\mathcal{A}}(\alpha) \otimes 1_{B'}) \circ F(1_A \otimes \beta) + (-1)^{|\alpha|} F(\alpha \otimes 1_{B'}) \circ F(1_A \otimes d_{\mathcal{B}}(\beta)) \\
&\quad = F(d_{\mathcal{A}}(\alpha) \otimes \beta) + (-1)^{|\alpha|} F(\alpha \otimes d_{\mathcal{B}}(\beta)),
\end{aligned}$$

so that the equality $(*)$ holds. $\square$

Example 1.2. If $\mathcal{A}$ is a dg category, then the following data give a dg functor $\mathcal{A}(?,?)\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathrm{Dg}-K$:

1) An assignment on objects $(A, A') \rightsquigarrow \mathcal{A}(A, A') = \mathrm{Hom}_{\mathcal{A}}(A, A')$.
2) If $\alpha\colon A \longrightarrow B$ and $\alpha'\colon A' \longrightarrow B'$ are homogeneous morphisms in $\mathcal{A}$, then $\mathcal{A}(\alpha^o \otimes \alpha')\colon \mathcal{A}(B, A') \longrightarrow \mathcal{A}(A, B')$ takes $f \rightsquigarrow (-1)^{|\alpha|(|\alpha'|+|f|)}\alpha' \circ f \circ \alpha$, for each homogeneous element $f \in \mathcal{A}(B, A')$.

Proof. For a fixed object A in $\mathcal{A}$, $\mathcal{A}(?, A) = A^\wedge\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}K = \mathrm{Dg}-K$ acts on morphisms as $A^\wedge(\alpha)(f) = (-1)^{|\alpha||f|} f \circ \alpha$ whenever f and α are composable homogeneous morphisms in $\mathcal{A}$. Then $\mathcal{A}(?, A)$ is what Keller calls the free right dg $\mathcal{A}$-module associated to A (see [10, Section 1.1]), today more commonly known as the *representable right* dg *$\mathcal{A}$-module* associated to A, and it is then a dg functor. Dually $A^\vee = \mathcal{A}(A, ?)\colon \mathcal{A} \longrightarrow C_{\mathrm{dg}}K = \mathrm{Dg} - K$ is the representable left dg $\mathcal{A}$-module, which acts on morphisms as $A^\vee(\alpha)(f) = \alpha \circ f$, and is then a dg functor. So conditions 2.a and 2.b of the last lemma hold.

On the other hand, if $\alpha\colon A \longrightarrow B$, $\alpha'\colon A' \longrightarrow A'$ and $f\colon B \longrightarrow A'$ are as in the statement, then one has

$$[\mathcal{A}(\alpha^o \otimes 1_{B'}) \circ \mathcal{A}(1_B^o \otimes \alpha')](f) = \mathcal{A}(\alpha^o \otimes 1_{B'})(\alpha' \circ f) = (-1)^{|\alpha|(|\alpha'|+|f|)}\alpha' \circ f \circ \alpha$$

while

$$\begin{aligned} &[\mathcal{A}(1_A^o \otimes \alpha') \circ \mathcal{A}(\alpha^o \otimes 1_{A'})](f) \\ &\quad = (-1)^{|\alpha||f|}\mathcal{A}(1_A^o \otimes \alpha')(f \circ \alpha) = (-1)^{|\alpha||f|}\alpha' \circ f \circ \alpha. \end{aligned}$$

Therefore condition 2.c in last lemma also holds. □

Example 1.3. Let $F\colon \mathcal{A} \longrightarrow \mathcal{A}'$ and $G\colon \mathcal{B} \longrightarrow \mathcal{B}'$ be dg functor between dg categories. The following data define a dg functor $F \otimes G\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A}' \otimes \mathcal{B}'$:

1) On objects one defines $(F \otimes G)(A, B) = (F(A), G(B))$.
2) If $\alpha\colon A_1 \longrightarrow A_2$ and $\beta\colon B_1 \longrightarrow B_2$ are morphisms in $\mathcal{A}$ and $\mathcal{B}$, respectively, then

$$\begin{tikzcd} (\mathcal{A} \otimes \mathcal{B})[(A_1, B_1), (A_2, B_2)] \arrow[d, equal] & (\mathcal{A}' \otimes \mathcal{B}')[(F \otimes G)(A_1, B_1), (F \otimes G)(A_2, B_2)] \arrow[d, equal] \\ \mathcal{A}(A_1, A_2) \otimes \mathcal{B}(B_1, B_2) \arrow[r] & \mathcal{A}'(F(A_1), F(A_2)) \otimes \mathcal{B}'(G(B_1), G(B_2)) \end{tikzcd}$$

is the map given by $(F \otimes G)(\alpha \otimes \beta) = F(\alpha) \otimes G(\beta)$.

Proof. We do not need to use Lemma 1.1, but the definition of the composition of morphisms in the tensor product dg category. Then a direct proof is easy and left as an exercise. □

With each dg category $\mathcal{A}$, one canonically associates its 0-*cycle category* $Z^0\mathcal{A}$ and its 0-*homology category* $\mathcal{H}^0\mathcal{A}$. Both of them have the same objects as $\mathcal{A}$, and as morphisms one puts $\mathrm{Hom}_{Z^0\mathcal{A}}(A, A') = Z^0(\mathcal{A}(A, A'))$ and $\mathrm{Hom}_{H^0\mathcal{A}}(A, A') = H^0(\mathcal{A}(A, A'))$. In both cases, the composition of morphisms is induced from that of $\mathcal{A}$. If $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is any dg functor, the fact that it induces a morphism $\mathcal{A}(A, A') \longrightarrow \mathcal{B}(F(A), F(A'))$ of graded K-modules which commutes with the differentials implies that it also induces a morphism of K-modules

$$\begin{aligned}\mathrm{Hom}_{Z^0\mathcal{A}}(A, A') = Z^0(\mathcal{A}(A, A')) &\longrightarrow Z^0(\mathcal{B}(F(A), F(A')))\\ &= \mathrm{Hom}_{Z^0\mathcal{B}}(F(A), F(A'))\end{aligned}$$

resp.

$$\begin{aligned}\mathrm{Hom}_{H^0\mathcal{A}}(A, A') = H^0(\mathcal{A}(A, A')) &\longrightarrow H^0(\mathcal{B}(F(A), F(A')))\\ &= \mathrm{Hom}_{H^0\mathcal{B}}(F(A), F(A')),\end{aligned}$$

for all objects $A, A' \in \mathrm{Ob}(\mathcal{A})$. It immediately follows that these are the assigments on morphisms of well-defined K-linear functors $Z^0F\colon Z^0\mathcal{A} \longrightarrow Z^0\mathcal{B}$ and $H^0F\colon H^0\mathcal{A} \longrightarrow H^0\mathcal{B}$.

The following concepts will be useful in the sequel.

Definition 1.4. Let $F, G\colon \mathcal{A} \longrightarrow \mathcal{B}$ be dg functors between dg categories. A natural transformation of dg functors $\tau\colon F \longrightarrow G$ is a natural transformation of K-linear functors such that $\tau_A\colon F(A) \longrightarrow G(A)$ is a homogeneous morphism of zero degree in $\mathcal{B}$, for each object $A \in \mathcal{A}$. We will say that F is a *homological natural transformation of* dg *functors* when, in addition, $\tau_A \in Z^0\mathcal{B}(F(A), G(A))$, for each $A \in \mathcal{A}$.

A *natural isomorphism of* dg *functors* is a homological natural transformation $\tau\colon F \longrightarrow G$ which is pointwise an isomorphism.

2. Dg algebras with enough idempotents and their categories of right dg modules

With 'algebra' instead of 'ring', the following concept is well-known (see [24, Chapter 10, Section 49]). Note that we use the term 'distinguished family' instead of the term 'complete family' used in this reference.

Definition 2.1. An *algebra with enough idempotents* is an algebra A which admits a family of nonzero orthogonal idempotents $(e_i)_{i\in I}$ such that $\bigoplus_{i\in I} e_iA = A = \bigoplus_{i\in I} Ae_i$. This family $(e_i)_{i\in I}$ will be called a *distinguished family of orthogonal idempotents*. A *graded algebra with enough idempotents* is an algebra with enough idempotents together with a grading $A = \bigoplus_{n\in\mathbb{Z}} A^n$ on it such that A admits a distinguished family of orthogonal idempotents consisting of homogeneous elements of degree 0. Without further remark, on a graded algebra with enough idempotents we only consider distinguished families consisting of degree zero homogeneous idempotents.

Note that, for A as above, to say that a right (resp. left) A-module is unitary is equivalent to say that we have an internal decomposition $M = \bigoplus_{i\in I} Me_i$ (resp. $M = \bigoplus_{i\in I} e_iM$) as K-module. Recall that all our modules will be unitary, unless explicitly said otherwise.

The crucial concept for us is the following:

Definition 2.2. A *differential graded (*dg*) algebra with enough idempotents* is a pair (A, d), where A is a graded algebra with enough idempotents and $d\colon A \longrightarrow A$ is a morphism of degree $+1$ of graded K-modules, called the *differential*, satisfying the following conditions: i) $d\circ d = 0$; ii) $d(e_i) = 0$ for all $i \in I$; and iii) (*Leibniz rule*) $d(ab) = d(a)b + (-1)^{|a|}ad(b)$, for all homogeneous elements $a, b \in A$.

Given a dg algebra with enough idempotents $A = (A, d)$, the usual opposite algebra has a canonical structure of graded algebra. However, the differential d would not satisfy Leibniz rule when viewed as a map $d^o\colon A^{\mathrm{op}} \longrightarrow A^{\mathrm{op}}$. This forces to redefine the concept of *opposite graded algebra with enough idempotents* A^{op} as the one having the same underlying graded K-module as A, but where the multiplication of homogeneous elements is defined by $a^o \cdot b^o := (-1)^{|a||b|}(ba)^o$, for all $a, b \in A$. Here we use the upper index ^{o} to indicate that we are viewing the element as one of the opposite graded algebra. The following is now routine:

Exercise 2.3. If (A, d) is a dg algebra with enough idempotents and A^{op} is the opposite graded algebra in the above sense, then $d^o\colon A^{\mathrm{op}} \longrightarrow A^{\mathrm{op}}$ is a differential making the pair (A^{op}, d^o) to be a dg algebra with enough idempotents (with the same distinguished family of homogeneous idempotents as A).

The following gives the definition of the *tensor product of two* dg *algebras with enough idempotents*.

Lemma 2.4. *Let $A = (A, d)$ and $B = (B, d)$ be two* dg *algebras with enough idempotents and let $A \otimes B$ their tensor product in* $\mathrm{Dg} - K$. *When* *one defines the multiplication of homogeneous tensors by the rule that* $(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} ac \otimes bd$*,* $A \otimes B$ *becomes a* dg *algebra with enough idempotents, with the same differential as in* $\mathrm{Dg} - K$*.*

Proof. It is routine to check the associativity, so that $A \otimes B$ becomes an associative graded algebra with the given multiplication. Moreover, if $(e_i)_{i \in I}$ and $(e'_j)_{j \in J}$ are distinguished families of homogeneous idempotents of degree 0 in A and B, respectively, then $(e_i \otimes e'_j)_{(i,j) \in I \times J}$ is a distinguished family of homogeneous orthogonal idempotents of degree 0 in $A \otimes B$. On the other hand, the differential $d \colon A \otimes B \longrightarrow A \otimes B$ vanishes on each $e_i \otimes e'_j$ It remains to check Leibniz rule. It is also routine, but for the convenience of the reader we explicitly give the calculations:

$$\begin{aligned}
d[(a_1 \otimes b_1) \cdot (a_2 \otimes b_2)] &= (-1)^{|b_1||a_2|} d[(a_1 a_2) \otimes (b_1 b_2)] \\
&= (-1)^{|b_1||a_2|}[d(a_1 a_2) \otimes (b_1 b_2) + (-1)^{|a_1|+|a_2|}(a_1 a_2) \otimes d(b_1 b_2)] \\
&= (-1)^{|b_1||a_2|}[(d(a_1)a_2 + (-1)^{|a_1|} a_1 d(a_2)) \otimes (b_1 b_2)] \\
&\quad + (-1)^{|b_1||a_2|+|a_1|+|a_2|}[(a_1 a_2) \otimes (d(b_1)b_2 + (-1)^{|b_1|} b_1 d(b_2))] \\
&= (-1)^{|b_1||a_2|} d(a_1)a_2 \otimes b_1 b_2 \\
&\quad + (-1)^{|b_1||a_2|+|a_1|} a_1 d(a_2) \otimes b_1 b_2 \\
&\quad + (-1)^{|b_1||a_2|+|a_1|+|a_2|} a_1 a_2 \otimes d(b_1) b_2 \\
&\quad + (-1)^{|b_1||a_2|+|a_1|+|a_2|+|b_1|} a_1 a_2 \otimes b_1 d(b_2).
\end{aligned}$$

while, on the other side, we have:

$$\begin{aligned}
d(a_1 \otimes b_1) &\cdot (a_2 \otimes b_2) + (-1)^{|a_1|+|b_1|}(a_1 \otimes b_1) \cdot d(a_2 \otimes b_2) \\
&= [d(a_1) \otimes b_1 + (-1)^{|a_1|} a_1 \otimes d(b_1)] \cdot (a_2 \otimes b_2) \\
&\quad + (-1)^{|a_1|+|b_1|}(a_1 \otimes b_1) \cdot [d(a_2) \otimes b_2 + (-1)^{|a_2|} a_2 \otimes d(b_2)] \\
&= (-1)^{|b_1||a_2|} d(a_1) a_2 \otimes b_1 b_2 \\
&\quad + (-1)^{|a_1|}(-1)^{(|b_1|+1)|a_2|} a_1 a_2 \otimes d(b_1) b_2 \\
&\quad + (-1)^{|a_1|+|b_1|}(-1)^{|b_1|(|a_2|+1)} a_1 d(a_2) \otimes b_1 b_2 \\
&\quad + (-1)^{|a_1|+|b_1|}(-1)^{|a_2|}(-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 d(b_2).
\end{aligned}$$

Therefore Leibniz rule holds for the given multiplication in $A \otimes B$. $\square$

Associated with any graded algebra with enough idempotents A, we have the category $\mathrm{Gr} - A$ of graded right A-modules, where the morphisms between two objects M and N are the homomorphisms of right A-modules

$f: M \longrightarrow N$ such $f(M^n) \subseteq N^n$, for all $n \in \mathbb{Z}$. The category $\mathrm{Gr} - A$ comes with a *shift functor* $?[1]: \mathrm{Gr} - A \longrightarrow \mathrm{Gr} - A$. For each graded right A-module M, $M[1]$ has the same underlying (ungraded) A-module as M, but the grading on $M[1]$ is given by $M[1]^n = M^{n+1}$, for all $n \in \mathbb{Z}$. The action of $?[1]$ on morphisms is the identity. It is clear that $?[1]$ is an equivalence of categories, which allows to define the iterated powers $?[n] = (?[1])^n$, for all $n \in \mathbb{Z}$. We then form the graded category $\mathrm{GR}-A$. Its objects are the same as in $\mathrm{Gr} - A$ and, given two graded right A-modules M and N, we define

$$\mathrm{HOM}_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}-A}(M, N[n])$$

as space of morphisms in $\mathrm{GR} - A$. On this space of morphisms we have an obvious grading given by $\mathrm{HOM}^n_A(M, N) := \mathrm{Hom}_{\mathrm{Gr}-A}(M, N[n])$, for each $n \in \mathbb{Z}$. The composition $g \circ f$ in $\mathrm{GR} - A$ of two homogeneous morphisms $f: M \longrightarrow N[n]$ and $g: N \longrightarrow P[p]$ is defined as the composition $g[n] \circ f$ in $\mathrm{Gr} - A$.

Definition 2.5. Let $A = (A, d)$ be a dg algebra with enough idempotents. A *right (resp. left) differential graded (*dg*) A-module* is a pair (M, d_M) consisting of a graded right (resp. left) A-module $M = \bigoplus_{n \in \mathbb{Z}} M^n$ together with a morphism $d_M: M \longrightarrow M[1]$ in $\mathrm{Gr} - K$ such $d_M \circ d_M = 0$ and $d_M(xa) = d_M(x)a + (-1)^{|x|} x d(a)$ (resp. $d_M(ax) = d(a)x + (-1)^{|a|} a d_M(x)$), for all homogeneous elements $x \in M$ and $a \in A$.

Suppose that A is a dg algebra with enough idempotents and that M is a right dg A-module. The graded right A-module $M[1]$ with its differential $d_{M[1]} = -d_M$ as dg K-module (see Section 1) becomes a right dg A-module. Indeed, if one has $x \in M[1]^n = M^{n+1}$ and $a \in A^p$, then

$$\begin{aligned} d_{M[1]}(xa) &= -d_M(xa) = -[d(x)a + (-1)^{n+1} x d(a)] \\ &= -d(x)a + (-1)^n x d(a) = d_{M[1]}(x)a + (-1)^{|x|} x d(a), \end{aligned}$$

where $|x| = n$ is the degree of x in $M[1]$. In this way, we get a functor $?[1]: \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - A$ which is 'almost' a dg functor, in the sense that if $d: \mathrm{HOM}_A(M, N) \longrightarrow \mathrm{HOM}_A(M, N)$ and $\delta: \mathrm{HOM}_A(M[1], N[1]) \longrightarrow \mathrm{HOM}_A(M[1], N[1])$ are the respective differentials on Hom spaces, then $\delta(f[1]) = -d(f)[1]$, for each homogeneous morphism $f \in \mathrm{HOM}_A(M, N)$. The reader is referred to section 4 to see that the corresponding functor for left dg modules produces suprising effects.

Proposition 2.6. *Let A be a* dg *algebra with enough idempotents. The following data give a* dg *category* $\mathrm{Dg}-A$*, the* dg *category of right* dg A*-modules:*

- *The objects of* $\mathrm{Dg}-A$ *are the right* dg A*-modules (see Definition 2.5);*
- *The morphisms in* $\mathrm{Dg}-A$ *and the composition of them is defined as in the category* $\mathrm{GR}-A$*.*
- *For each pair* (M,N) *of objects, the differential*

$$d\colon\ \mathrm{HOM}_A(M,N)\longrightarrow \mathrm{HOM}_A(M,N)$$

on Hom *spaces is defined by the rule* $d(f)=d_N\circ f-(-1)^{|f|}f\circ d_M$*, for each homogeneous morphism* f*.*

Proof. We first need to check that the differential on Hom spaces is well-defined, i.e. that $d(f)$ is a homomorphism of graded right A-modules, which is homogeneous of degree $|f|+1$, whenever $f\in\mathrm{HOM}_A(M,N)$ is homogeneous. Indeed if $x\in M$ and $a\in A$ are homogeneous elements, then we have:

$$\begin{aligned}
d(f)(xa) &= [d_N\circ f-(-1)^{|f|}f\circ d_M](xa)\\
&= d_N(f(x)a)-(-1)^{|f|}f(d_M(xa))\\
&= d_N(f(x))a+(-1)^{|f|+|x|}f(x)d(a)-(-1)^{|f|}f(d_M(x)a\\
&\quad +(-1)^{|x|}xd(a))\\
&= (d_N\circ f)(x))a+(-1)^{|f|+|x|}f(x)d(a)-(-1)^{|f|}(f\circ d_M)(x)a\\
&\quad -(-1)^{|f|+|x|}f(x)d(a)\\
&= (d_N\circ f)(x))a-(-1)^{|f|}(f\circ d_M)(x)a\\
&= d(f)(x)a.
\end{aligned}$$

Then $d(f)$ is a homogeneous morphism in $\mathrm{GR}-A$, clearly of degree $|f|+1$.

Given the fact that the differential on $\mathrm{HOM}_A(M,N)$ is the restriction of the differential on $\mathrm{HOM}_K(M,N)$ and that the composition of morphism in $\mathrm{Dg}-A$ is defined as in $\mathrm{Dg}-K$, and the latter is a dg category, the equality

$$d(g\circ f)=d(g)\circ f+(-1)^{|g|}g\circ d(f), \qquad (*)$$

holds for all homogeneous morphisms $f\in\mathrm{HOM}_A(M,N)$ and $g\in\mathrm{HOM}_A(N,P)$. Then $\mathrm{Dg}-A$ is also a dg category.

□

3. Dg algebras with enough idempotents versus small dg categories

Let $A = (A, d)$ be a dg algebra with enough idempotents on which we fix a distinguished family of orthogonal idempotents $(e_i)_{i \in I}$, which are homogeneous of degree zero and such that $d(e_i) = 0$, for all $i \in I$. We can view A as a small dg category as follows:

- The set of objects is $\mathrm{Ob}(A) = I$;
- If $i, j \in A$, the set of morphisms of degree n from i to j is $A^n(i, j) := e_j A^n e_i$, for all $n \in \mathbb{Z}$;
- The composition map $A(j,k) \times A(i,j) = e_k A e_j \times e_j A e_i \longrightarrow e_k A e_i = A(i,k)$ is the multiplication map.
- The differential $d\colon A(i,j) = e_j A e_i \longrightarrow e_j A e_i = A(i,j)$ is the differential of A as a dg algebra, for all $i, j \in I$.

It is routine to check that the data above make A into a small dg category. Conversely, let $\mathcal{A}$ be a small dg category. We can view $\mathcal{A}$ as a dg algebra with enough idempotents as follows:

- The elements of $\mathcal{A}$ are those of $\bigoplus_{A,B \in \mathrm{Ob}(\mathcal{A})} \mathcal{A}(A, B)$, and we put
$$\mathcal{A}^n = \bigoplus_{A,B \in \mathrm{Ob}(\mathcal{A})} \mathcal{A}^n(A, B)$$
for the K-module of elements of degree n in $\mathcal{A}$, for all $n \in \mathbb{Z}$.
- The multiplication in $\mathcal{A}$ extends by K-linearity the composition of morphisms in $\mathcal{A}$.
- The differential $d\colon \mathcal{A} \longrightarrow \mathcal{A}$ is the direct sum of the differentials $d_{A,B}\colon \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, B)$, as A, B vary on the set of objects of $\mathcal{A}$.

It is routine to see that the data above make $\mathcal{A}$ into a dg algebra with enough idempotents, where the identities $e_A := 1_A$ $(A \in \mathrm{Ob}(\mathcal{A}))$ form a distinguished family of orthogonal idempotents of degree zero. Note that we have $\mathcal{A}(A, B) = e_B \mathcal{A} e_A$.

The processes explained above of passing from dg algebras with enough idempotents to small dg categories, and viceversa, are clearly inverse to each other. This allows us to pass freely from one language to the other. Note, in particular, that the opposite dg algebra with enough idempotent corresponds to the opposite dg category by this bijective correspondence.

To be consistent with our notation in the previous section, we shall denote by $\mathrm{Gr} - K$ the category of graded K-modules with degree zero morphisms and $\mathrm{GR} - K$ the graded category with the same objects and where, for each pair (V, W) of objects, the graded K-module of morphisms is

$$\mathrm{HOM}_K(V,W)=\bigoplus_{p\in\mathbb{Z}}\mathrm{HOM}^p_K(V,W),$$

where $\mathrm{HOM}^p(V,W)=\mathrm{Hom}_{\mathrm{Gr}-K}(V,W[p])$ consists of those morphisms of K-modules $f\colon V\longrightarrow W$ such that $f(V^n)\subseteq V^{n+p}$, for all $n\in\mathbb{Z}$. Note that $\mathrm{GR}-K$ is just the underlying graded category of the dg category $\mathrm{Dg}-K$.

Given a small dg category $\mathcal{A}$, a *graded right* $\mathcal{A}$-module was defined in [10] as a graded functor $M\colon\mathcal{A}^{\mathrm{op}}\longrightarrow\mathrm{GR}-K$. The category $\mathcal{GA}$ has as objects the graded right $\mathcal{A}$-modules and as morphisms their natural transformations. Note that, by definition, if $f\colon M\longrightarrow N$ is a morphism in $\mathcal{GA}$ then $f_A\colon M(A)\longrightarrow N(A)$ is a morphism in $\mathrm{Gr}-K$, for each $A\in\mathrm{Ob}(\mathcal{A})$. The category $\mathcal{GA}$ comes with a graded functor $?[1]\colon\mathcal{GA}\longrightarrow\mathcal{GA}$ given on objects by the rule $M[1](A)=M(A)[1]$. If $a^o\in\mathcal{A}^{\mathrm{op}}(A,B)=\mathcal{A}(B,A)$ is a homogeneous element, then $M[1](a^o)\colon M(A)[1]\longrightarrow M(B)[1]$ is the map $(-1)^{|a|}M(a^o)\colon M(A)\longrightarrow M(B)$. We claim that with this definition we have a well-defined graded right $\mathcal{A}$-module. Indeed, if $b^o\in\mathcal{A}^{\mathrm{op}}(B,C)=\mathcal{A}(C,B)$ is another homogeneous element, then we have

$$\begin{aligned}M[1](b^o\circ a^o)&=(-1)^{|a||b|}M[1]((a\circ b)^o)=(-1)^{|a||b|}(-1)^{|a|+|b|}M((a\circ b)^o)\\&=(-1)^{|a|+|b|}M[(-1)^{|a||b|}(a\circ b)^o]=(-1)^{|a|+|b|}M(b^o\circ a^o)\end{aligned}$$

while

$$M[1](b^o)\circ M[1](a^o)=(-1)^{|a|+|b|}M(b^o)\circ M(a^o).$$

Therefore $M[1]$ is a well-defined graded right $\mathcal{A}$-module. Note the discrepancy of the definition of $M[1]$ with the definition in [10]. The assignment $M\rightsquigarrow M[1]$ extends to an auto-equivalence of categories $\mathcal{GA}\longrightarrow\mathcal{GA}$ which acts as the identity on morphisms. Then the graded category $\mathrm{Gra}\mathcal{A}$ was defined in [10] as the one having the same objects as $\mathcal{GA}$ and as graded K-module of morphisms $\mathrm{Hom}_{\mathrm{Gra}\mathcal{A}}(M,N)=\bigoplus_n\mathrm{Hom}_{\mathcal{GA}}(M,N[n])$, where the composition of homogeneous element is given as $g\circ f=g[p]\circ f$, provided $|f|=p$.

A dg functor $M\colon\mathcal{A}^{\mathrm{op}}\longrightarrow\mathrm{Dg}-K$ is called a *right* dg $\mathcal{A}$-*module*. It becomes an object of $\mathrm{Gra}\mathcal{A}$ when considering the composition $\mathcal{A}^{\mathrm{op}}\xrightarrow{M}\mathrm{Dg}-K\xrightarrow{forgetful}\mathrm{GR}-K$. The category $\mathcal{C}_{\mathrm{dg}}\mathcal{A}$ (see [12]), denoted $\mathrm{Dif}\mathcal{A}$ in [10], has as objects the right dg $\mathcal{A}$-modules with spaces of morphisms $\mathrm{Hom}_{\mathcal{C}_{\mathrm{dg}}\mathcal{A}}(M,N)=\mathrm{Hom}_{\mathrm{Gra}\mathcal{A}}(M,N)$, where the differential $d\colon\mathrm{Hom}_{\mathcal{C}_{\mathrm{dg}}\mathcal{A}}(M,N)\longrightarrow\mathrm{Hom}_{\mathcal{C}_{\mathrm{dg}}\mathcal{A}}(M,N)$ acts as $d(f)=d_N\circ f-(-1)^{|f|}f\circ d_M$. Note that one extends $?[1]$ from $\mathrm{Gra}\mathcal{A}$ to $?[1]\colon\mathcal{C}_{\mathrm{dg}}\mathcal{A}\longrightarrow\mathcal{C}_{\mathrm{dg}}\mathcal{A}$, by defining the differential as $d_{M[1]}=-d_M[1]$.

Theorem 3.1. *Let $A = (A, d)$ be a graded algebra with enough idempotents, where $(e_i)_{i\in I}$ is a fixed distinguished family of orthogonal idempotents, all of them homogeneous of zero degree and annihilated by d. We also view A as a* dg *category with $Ob(A) = I$ as described above. Let M, N be arbitrary objects of* $\mathrm{Dg} - A$ *and $f \colon M \longrightarrow N$ be a homogeneous morphism in this category. The following assertions hold:*

1) *The assignments $i \rightsquigarrow \widetilde{M}(i) := Me_i$, and*

$$e_i A^n e_j \longrightarrow \mathrm{Hom}^n_{\mathrm{GR}-K}(Me_i, Me_j),$$

   $a^o \rightsquigarrow \widetilde{M}(a^o)$, where $\widetilde{M}(a^o)(x) = (-1)^{|a||x|}xa$ for each $x \in Me_i$ homogeneous, define a dg *functor $\widetilde{M} \colon A^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}K$ and, hence, an object of $\mathcal{C}_{\mathrm{dg}}A$.*

2) *If $\widetilde{f} = (\widetilde{f}_i)_{i\in I}$, where $\widetilde{f}_i := f_{|Me_i} \colon \widetilde{M}(i) = Me_i \longrightarrow Ne_i = \widetilde{N}(i)$, for each $i \in I$, then $\widetilde{f}$ is a morphism $\widetilde{M} \longrightarrow \widetilde{N}$ of degree $|f|$ in $\mathcal{C}_{\mathrm{dg}}A$.*
3) *The assignments $M \rightsquigarrow \widetilde{M}$ and $f \rightsquigarrow \widetilde{f}$ of the two previous assertions define an equivalence of* dg *categories* $\mathrm{Dg} - A \xrightarrow{\cong} \mathcal{C}_{\mathrm{dg}}A$.

Proof. 1) We first prove that $\widetilde{M}$ is a graded functor between the underlying graded categories of A and $\mathcal{C}_{\mathrm{dg}}K = \mathrm{Dg} - K$, for which we just need to check that $\widetilde{M}(b^o \circ a^o) = \widetilde{M}(b^o) \circ \widetilde{M}(a^o)$ whenever $a^o \in A^{\mathrm{op}}(i, j) = e_iAe_j$ and $b^o \in A^{\mathrm{op}}(j, k) = e_jAe_k$ are homogeneous elements, where $i, j, k \in I$. Note that both sides of the desired equality are then K-linear maps $\widetilde{M}(i) = Me_i \longrightarrow Me_k = \widetilde{M}(k)$. When applying them to a homogeneous element $x \in Me_i$, we have:

$$\begin{aligned}\widetilde{M}(b^o \circ a^o)(x) &= (-1)^{|a||b|}\widetilde{M}((ab)^o)(x) = (-1)^{|a||b|}(-1)^{|ab||x|}x(ab)\\ &= (-1)^{|a||b|+|a||x|+|b||x|}x(ab) = (-1)^{|a||x|}(-1)^{|b||xa|}(xa)b\\ &= (-1)^{|a||x|}\widetilde{M}(b^o)(xa) = \widetilde{M}(b^o)[(-1)^{|a||x|}xa] = (\widetilde{M}(b^o) \circ \widetilde{M}(a^o))(x)\end{aligned}$$

The desired equality then holds due to the fact that M is a right A-module.

In order to see that $\widetilde{M}$ is a dg functor, we need to check that it commutes with the differentials on Hom spaces of A^{op} and $\mathcal{C}_{\mathrm{dg}}K$. That is, that if $a \in e_iAe_j = A^{\mathrm{op}}(i, j)$ is a homogeneous element, then $\widetilde{M}(d(a)^o) = d_{\mathrm{HOM}_K(\widetilde{M}(i),\widetilde{M}(j))}(\widetilde{M}(a))$. To check this equality, we evaluate the two maps on a homogeneous element $x \in \widetilde{M}(i) = Me_i$. We then have:

$$\begin{aligned}
&d_{\mathrm{HOM}_K(\widetilde{M}(i),\widetilde{M}(j))}(\widetilde{M}(a^o))(x)\\
&\quad = [d_{Me_j} \circ \widetilde{M}(a^o) - (-1)^{|\widetilde{M}(a^o)|}\widetilde{M}(a^o) \circ d_{Me_i}](x)\\
&\quad = d_M((-1)^{|a||x|}xa) - (-1)^{|a|}(-1)^{|a||d_M(x)|}d_M(x)a\\
&\quad = (-1)^{|a||x|}(d_M(xa) - d_M(x)a) = (-1)^{|a||x|}(-1)^{|x|}xd(a)\\
&\quad = (-1)^{(|a|+1)|x|}xd(a) = (-1)^{|d(a)||x|}xd(a) = \widetilde{M}(d(a)^o)(x),
\end{aligned}$$

as desired.

2) In order to prove this assertion, we first show that $\widetilde{M[1]}$ is isomorphic to $\widetilde{M}[1]$. On objects, we have $\widetilde{M[1]}(i) = M[1]e_i = Me_i[1] = \widetilde{M}[1](i)$, for each $i \in I$. On the other hand, if $a^o \in A^{\mathrm{op}}(i,j) = e_iAe_j$ and $x \in \widetilde{M[1]}(e_i) = Me_i$ are homogeneous elements, then we have that $\widetilde{M[1]}(a^o)(x) = (-1)^{|a||x|_{M[1]}}xa$, where $|x|_{M[1]}$ denotes the degree of x as an element of $M[1]$. We know that $|x|_{M[1]} = |x| - 1$, where $|x|$ is the degree of x as an element of M. Therefore we have $\widetilde{M[1]}(a^o)(x) = (-1)^{|a|(|x|-1)}xa$. On the other side, by definition of the shift in $Gra - K$, we have that $\widetilde{M}[1](a^o) = (-1)^{|a|}\widetilde{M}(a^o)$. It follows that

$$(\widetilde{M}[1])(a^o)(x) = (-1)^{|a|}\widetilde{M}(a^o)(x) = (-1)^{|a|}(-1)^{|a||x|}xa.$$

As a consequence, we have that $\widetilde{M[1]}(a^o)(x) = (\widetilde{M}[1])(a^o)(x)$.

Let $a^o \in A^{\mathrm{op}}(i,j) = e_iAe_j$ be homogeneous and assume that $|f| = n$, that is, that $f\colon M \longrightarrow N[n]$ is a morphism in $\mathrm{Gr} - A$. We need to prove that the following diagram in $\mathrm{Gr} - K$ is commutative:

$$\begin{tikzcd}
\widetilde{M}(i) \arrow[r, "\widetilde{f}_i"] \arrow[d, "\widetilde{M}(a^o)"'] & \widetilde{N}[n](i) \arrow[d, "(\widetilde{N}[n])(a^o)"] \\
\widetilde{M}(j) \arrow[r, "\widetilde{f}_j"] & \widetilde{N}[n](j)
\end{tikzcd}$$

Indeed, for each $x \in \widetilde{M}(i) = Me_i$ homogeneous, we have

$$(\widetilde{f}_j \circ \widetilde{M}(a^o))(x) = \widetilde{f}_j((-1)^{|a||x|_M}xa) = (-1)^{|a||x|_M}f(xa),$$

and, using the previous paragraph, we also have

$$\begin{aligned}
(\widetilde{N}[n](a^o) \circ \widetilde{f}_i)(x) &= (\widetilde{N[n]}(a^o) \circ \widetilde{f}_i)(x)\\
&= \widetilde{N[n]}(a^o)(f(x)) = (-1)^{|a||f(x)_{N[n]}|}f(x)a.
\end{aligned}$$

Note that $|f(x)|_{N[n]} = |x|_M$ because $f\colon M \longrightarrow N[n]$ is a morphism of degree zero. On the other hand, the product $f(x)a$ is considered in the graded right A-module $N[n]$. The commutativity of the desired diagram follows from that fact that $f\colon M \longrightarrow N[n]$ is a morphism of right A-modules.

3) Since the definition of $\widetilde{(?)}$ on morphisms is the 'identity', i.e. $\widetilde{f}_i\colon \widetilde{M}(i) = Me_i \longrightarrow \widetilde{N}(i) = Ne_i$ is just the restriction of f to Me_i, we readily see that we have a well-defined K-linear functor between the underlying graded categories $\widetilde{(?)}\colon \mathrm{GR} - A \longrightarrow Gra - A$. This functor is clearly graded since $|\widetilde{f}| = |f|$ for each homogeneous morphism f in $\mathrm{GR} - A$. Moreover the differential of the graded K-module $Me_i = \widetilde{M}(i)$ is the same when coming from M that when coming from $\widetilde{M}$. Again the fact that the action of $\widetilde{(?)}$ on morphisms is the identity implies that $\widetilde{(?)}$ commutes with the differentials on Hom spaces. That is, it is actually a dg functor $\mathrm{Dg} - A \longrightarrow \mathcal{C}_{\mathrm{dg}}A$.

On the other hand, the 'identity' condition on the action on morphisms immediately implies that $\widetilde{(?)}$ is a faithful functor. We shall now prove that $\widetilde{(?)}$ is full. If $\psi\colon \widetilde{M} \longrightarrow \widetilde{N}[n]$ is a morphism in $\mathcal{G}A$, then for each $a^o \in A^{\mathrm{op}}(i,j) = e_iAe_j$, we have that $\psi_j \circ \widetilde{M}(a^o) = \widetilde{N}[n](a^o) \circ \psi_i$. When applying both members of this equality to an element $x \in \widetilde{M}(i) = Me_i$, we get that

$$(\psi_j \circ \widetilde{M}(a^o))(x) = \psi_j[(-1)^{|a||x|_M} xa] = (-1)^{|a||x|_M}\psi_j(xa)$$

while

$$(\widetilde{N}[n](a^o) \circ \psi_i)(x) = (\widetilde{N[n]}(a^o) \circ \psi_i)(x) = (-1)^{|a||\psi_i(x)|_{N[n]}}\psi_i(x)a.$$

Bearing in mind that $|\psi_i(x)|_{N[n]} = |x|_M$, we conclude that $\psi_j(xa) = \psi_i(x)a$. This means that if we define $f\colon M = \bigoplus_{i\in I} Me_i \longrightarrow \bigoplus_{i\in I} N[n]e_i = N[n]$ as the direct sum of the $\psi_i\colon Me_i = \widetilde{M}(i) \longrightarrow \widetilde{N}[n](i) = N[n]e_i$, then f is a morphism of graded right A-modules such that $\widetilde{f} = \psi$. We showed that the functor $\widetilde{(?)}$ is also full.

It remains to check that $\widetilde{(?)}$ is a dense functor. Let F be an object of $\mathcal{C}_{\mathrm{dg}}A$ and consider the dg K-module $M_F := \bigoplus_{i\in I} F(i)$. We endow M_F with a structure of graded right A-module as follows. Given $x \in F(i)$ and $a^o \in A^{\mathrm{op}}(j,k) = e_jAe_k$, we define $xa := (-1)^{|a||x|}\delta_{ij}F(a^o)(x)$, where δ_{ij} is the Kronecker symbol. Then one extends this multiplication by K-linearity. In order to see that this rule gives $M := M_F$ the structure of a graded right A-module, we just need to consider $x \in F(i) = Me_i$, $a \in e_iAe_j$ and $b \in e_jAe_k$ homogeneous elements and check that $x(ab) = (xa)b$.

Indeed, bearing in mind that $(ab)^o = (-1)^{|a||b|}(b^o \circ a^o)$ when looking at the elements of A as morphism in the underlying graded K-category, we have an equality

$$\begin{aligned} x(ab) &= (-1)^{|ab||x|}F((ab)^o)(x) = (-1)^{|ab||x|}(-1)^{|a||b|}F(b^o \circ a^o)(x) \\ &= (-1)^{|a||x|+|b||x|+|a||b|}[F(b^o) \circ F(a^o)](x) \\ &= (-1)^{|b||xa|}(-1)^{|a||x|}F(b^o)(F(a^o)(x)) \\ &= (-1)^{|b||xa|}F(b^o)(xa) = (xa)b, \end{aligned}$$

which shows that $x(ab) = (xa)b$ as desired.

We claim that the differential $d = \oplus d_{F(i)} \colon M = \oplus_{i \in I} F(i) \longrightarrow \oplus_{i \in I} F(i) = M$ satisfies Libniz rule, thus making M into a right dg A-module. To see this, note that since $F \colon A^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}K$ is a dg functor we have an equality $\delta(F(a^o)) = F(d(a)^o)$, for any morphism $a^o \in A^{\mathrm{op}}(i,j) = e_iAe_j$, where $d \colon e_iAe_j \longrightarrow e_iAe_j$ is the restriction of the differential of A and $\delta \colon \mathrm{HOM}_K(F(i), F(j)) = \mathrm{HOM}_K(Me_i, Me_j) \longrightarrow \mathrm{HOM}_K(Me_i, Me_j)$ is the differential on Hom spaces of $\mathcal{C}_{\mathrm{dg}}K$. We then have that $F(d(a)^o) = d_{Me_j} \circ F(a^o) - (-1)^{|F(a^o)|}F(a^o) \circ d_{Me_i}$. Bearing in mind that $|F(a^o)| = |a|$, when making act both members of the last equality on a homogeneous element $x \in Me_i$, we have

$$(-1)^{|d(a)||x|}xd(a) = (-1)^{|a||x|}d_{Me_j}(xa) - (-1)^{|a|}(-1)^{|d_{Me_i}(x)||a|}d_{Me_i}(x)a.$$

Cancelling $(-1)^{|a||x|}$ from this equality, we get that

$$(-1)^{|x|}xd(a) = d_M(xa) - d_M(x)a,$$

from which Leibniz rule immediately follows.

The fact that $\widetilde{M_F} \cong F$ follows in a straightforward way, and hence $\widetilde{(?)}$ is a dense functor. $\square$

Remark 3.2. Note that the equivalence of categories $\widetilde{(?)} \colon \mathrm{Dg} - A \longrightarrow \mathcal{C}_{\mathrm{dg}}A$ given by Theorem 3.1 takes e_iA to the representable dg A-module $i^\wedge \colon A(?, i) \colon A^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}K = \mathrm{Dg} - K$ (see Example 1.2 and its proof).

4. Right versus left dg modules

From the definition of left dg A-module we get the following:

Lemma 4.1. *If (M, d_M) is a left* dg *A-module, then it is a right* dg *A^{op}-module with the multiplication map $M \otimes A^{\mathrm{op}} \longrightarrow M$ defined as*

$(x, a^o) \rightsquigarrow xa^o := (-1)^{|a||x|}ax$, *for all homogeneous elements* $a \in A$ *and* $x \in M$. *Conversely, if* (M, d_M) *is a right* dg A^{op}*-module, then it is a left* dg A*-module, where the multiplication map* $A \otimes M \longrightarrow M$ *takes* $(a, x) \rightsquigarrow ax := (-1)^{|a||x|}xa^o$*, for* a *and* x *as above.*

Proof. We just prove the first implication, the reverse one being then clear. Given $x \in M$ and $a, b \in A$ homogeneous elements, we have:

$$(xa^o)b^o = (-1)^{|xa^o||b|}b(xa^o) = (-1)^{(|x|+|a|)|b|)}(-1)^{|x||a|}b(ax)$$

and

$$\begin{aligned} x(a^ob^o) &= (-1)^{|a||b|}x(ba)^o = (-1)^{|a||b|}(-1)^{|x||ba|}(ba)x \\ &= (-1)^{|a||b|}(-1)^{|x|(|b|+|a|)}(ba)x. \end{aligned}$$

Therefore we have $(xa^o)b^o = x(a^ob^o)$. Since up to here the differential has played no role, we have actually proved that any graded left A-module is a graded right A^{op}-module.

We next check that the differential of M as a left A-module satisfies Leibniz rule as a right A^{op}-module. We have that

$$d_M(xa^o) = (-1)^{|x||a|}d_M(ax) = (-1)^{|x||a|}[d(a)x + (-1)^{|a|}ad_M(x)],$$

while we have

$$\begin{aligned} &d_M(x)a^o + (-1)^{|x|}xd(a)^o \\ &\quad = (-1)^{(|x|+1)|a|}ad_M(x) + (-1)^{|x|}(-1)^{|x|(|a|+1)}d(a)x, \end{aligned}$$

so that $d_M(xa^o) = d_M(x)a^o + (-1)^{|x|}xd(a)$, for all homogeneous elements $x \in M$ and $a^o \in A^{\mathrm{op}}$. □

As with graded right A-modules, one first defines the category $A - \mathrm{Gr}$ of graded left A-modules, where morphisms are the graded morphisms of zero degree. By the sign trick of the previous lemma this category should be canonically identified with $\mathrm{Gr} - A^{\mathrm{op}}$. We next need to define a shift functor $?[1]\colon A - \mathrm{Gr} \longrightarrow A - \mathrm{Gr}$ which, viewed as a functor $\mathrm{Gr} - A^{\mathrm{op}} \longrightarrow \mathrm{Gr} - A^{\mathrm{op}}$, coincides with the shift for graded right modules (see the paragraph after the proof of Lemma 2.4). This forces the definition of the multiplication map $A \otimes M[1] \longrightarrow M[1]$ $((a, x) \rightsquigarrow a \cdot x)$. Indeed we will have $a \cdot x = (-1)^{|a||x|_{M[1]}}xa^o$. But the multiplication xa^o is the same in $M[1]$ and M, due the the definition of $?[1]$ for graded right A^{op}-modules. Then in M we have $xa^o = (-1)^{|a||x|_M}ax$, where ax is given by the multiplication $A \otimes M \longrightarrow M$. We then get that the multiplication map in $M[1]$ is given by

$$a \cdot x = (-1)^{|a||x|_{M[1]}}(-1)^{|a||x|_M} ax = (-1)^{|a|} ax,$$

where ax is the multiplication in M.

This readily gives a graded K-category $A - \mathrm{GR}$ whose objects are the objects of $A - \mathrm{Gr}$ and where the space of morphisms $\mathrm{Hom}_{A^{\mathrm{op}}}(M, N)$ between two objects M and N is graded in such a way that the n-th homogeneous component is $\mathrm{HOM}^n_{A^{\mathrm{op}}}(M, N) = \mathrm{Hom}_{A-\mathrm{Gr}}(M, N[n])$. Note that, viewing an element $f \in \mathrm{Hom}_{A-\mathrm{Gr}}(M, N[n])$, as a morphism $f \colon M \longrightarrow N$ of degree n, we have

$$f(ax) = ((-1)^{|a|})^n af(x) = (-1)^{|f||a|} af(x),$$

for all homogeneous elements $a \in A$ and $x \in M$. This is due to the fact that the multiplication map $A \otimes N[n] \longrightarrow N[n]$ acts as

$$a \cdot y = ((-1)^{|a|})^n ay = (-1)^{na} ay,$$

for all homogeneous elements $a \in A$ and $y \in N[n]$, where ay is the product in N.

Remark 4.2. We have an obvious forgetful functor $A - \mathrm{Gr} \longrightarrow A - \mathrm{Mod}$ acting as the identity on objects and morphisms. However, we have such a functor for the category $A - \mathrm{GR}$ only in case A is evenly graded (i.e. $A^{2k+1} = 0$, for all $k \in \mathbb{Z}$). In the general case one has a forgetful 'pseudo-functor' $A - \mathrm{GR} \longrightarrow A - \mathrm{Mod}$. It acts as the identity on objects, but takes $f \rightsquigarrow \hat{f}$, where $\hat{f}(x) = (-1)^{|f||x|} f(x)$, for all homogeneous elements $f \in \mathrm{HOM}_{A^{\mathrm{op}}}(M, N)$ and $x \in M$ (note that $\hat{f}$ is a morphism in $\mathrm{Mod} - A$). This assignment satisfies the equality $\widehat{g \circ f} = (-1)^{|f||g|} \hat{g} \circ \hat{f}$, for all homogeneous morphisms f, g in $A - \mathrm{GR}$. The ultimate reason for this disruption is that A^{op} is not the opposite algebra of A as an ungraded algebra.

We have essentially proved the following expected result.

Proposition 4.3. *Let A be a* dg *algebra with enough idempotents. The following data give a* dg *category $A - \mathrm{Dg}$ which is equivalent to the* dg *category $\mathrm{Dg} - A^{\mathrm{op}}$:*

- *The objects of $A - \mathrm{Dg}$ are the left* dg *A-modules (see Definition 2.5);*
- *The morphisms in $A - \mathrm{Dg}$ are defined as in the category $A - \mathrm{GR}$;*

- *For each pair (M,N) of objects, the differential* $d\colon \mathrm{Hom}_{A^{\mathrm{op}}}(M,N) \to \mathrm{Hom}_{A^{\mathrm{op}}}(M,N)$ *on* Hom *spaces is defined by the rule* $d(f) := d_N \circ f - (-1)^{|f|} f \circ d_M$.

Proof. Once we identify the objects of $A-\mathrm{Dg}$ with those of $\mathrm{Dg}-A^{\mathrm{op}}$, we only need to identify the spaces of morphisms in $A-\mathrm{GR}$ and $\mathrm{GR}-A^{\mathrm{op}}$ for the differential $d\colon \mathrm{Hom}_{A^{\mathrm{op}}}(M,N) \longrightarrow \mathrm{Hom}_{A^{\mathrm{op}}}(M,N)$ is just the restriction of that of the dg K-module $\mathrm{HOM}_K(M,N)$. Indeed, if $f\colon M \longrightarrow N$ is a homogeneous morphism in $A-\mathrm{GR}$, then

$$\begin{aligned} f(xa^o) &= (-1)^{|a||x|} f(ax) = (-1)^{|a||x|}(-1)^{|a||f|} af(x) \\ &= (-1)^{|a||f(x)|} af(x) = f(x)a^o. \end{aligned} \qquad \square$$

As in the case of right modules, the shift functor $?[1]\colon A-\mathrm{Gr} \longrightarrow A-\mathrm{Gr}$ extends to a functor $?[1]\colon A-\mathrm{Dg} \longrightarrow A-\mathrm{Dg}$ such that $d(f[1]) = -d(f)[1]$, for all homogeneous $f \in \mathrm{HOM}_{A^{\mathrm{op}}}(M,N)$, where d is the differential on Hom spaces.

5. Dg bimodules

Definition 5.1. Let A and B be dg algebras with enough idempotents. A *graded $A-B$-bimodule* is graded K-module M together with the following data:

1) A morphism of graded K-vector spaces $\mu_{left}\colon A \otimes M \longrightarrow M$ ($a \otimes x \rightsquigarrow ax$) making M into a graded left A-module,
2) and a morphism of graded K-vector spaces $\mu_{right}\colon M \otimes B \longrightarrow M$ ($x \otimes b \rightsquigarrow xb$) making M into a graded right B-module,
3) such that $(ax)b = a(xb)$, for all $(a,x,b) \in A \times M \times B$.

A *differential graded (*dg*) $A-B$-bimodule* is a pair (M, d_M) consisting of a graded $A-B$-bimodule M and a morphism $d_M\colon M \longrightarrow M$ in $\mathrm{GR}-K$ of degree $+1$, called the differential, such that $d_M \circ d_M = 0$ and

$$d_M(axb) = d_A(a)xb + (-1)^{|a|} a d_M(x) b + (-1)^{|a|+|x|} a x d_B(b),$$

for all homogeneous elements $a \in A$, $x \in M$ and $b \in B$. This latter formula is called Leibniz rule (for the the dg bimodule).

As in the case of right or left dg modules, we successively consider the category $A-\mathrm{Gr}-B$ of graded $A-B-$bimodules with morphisms of zero degree, the graded category $A-\mathrm{GR}-B$, where $\mathrm{HOM}_{A-B}(M,N) := \mathrm{Hom}_{A-\mathrm{GR}-B}(M,N)$ is the graded K-module with n-th homogeneous component $\mathrm{HOM}^n_{A-B}(M,N) = \mathrm{Hom}_{A-\mathrm{Gr}-B}(M,N[n])$, for each $n \in \mathbb{Z}$, and where $g \circ f := g[p] \circ f$ in case f and g are homogeneous elements of $\mathrm{HOM}_{A-B}(M,N)$, with $|f| = p$. Finally, $A - \mathrm{Dg} - B$ will denote the dg category whose objects are the dg $A-B-$bimodules with morphisms as in $A - \mathrm{GR} - B$.

Proposition 5.2. *Let A and B be* dg *algebras with enough idempotents. The following three terms 'are' synonymous:*

1) Dg $A-B$*-bimodule;*
2) *Right* dg $B \otimes A^{\mathrm{op}}$*-module;*
3) *Left* dg $A \otimes B^{\mathrm{op}}$*-module.*

In particular, there are equivalences of dg *categories*

$$\mathrm{Dg} - (B \otimes A^{\mathrm{op}}) \cong A - \mathrm{Dg} - B \cong (A \otimes B^{\mathrm{op}}) - \mathrm{Dg}.$$

Proof. We define the map $\Phi\colon A \otimes B^{\mathrm{op}} \longrightarrow (B \otimes A^{\mathrm{op}})^{\mathrm{op}}$ by the rule $\Phi(a \otimes b^o) = (-1)^{|a||b|}(b \otimes a^o)^o$, for all homogeneous elements $a \in A$ and $b \in B$. We will prove that Φ is an isomorphism of dg algebras, which will imply that left dg $A \otimes B^{\mathrm{op}}$-module is synonymous of right dg $B \otimes A^{\mathrm{op}}$-module using Proposition 4.3. We clearly have that Φ a morphism (of zero degree) of graded K-modules. Moreover, if $a_1, a_2 \in A$ and $b_1, b_2 \in B$ are homogeneous elements, then we have equalities

$$\begin{aligned}\Phi[(a_1 \otimes b_1^o) \cdot (a_2 \otimes b_2^o)] &= (-1)^{|b_1||a_2|}\Phi(a_1a_2 \otimes b_1^ob_2^o)\\ &= (-1)^{|b_1||a_2|+|b_1||b_2|}\Phi(a_1a_2 \otimes (b_2b_1)^o)\\ &= (-1)^{|b_1||a_2|+|b_1||b_2|+(|a_1|+|a_2|)(|b_1|+|b_2|)}[b_2b_1 \otimes (a_1a_2)^o]^o\end{aligned}$$

and

$$\begin{aligned}&\Phi(a_1 \otimes b_1^o) \cdot \Phi(a_2 \otimes b_2^o)\\ &\quad = (-1)^{|a_1||b_1|+|a_2||b_2|}(b_1 \otimes a_1^o)^o \cdot (b_2 \otimes a_2^o)^o\\ &\quad = (-1)^{|a_1||b_1|+|a_2||b_2|+(|b_1|+|a_1|)(|b_2|+|a_2|)}[(b_2 \otimes a_2^o)(b_1 \otimes a_1^o)]^o\\ &\quad = (-1)^{|a_1||b_1|+|a_2||b_2|+(|b_1|+|a_1|)(|b_2|+|a_2|)+|a_2||b_1|}[b_2b_1 \otimes a_2^oa_1^o]^o\\ &\quad = (-1)^{|a_1||b_1|+|a_2||b_2|+(|b_1|+|a_1|)(|b_2|+|a_2|)+|a_2||b_1|+|a_1||a_2|}[b_2b_1 \otimes (a_1a_2)^o]^o.\end{aligned}$$

We compare the signs of the two expressions.

$$\begin{aligned}&|a_2||b_1| + |b_1||b_2| + |a_1||b_1| + |a_1||b_2| + |a_2||b_1| + |a_2||b_2|\\ &\quad = |b_1||b_2| + |a_1||b_1| + |a_1||b_2| + |a_2||b_2|\\ &\quad = |a_1||b_1| + |a_2||b_2| + |b_1||b_2| + |b_1||a_2| + |a_1||b_2| + |a_1||a_2|\\ &\qquad + |a_2||b_1| + |a_1||a_2|\end{aligned}$$

We conclude that Φ is a (clearly bijective) homomorphism of graded algebras. In order to prove that it is actually an isomorphism of dg algebras, we need to check that it is compatible with the differentials. Indeed, if $a \in A$ and $b \in B$ are homogeneous elements, then we have

$$\begin{aligned}
[d \circ \Phi](a \otimes b^o) &= (-1)^{|a||b|} d[(b \otimes a^o)^o] \\
&= (-1)^{|a||b|}[(d(b) \otimes a^o)^o + (-1)^{|b|}(b \otimes d(a)^o)^o] \\
&= (-1)^{|a||b|}(d(b) \otimes a^o)^o + (-1)^{(|a|+1)|b|}(b \otimes d(a)^o)^o \\
&= (-1)^{(|a|+1)|b|}(b \otimes d(a)^o)^o + (-1)^{|a|+|a|(|b|+1)}(d(b) \otimes a^o)^o \\
&= \Phi[d(a) \otimes b^o + (-1)^{|a|} a \otimes d(b)^o] = [\Phi \circ d](a \otimes b^o)
\end{aligned}$$

which shows that $\Phi \circ d = d \circ \Phi$ and, hence, that Φ is an isomorphism of dg algebras.

If M is a dg $A-B$-bimodule, then we define a multiplication map $M \otimes (B \otimes A^{\mathrm{op}}) \longrightarrow M$ which takes $x \otimes b \otimes a^o \rightsquigarrow x(b \otimes a^o) := (-1)^{(|x|+|b|)|a|} axb$, whenever $x \in M$, $b \in B$ and $a \in A$ are homogeneous elements. We claim that this map endows M with a structure of graded right $B \otimes A^{\mathrm{op}}$-module. For this we just need to check the equality $x[(b_1 \otimes a_1^o)(b_2 \otimes a_2^o)] = [x(b_1 \otimes a_1^o)](b_2 \otimes a_2^o)$, for all homogeneous elements $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $x \in M$. Indeed we have:

$$\begin{aligned}
x[(b_1 \otimes a_1^o)(b_2 \otimes a_2^o)] &= (-1)^{|a_1||b_2|} x[b_1 b_2 \otimes a_1^o a_2^o] \\
&= (-1)^{|a_1||b_2|+|a_1||a_2|} x[b_1 b_2 \otimes (a_2 a_1)^o] \\
&= (-1)^{|a_1||b_2|+|a_1||a_2|+(|x|+|b_1|+|b_2|)(|a_1|+|a_2|)} a_2 a_1 x b_1 b_2 \\
&= (-1)^{(|x|+|b_1|)|a_1|+(|a_1|+|x|+|b_1|+|b_2|)|a_2|} a_2 a_1 x b_1 b_2 \\
&= (-1)^{(|x|+|b_1|)|a_1|} a_1 x b_1 (b_2 \otimes a_2^o) = [x(b_1 \otimes a_1^o)](b_2 \otimes a_2^o)
\end{aligned}$$

We conclude that the above mentioned multiplication map endows M with a structure of graded right $B \otimes A^{\mathrm{op}}$-module. We finally check that the differential $d_M \colon M \longrightarrow M$ satisfies Leibniz rule $d_M[x(b \otimes a^o)] = d_M(x)(b \otimes a^o) + (-1)^{|x|} x d(b \otimes a^o)$, for all homogeneous elements $x \in M$, $b \in B$ and $a \in A$. Indeed we have an equality

$$\begin{aligned}
& d_M(x)(b \otimes a^o) + (-1)^{|x|} x d(b \otimes a^o) \\
&\quad = (-1)^{(|x|+1+|b|)|a|} a d_M(x) b + (-1)^{|x|} x[d(b) \otimes a^o + (-1)^{|b|} b \otimes d(a)^o] \\
&\quad = (-1)^{(|x|+1+|b|)|a|} a d_M(x) b + (-1)^{|x|}(-1)^{(|x|+|b|+1)|a|} a x d(b) \\
&\qquad + (-1)^{|x|+|b|}(-1)^{(|x|+|b|)(|a|+1)} d(a) x b
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{(|x|+|b|)|a|}[(-1)^{|a|}ad_M(x)b + (-1)^{|x|+|a|}axd(b) \\
&\quad + (-1)^{2(|x|+|b|)}d(a)xb] \\
&= (-1)^{(|x|+|b|)|a|}d(axb) = d_M[x(b \otimes a^o)]
\end{aligned}$$

where the last equation holds by the definition of right $B \otimes A^{\mathrm{op}}$-module structure on M. Then (M, d_M) is a right dg $B \otimes A^{\mathrm{op}}$-module.

Obviously, one can reverse the arguments step by step, so that if X is a right dg $B \otimes A^{\mathrm{op}}$-module, then it is also a dg $A - B$-bimodule when taken with multiplication $axb = (-1)^{(|x|+|b|)|a|}x(b \otimes a^o)$, for all homogeneous elements $a \in A$, $x \in M$ and $b \in B$. □

Remark 5.3. We emphasize the structures of right dg $B \otimes A^{\mathrm{op}}$-module and left dg $A \otimes B^{\mathrm{op}}$-module coming from Proposition 5.2 and its proof. If M is a dg $A - B$-bimodule and $a \in A$, $b \in B$ and $x \in M$ are homogeneous elements, then we have:
1) $x(b \otimes a^o) = (-1)^{(|x|+|b|)|a|}axb$;
2) $(a \otimes b^o)x = (-1)^{|b||x|}axb$.
Indeed the first equality appears in the proof of Proposition 5.2 and then, by Proposition 4.3, we have a structure of left dg $(B \otimes A^{\mathrm{op}})^{\mathrm{op}}$-module on M. Using then the isomorphism $\Phi\colon A \otimes B^{\mathrm{op}} \longrightarrow (B \otimes A^{\mathrm{op}})^{\mathrm{op}}$ from the proof of , we make M into a left dg $A \otimes B^{\mathrm{op}}$-module. The reader can check that this module structure is given by equality 2.

Example 5.4. If A is a dg algebra with enough idempotents, then it is a dg $A - A$-bimodule with its canonical multiplication and differential. We will call it the *regular* dg *bimodule.*

6. Homotopy category and derived category

As in the case of small dg categories, given a dg algebra with enough idempotents A, the 0-cycle category $Z^0(\mathrm{Dg} - A)$, denoted by $\mathcal{C}(A)$ in the sequel, has two structures to take into account. It is a bicomplete abelian category, where the exact sequences are those sequences $0 \to L \longrightarrow M \longrightarrow N \to 0$ of morphisms in $\mathcal{C}(A)$ which are exact as sequences in $\mathrm{Gr} - A$. But, even more relevant to us, it has a Quillen exact structure where the conflations (=admissible short exact sequences) are the short exact sequences which split in $\mathrm{Gr} - A$ (see [2] and [11] for the axioms and details about exact categories). It is called the *semi-split exact structure.* We are now going to give an explicit description of the projective (=injective) objects for this exact structure.

Lemma 6.1. *Any conflation in $\mathcal{C}(A)$ is isomorphic to one whose underlying exact sequence in* $\mathrm{Gr}-A$ *is* $0 \to L \xrightarrow{\binom{1}{0}} L \oplus N \xrightarrow{(0\ 1)} N \to 0$, *where the differential of $L \oplus N$ is of the form $\delta = \begin{pmatrix} d_L & s \\ 0 & d_N \end{pmatrix}$, for some morphism $s\colon N \longrightarrow L$ of degree* 1 *in* $\mathrm{GR}-A$ *such that $d_L \circ s + s \circ d_N = 0$.*

Proof. By the definition of conflations, the underlying exact sequence in $\mathrm{Gr}-A$ of such a conflation is always as indicated, where L and N are right dg A-modules. We initially put $\delta = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ $(\binom{x}{y} \rightsquigarrow \begin{pmatrix} d_{11}(x)+d_{12}(y) \\ d_{21}(x)+d_{22}(y) \end{pmatrix})$. Since $L \xrightarrow{\binom{1}{0}} L \oplus N$ should be an element of $Z^0(\mathrm{Hom}_A(L, L \oplus N))$, we should have $\delta \circ \binom{1}{0} - \binom{1}{0} \circ d_L = 0$. From this equality we get that $d_{21} = 0$ and $d_{11} = d_L$. Using that $L \oplus N \xrightarrow{(0\ 1)} N$ is in $Z^0(\mathrm{Hom}_A(L \oplus N, N))$ we then get $d_N \circ (0\ 1) - (0\ 1) \circ \delta$, from which we get that $d_{22} = d_N$. Finally, from the equality $\delta \circ \delta = 0$ we get that $d_L \circ s + s \circ d_N = 0$, with $s = d_{12}$. □

Remark 6.2. If $f\colon M \longrightarrow N$ is a morphism in $\mathcal{C}(A)$, then one can consider the split exact sequence $0 \to N \xrightarrow{\binom{1}{0}} N \oplus M[1] \xrightarrow{(0\ 1)} M[1] \to 0$ in $\mathrm{Gr}-A$. Viewing f as morphism of degree $+1$ from $M[1]$ to N, Lemma 6.1 makes $N \oplus M[1]$ into a right dg A-module with differential $\delta = \begin{pmatrix} d_N & f \\ 0 & d_{M[1]} \end{pmatrix}$. This dg A-module is known as the *cone of f* and will be denoted by $C(f)$ in the sequel. Note that we have an associated conflation $0 \to N \longrightarrow C(f) \longrightarrow M[1] \to 0$ in $\mathcal{C}(A)$.

Proposition 6.3. *For a right* dg *A-module P, the following assertions are equivalent:*

1) *P is projective with respect to the semi-split exact structure;*
2) *P is injective with respect to the semi-split exact structure;*
3) *P is isomorphic to a direct summand of a cone $C(1_M)$, for a right* dg *A-module M.*

Such a P is acyclic. In particular $\mathcal{C}(A)$ is a Frobenius exact category with the semi-split structure.

Proof. The acyclic condition $C(1_M)$ is well-known. It is then enough to check that $C(1_M)$ is projective and injective, for each right dg A-module M. Once this is proved, if P is projective (resp. injective) object, then the canonical conflation $0 \to P[-1] \longrightarrow C(1_{P[-1]}) \to P \to 0$ (resp. $0 \to P \longrightarrow C(1_P) \to P[1] \to 0$) must split in $\mathcal{C}(A)$, and the rest of the proof will be trivial.

By Lemma 6.1, any deflation (=admissible epimorphism) in $\mathcal{C}(A)$ can be identified with $\begin{pmatrix} 0 & 1 \end{pmatrix} : L \oplus N \longrightarrow N$, where $L \oplus N$ is made into a right dg A-module with the differential $\delta = \begin{pmatrix} d_L & s \\ 0 & d_N \end{pmatrix}$ described there. If now $f\colon C(1_M) \longrightarrow N$ is any morphism in $\mathcal{C}(A)$, then, viewed in $\mathrm{Gr} - A$, it is a morphism $f = \begin{pmatrix} u & v \end{pmatrix} : M \oplus M[1] \longrightarrow N$ such that $\begin{pmatrix} u & v \end{pmatrix} \circ \begin{pmatrix} d_M & 1_{M[1]} \\ 0 & d_{M[1]} \end{pmatrix} = d_N \circ \begin{pmatrix} u & v \end{pmatrix}$. The second component of this equality gives that $u + v \circ d_{M[1]} = d_N \circ v$, which we express as $u = \hat{d}(v)$, where v is viewed as a morphism of degree -1 in $\mathrm{GR} - A$ and $\hat{d}$ is the internal differential $\hat{d}\colon \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(M,N)$ in the dg category $\mathrm{Dg} - A$. We will now check that the morphism in $\mathrm{Gr} - A$ given in matrix form as $\alpha = \begin{pmatrix} s \circ v & 0 \\ \hat{d}(v) & v \end{pmatrix} : M \oplus M[1] \longrightarrow L \oplus N$ is a morphism in $\mathcal{C}(A)$, $\alpha\colon C(1_M) \longrightarrow (L \oplus N, \delta)$, such that $\begin{pmatrix} 0 & 1 \end{pmatrix} \circ \alpha = \begin{pmatrix} u & v \end{pmatrix} = f$. In order to see that α is a morphism in $\mathcal{C}(A)$, we just need to check the equality

$$\begin{pmatrix} s \circ v & 0 \\ \hat{d}(v) & v \end{pmatrix} \circ \begin{pmatrix} d_M & 1_{M[1]} \\ 0 & d_{M[1]} \end{pmatrix} = \begin{pmatrix} d_L & s \\ 0 & d_N \end{pmatrix} \circ \begin{pmatrix} s \circ v & 0 \\ \hat{d}(v) & v \end{pmatrix}.$$

We check the equality entry by entry:

(11) We need to check that $s \circ v \circ d_M = d_L \circ s \circ v + s \circ \hat{d}(v)$. But we have $s \circ \hat{d}(v) = s \circ (d_N \circ v - (-1)^{|v|} v \circ d_M) = s \circ d_N \circ v + s \circ v \circ d_M$ and, using the fact that $s \circ d_N = -d_L \circ s$, the desired equality follows.

(12) The equality $s \circ v = s \circ v$ is clear.

(21) We need to check that $\hat{d}(v) \circ d_M = d_N \circ \hat{d}(v)$. But this is an immediate consequence of the fact that $0 = \hat{d}(\hat{d}(v)) = d_N \circ \hat{d}(v) - (-1)^{|\hat{d}(v)|} \hat{d}(v) \circ d_M = d_N \circ \hat{d}(v) - \hat{d}(v) \circ d_M$.

(22) We need to check that $\hat{d}(v) + v \circ d_{M[1]} = d_N \circ v$. This is a direct consequence of the equality $\hat{d}(v) = d_N \circ v - (-1)^{|v|} v \circ d_M = d_N \circ v + v \circ d_M$ and the fact that $d_{M[1]} = -d_M$.

Once we know that α is a morphism in $\mathcal{C}(A)$, it is clear that $\begin{pmatrix} 0 & 1 \end{pmatrix} \circ \alpha = \begin{pmatrix} \hat{d}(v) & v \end{pmatrix} = f$. Therefore $C(1_M)$ is projective with respect to the semi-split exact structure of $\mathcal{C}(A)$. That it is also injective can be proved using a dual argument. $\square$

Definition 6.4. A right dg A-module P is called *contractible* when it satisfies any one of the equivalent conditions of Proposition 6.3.

Corollary 6.5. *When $\mathcal{C}(A)$ is considered with its semi-split (Frobenius) exact structure, its stable category $\underline{\mathcal{C}(A)} =: \mathcal{H}(A)$ is a triangulated category with arbitrary (set-indexed) coproducts, where the suspension functor is induced by the shift functor* $?[1]$ *of* $\mathrm{Dg} - A$ *and where the triangles are,* *up to isomorphism, the images by the quotient functor* $\mathcal{C}(A) \longrightarrow \mathcal{H}(A)$ *of conflations in* $\mathcal{C}(A)$*. Moreover,* $\mathcal{H}(A)$ *is equivalent to the* 0*-homology category* $H^0(\mathrm{Dg} - A)$*.*

Proof. It is a standard fact (see [7, Section I.2]) that the stable category $\underline{\mathcal{E}}$ of a Frobenius exact category $\mathcal{E}$ is a triangulated category whose suspension functor is the cosyzygy functor and whose triangles are, up to isomorphism, the images by the projection functor $\mathcal{E} \longrightarrow \underline{\mathcal{E}}$ of conflations in $\mathcal{E}$. But, for each object M of $\mathcal{C}(A)$, we have a conflation $0 \to M \longrightarrow C(1_M) \longrightarrow M[1] \to 0$, where $C(1_M)$ is contractible. It follows that the shift functor $?[1]\colon \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - A$ (or $?[1]\colon \mathcal{C}(A) \longrightarrow \mathcal{C}(A)$) induces the suspension functor of $\underline{\mathcal{C}(A)} =: \mathcal{H}(A)$.

For the last assertion, we need to prove that a morphism $f\colon M \longrightarrow N$ in $\mathcal{C}(A)$ factors through a contractible dg A-module if, and only if, it is in the image of the internal differential $d\colon \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(M,N)$. To avoid confusions, we will denote by $\hat{d}$ this internal differential. Indeed, the morphism f factors through a contractible dg A-module if, and only if, it factors through the canonical deflation (= admissible epimorphism) $C(1_{N[-1]}) \xrightarrow{(0\ 1)} N$. This happens if, and only if, there is a morphism $\sigma\colon M \longrightarrow N$ of degree -1 in $\mathrm{GR} - A$ such that $\left(\begin{smallmatrix} \sigma \\ f \end{smallmatrix}\right)\colon M \longrightarrow C(1_{N[-1]}) \equiv N[-1] \oplus N$ is a morphism in $\mathcal{C}(A)$. This in turn is equivalent to the existence of such a σ such that the matrix equality $\left(\begin{smallmatrix} d_{N[-1]} & 1_N \\ 0 & d_N \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} \sigma \\ f \end{smallmatrix}\right) = \left(\begin{smallmatrix} \sigma \\ f \end{smallmatrix}\right) \circ d_M$ holds. That is, f factors through a projective object if, and only if, there is morphism $\sigma\colon M \longrightarrow N$ of degree -1 in $\mathrm{GR} - A$ such that $d_{N[-1]} \circ \sigma + f = \sigma \circ d_M$, which is equivalent to saying that $f = \hat{d}(\sigma)$. □

Definition 6.6. The category $\mathcal{H}(A)$ of last corollary is called the *homotopy category of* A. A morphism $f\colon M \longrightarrow N$ in $\mathcal{C}(A)$ is called *null-homotopic* when it is a 0-boundary $f \in B^0(\mathrm{HOM}_A(M,N))$ (i.e. $f = d_N \circ \sigma + \sigma \circ d_M$, for some $\sigma \in \mathrm{HOM}_A^{-1}(M,N)$). This is equivalent to say that f is mapped onto zero by the projection functor $\mathcal{C}(A) \longrightarrow \mathcal{H}(A)$.

Note that if $f\colon M \longrightarrow N$ is a morphism in $\mathcal{C}(A)$ then we get induced morphisms of K-modules $Z^k(f) := f_{|Z^k(M)}\colon Z^k(M) \longrightarrow Z^k(N)$ and $B^k(f) := f_{|B^k(M)}\colon B^k(M) \longrightarrow B^k(N)$, for all $k \in \mathbb{Z}$. They give rise to functors $Z^k, B^k\colon \mathcal{C}(A) \longrightarrow \mathrm{Mod} - K$, for all $k \in \mathbb{Z}$, and, gathering all together, to functors $Z^*, B^*\colon \mathcal{C}(A) \longrightarrow \mathrm{Gr} - K$ given by $Z^*(M) = \oplus_{k\in\mathbb{Z}} Z^k(M)$ (resp. $B^*(M) = \oplus_{k\in\mathbb{Z}} B^k(M)$) and $Z^*(f) = \oplus_{k\in\mathbb{Z}} Z^k(f)$ (resp. $B^*(f) = \oplus_{k\in\mathbb{Z}} B^k(f)$). These functors are compatible with the inclusions $B^k(?) \hookrightarrow Z^k(?)$ and, hence, they give rise to functors $H^k\colon \mathcal{C}(A) \longrightarrow \mathrm{Mod}-K$ ($k \in \mathbb{Z}$) and $H^*\colon \mathcal{C}(A) \longrightarrow \mathrm{Gr}-K$, where $H^k(M) = Z^k(M)/B^k(M)$ and $H^*(M) = \oplus_{k\in\mathbb{Z}} H^k(M)$. We call H^k the *k-th homology functor* and, without mentioning the degree, we call H^* the *homology functor*. If f is null-homotopic, and hence $f = d_N \circ \sigma + \sigma \circ d_M$, for some morphism $\sigma\colon M \longrightarrow N[-1]$ in $\mathrm{Gr}-A$, then $\mathrm{Im}(Z^k(f)) \subseteq B^k(N)$, for all $k \in \mathbb{N}$. This implies that the functor H^k vanishes on null-homotopic morphisms, for all $k \in \mathbb{Z}$, which implies that we have a uniquely determined functor, still denoted and called the same, $H^k\colon \mathcal{H}(A) \longrightarrow \mathrm{Mod}-K$ such that the composition $\mathcal{C}(A) \xrightarrow{proj} \mathcal{H}(A) \xrightarrow{H^k} \mathrm{Mod}-K$ is the k-th homology functor. We also get a corresponding functor $H^*\colon \mathcal{H}(A) \longrightarrow \mathrm{Mod}-A$.

Definition 6.7. A *quasi-isomorphism* of dg modules is a morphism $f\colon M \longrightarrow N$ in $\mathcal{C}(A) = Z^0(\mathrm{Dg}-A)$ such that $H^*(f)$ is an isomorphism in $\mathrm{Gr}-K$. This is equivalent to saying that its cone $C(f)$ is an acyclic dg A-module (see Remark 6.2).

As in the case of small dg categories and their dg modules, the class of quasi-isomorphisms is a multiplicative system in $\mathcal{H}(A)$ compatible with the triangulation, in the sense of Verdier (see [23, Section II.2], where we refer the reader for the concepts and terminology concerning localization of triangulated categories used in this paper).

Definition 6.8. The localization of $\mathcal{H}(A)$ with respect to the class of quasi-isomorphisms, denoted by $\mathcal{D}(A)$, is called the *derived category of A*. It is a triangulated category with arbitrary coproducts and the canonical functor $q\colon \mathcal{H}(A) \longrightarrow \mathcal{D}(A)$ is a triangulated functor. The shift in $\mathcal{D}(A)$ is induced by that of $\mathcal{H}(A)$ and the triangles in $\mathcal{D}(A)$ are, up to isomorphism, the images by q of triangles in $\mathcal{H}(A)$.

Note that, by the universal property of the localized category, since the functor $H^*\colon \mathcal{H}(A) \longrightarrow \mathrm{Gr}-K$ takes quasi-isomorphisms to isomorphism, there is a uniquely determined functor, still denoted and called the same, $H^*\colon \mathcal{D}(A) \longrightarrow \mathrm{Gr}-K$ such that the composition $\mathcal{H}(A) \xrightarrow{q_A} \mathcal{D}(A) \xrightarrow{H^*} \mathrm{Gr}-K$ is the homology functor.

Remark 6.9. What we have done for A can be done also for A^{op} and for $B \otimes A^{\mathrm{op}}$, where B is another algebra with enough idempotents, obtaining the categories $\mathcal{C}(A^{\mathrm{op}})$, $\mathcal{H}(A^{\mathrm{op}})$ and $\mathcal{D}(A^{\mathrm{op}})$ (resp. $\mathcal{C}(B\otimes A^{\mathrm{op}})$, $\mathcal{H}(B\otimes A^{\mathrm{op}})$ and $\mathcal{D}(B \otimes A^{\mathrm{op}})$). Due to the equivalences of dg categories $A-\mathrm{Dg} \cong \mathrm{Dg}-A^{\mathrm{op}}$ (see Proposition 4.3) and $A-\mathrm{Dg}-B \cong \mathrm{Dg}(B\otimes A^{\mathrm{op}})$ (see

Proposition 5.2), we will look at $\mathcal{C}(A^{\mathrm{op}})$ (resp. $\mathcal{H}(A^{\mathrm{op}})$) and $\mathcal{C}(B \otimes A^{\mathrm{op}})$ (resp. $\mathcal{H}(B \otimes A^{\mathrm{op}})$) as the 0-cycle categories $Z^0(A-\mathrm{Dg})$ and $Z^0(A-\mathrm{Dg}-B)$ (resp. 0-homology categories $H^0(A-\mathrm{Dg})$ and $H^0(A-\mathrm{Dg}-B)$). In particular, the objects of $\mathcal{D}(A^{\mathrm{op}})$ are considered to be left dg A-modules and those of $\mathcal{D}(B \otimes A^{\mathrm{op}})$ as dg $A-B-$bimodules.

As in the case of the derived category of an abelian category (see [23]), we have:

Proposition 6.10. *Let A be a* dg *algebra with enough idempotents. The canonical composition functor $\mathcal{C}(A) \xrightarrow{p} \mathcal{H}(A) \xrightarrow{q} \mathcal{D}(A)$ takes short exact sequences in $\mathcal{C}(A)$ (for the abelian structure) to triangles in $\mathcal{D}(A)$.*

Proof. Let $0 \to L \xrightarrow{u} M \xrightarrow{v} N \to 0$ be a short exact sequence in $\mathcal{C}(A)$ and fix an inflation (=admissible monomorphism with respect to the semi-split exact structure of $\mathcal{C}(A)$) $j\colon L \longrightarrow I$, where I is a contractible dg A-module. If X is the lower right corner of the pushout of u and j, then we get the following commutative diagram whose rows are exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\binom{u}{j}"] \arrow[d, equal] & M \oplus I \arrow[r] \arrow[d, "(1{,}0)"] & X \arrow[r] \arrow[d, "\epsilon"] & 0 \\
0 \arrow[r] & L \arrow[r, "u"] & M \arrow[r, "v"] & N \arrow[r] & 0
\end{tikzcd}$$

It then follows that the right square of this diagram is bicartesian and, as a consequence, that $I = \mathrm{Ker}[\begin{pmatrix} 1 & 0 \end{pmatrix}] \cong \mathrm{Ker}(\epsilon)$. Since I is acyclic we get that ϵ is a quasi-isomorphism and, hence, the three vertical arrows of last diagram are quasi-isomorphisms. Then the images of the 'rows' of this diagram by the canonical functor $q \circ p\colon \mathcal{C}(A) \longrightarrow \mathcal{D}(A)$ are isomorphic. But the upper row of the diagram is a conflation (see [2, Proposition 2.12]), whose image by $q \circ p$ 'is' then a triangle in $\mathcal{D}(A)$(see Definition 6.8). □

Recall that if $\mathcal{D}$ is a triangulated category with (set-indexed) coproducts, then an object C of $\mathcal{D}$ is called *compact* when the functor $\mathrm{Hom}_{\mathcal{D}}(C, ?)\colon \mathcal{D} \longrightarrow Ab$ preserves coproducts. The category $\mathcal{D}$ is said to be *compactly generated* when there is a set $\mathcal{S}$ of compact objects such that $\bigcap_{n \in \mathbb{Z}, S \in \mathcal{S}} \mathrm{Ker}(\mathrm{Hom}_{\mathcal{D}}(S, ?[n]) = 0$, and $\mathcal{D}$ is said to be *algebraic* when it is triangle equivalent to the stable category of some Frobenius exact category (see [7, Section I.2]). As an immediate consequence of [10, Theorem 4.3], our Theorem 3.1 and its proof, we get:

Corollary 6.11. *For a triangulated category $\mathcal{D}$, the following assertions are equivalent:*

1) *$\mathcal{D}$ is compactly generated and algebraic.*
2) *$\mathcal{D}$ is triangle equivalent to $\mathcal{D}(A)$, for some* dg *algebra with enough idempotents A.*

7. Derived functors

We call the attention of the reader on the following fact, that we shall freely use.

Remark 7.1. If in Definition 1.4 one has $\mathcal{A} = \mathrm{Dg}-A$ and $\mathcal{B} = \mathrm{Dg}-B$, for some dg algebras with enough idempotents A and B, then the homological condition translate into the fact that τ_M commutes with the differentials. That is, that $d_{G(M)} \circ \tau_M = \tau_M \circ d_{F(M)}$, for each right dg A-module M.

We start with the following observation

Lemma 7.2. *Let A and B be* dg *algebras with enough idempotents and $F\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ be a* dg *functor. Then there is a natural isomorphism $\rho_{F,?}\colon F \circ (?[1]) \cong (?[1]) \circ F$ which is natural on F. That is, such that if $\tau\colon F \longrightarrow G$ is a natural transformation of* dg *functors, then the following diagram in* $\mathrm{Dg}-B$ *is commutative, for each right* dg *A-module M:*

$$\begin{tikzcd} F(M[1]) \arrow[r, "\xi_{F,M}"] \arrow[d, "\tau_{M[1]}"'] & F(M)[1] \arrow[d, "\tau_M[1]"] \\ G(M[1]) \arrow[r, "\xi_{G,M}"] & G(M)[1] \end{tikzcd}$$

Proof. We consider the morphism

$$1_M^- \in \mathrm{HOM}_A^{-1}(M, M[1]) = \mathrm{Hom}_{\mathrm{Gr}-A}(M, M)$$

given by $1_M^- = 1_M$. Then

$$F(1_M^-) \in \mathrm{HOM}_B^{-1}(F(M), F(M[1])) = \mathrm{HOM}_{\mathrm{Gr}-B}(F(M), F(M[1])[-1]),$$

and hence $F(1_M^-)[1] \in \mathrm{Hom}_{\mathrm{Gr}-B}(F(M)[1], F(M[1]))$.

Similarly, we have $1_M^+ \in \mathrm{HOM}_A^1(M[1], M) = \mathrm{HOM}_{\mathrm{Gr}-A}(M[1], M[1])$ given by $1_M^+ = 1_{M[1]}$, so that

$$F(1_M^+) \in \mathrm{HOM}_B^1(F(M[1]), F(M)) = \mathrm{Hom}_{\mathrm{Gr}-B}(F(M[1]), F(M)[1]).$$

We then get that

$$1_{F(M[1])} = F(1_{M[1]}) = F(1^-_M \circ 1^+_M) = F(1^-_M) \circ F(1^+_M).$$

By the definition of the composition of morphisms in $\mathrm{GR}-B$, we then get that $1_{F(M[1])}$ is equal to the composition $F(M[1]) \stackrel{F(1^+_M)}{\longrightarrow} F(M)[1] \stackrel{F(1^-_M)[1]}{\longrightarrow} F(M[1])$. On the other hand, we have $F(1^+_M) \circ (F(1^-_M)[1]) = (F(1^+_M)[-1] \circ F(1^-_M))[1]$. But due to the definition of the composition of morphisms in $\mathrm{GR}-B$, we have

$$F(1^+_M)[-1] \circ F(1^-_M) = F(1^+_M) \circ F(1^-_M)) = F(1^+_M \circ 1^-_M) = F(1_M) = 1_{F(M)}.$$

We then have

$$F(1^+_M) \circ (F(1^-_M)[1]) = 1_{F(M)}[1] = 1_{F(M)[1]},$$

which shows that $F(1^+_M)$ and $(F(1^-_M)[1]$ are mutually inverse isomorphisms.

We define $\rho_{F,M} = F(1^+_M)\colon F(M[1]) \longrightarrow F(M)[1]$, for each right dg A-module M. Note that if $\alpha\colon M \longrightarrow N$ is any homogeneous morphism in $\mathrm{Dg}-A$, then the compositions $M[1] \stackrel{\alpha[1]}{\longrightarrow} N[1] \stackrel{1^+_N}{\longrightarrow} N$ and $M[1] \stackrel{1^+_M}{\longrightarrow} M \stackrel{\alpha}{\longrightarrow} N$ coincide in $\mathrm{GR}-A$. It follows that

$$\begin{aligned} F(\alpha)[1] \circ \rho_{F,M} &= F(\alpha) \circ F(1^+_M) = F(\alpha \circ 1^+_M) = F(1^+_N \circ \alpha[1]) \\ &= F(1^+_N) \circ F(\alpha[1]) = \rho_{F,N} \circ F(\alpha[1]), \end{aligned}$$

when we interpret $\alpha[1]$ as an element of $\mathrm{HOM}^{-1}_A(M[1], N)$, using the definition of the composition of morphisms in $\mathrm{GR}-A$ and $\mathrm{GR}-B$ and the functoriality of F. It follows that $\rho = (\rho_{F,N})_{N \in \mathrm{Dg}-A}$ defines a natural isomorphism $F \circ (?[1]) \cong (?[1]) \circ F$.

It remains to check the commutativity of the diagram in the statement, whenever $\tau\colon F \longrightarrow G$ is a natural transformation of dg functors. But we have $\tau_M[1] \circ \rho_{F,M} = \tau_M[1] \circ F(1^+_M) = \tau_M \circ F(1^+_M)$, when viewing $F(1^+_M)$ as an element of $\mathrm{HOM}^1_B(M[1], M)$. The naturality of τ then gives that $\tau_M[1] \circ \rho_{F,M} = G(1^+_M) \circ \tau_{M[1]} = \rho_{G,M} \circ \tau_{M[1]}$, as desired. □

Proposition 7.3. *Let A be a* dg *algebra with enough idempotents. The canonical functor $q = q_A\colon \mathcal{H}(A) \longrightarrow \mathcal{D}(A)$ has a left adjoint and a right adjoint, both of them triangulated and fully faithful.*

Proof. Keller proved (see [10, Theorems 3.1 and 3.2]) that, for each object M of $\mathcal{H}(A)$, we have quasi-isomorphisms $\pi = \pi_M\colon P_M \longrightarrow M$ and $\iota = \iota_M\colon M \longrightarrow I_M$ in $\mathcal{H}(A)$, where P_M and I_M are right dg A-modules such that the functors $\mathrm{Hom}_{\mathcal{H}(A)}(P_M, ?)$ and $\mathrm{Hom}_{\mathcal{H}(A)}(?, I_M)$ vanish on acyclic complexes. By a standard argument, one sees that this last property implies that the maps $\mathrm{Hom}_{\mathcal{H}(A)}(P_M, N) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(P_M, N)$ and $\mathrm{Hom}_{\mathcal{H}(A)}(N, I_M) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(N, I_M)$ defined by q are both bijective, for any object N of $\mathcal{H}(A)$.

We now define the left adjoint $\Pi\colon \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ as follows. For each right dg A-module M, we fix a quasi-isomorphism $\pi_M\colon P_M \longrightarrow M$ as above, and define $\Pi(M) = P_M$ on objects. If now $f\colon M \longrightarrow N$ is a morphism in $\mathcal{D}(A)$, then $\pi_N^{-1} \circ f \circ \pi_M \in \mathrm{Hom}_{\mathcal{D}(A)}(P_M, P_N)$. By the last paragraph, we then get a unique morphism $\alpha\colon P_M \longrightarrow P_N$ in $\mathcal{H}(A)$ such that $q(\alpha) = \pi_N^{-1} \circ f \circ \pi_M$. We define $\Pi(f) = \alpha$. It is routine to check that in this way we have defined a functor $\Pi\colon \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$. Moreover the map $\mathrm{Hom}_{\mathcal{H}(A)}(\Pi(M), N) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(M, N) = \mathrm{Hom}_{\mathcal{D}(A)}(M, q(N))$ taking $\beta \rightsquigarrow q(\beta) \circ \pi_M^{-1}$ is bijective and natural on both arguments. Then Π is left adjoint to q. The co-unit of this adjunction is just $\pi\colon \Pi \circ q \longrightarrow 1_{\mathcal{H}(A)}$, where $\pi_M\colon (\Pi \circ q)(M) = P_M \longrightarrow M$ is the quasi-isomorphism fixed above. The unit $\lambda\colon 1_{\mathcal{D}(A)} \longrightarrow q \circ \Pi$ is given by $\lambda_M = \pi_M^{-1}\colon M \longrightarrow (q \circ \Pi)(M) = P_M$. It follows that λ is a natural isomorphism, which implies that Π is fully faithful (see [8, Proposition II.7.5]). On the other hand, it is well-known that the left adjoint of a triangulated functor is also triangulated (see [16, Lemma 5.3.6]).

The existence of a right adjoint $\Upsilon\colon \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ acting on objects as $\Upsilon(M) = I_M$ is proved by an argument dual to the one in the previous paragraphs. $\square$

Definition 7.4. A right dg A-module P (resp. I) is called *homotopically projective (resp. injective)* if the functor $\mathrm{Hom}_{\mathcal{H}(A)}(P, ?)\colon \mathcal{H}(A) \longrightarrow \mathrm{Mod} - K$ (resp. $\mathrm{Hom}_{\mathcal{H}(A)}(?, I) = \mathcal{H}(A)^{\mathrm{op}} \longrightarrow \mathrm{Mod} - K$) vanishes on acyclic complexes. By the proof of Proposition 7.3, if Π and Υ are the left and right adjoints of $q\colon \mathcal{H}(A) \longrightarrow \mathcal{D}(A)$, respectively, then the essential image $Im(\Psi)$ (resp. $Im(\Upsilon)$) consists of homotopically projective (resp. injective) objects. We will call Π and Υ the *homotopically projective resolution functor* and *homotopically injective resolution functor*, respectively. Given a right dg A-module M, a *homotopically projective resolution* (resp. *homotopically injective resolution*) of M will be a quasi-isomorphism $\pi\colon P \longrightarrow M$ (resp. $\iota\colon M \longrightarrow I$), where P is a homotopically projective (resp. injective) right dg A-module.

Remark 7.5. Note that we have

$$\begin{aligned} H^k(\mathrm{HOM}_A(M,N)) &= (H^0 \circ (?[k])(\mathrm{HOM}_A(M,N)) \\ &= H^0(\mathrm{HOM}_A(M,N[k])) = \mathrm{Hom}_{\mathcal{H}(A)}(M,N[k]), \end{aligned}$$

for all $k \in \mathbb{Z}$. Then saying that P (resp. Y) is a homotopically projective (resp. homotopically injective) dg A-module is equivalent to saying that the dg K-module $\mathrm{HOM}_A(P,N)$ (resp. $\mathrm{HOM}_A(N,Y)$) is acyclic whenever N is an acyclic dg A-module.

Example 7.6. If $(e_i)_{i\in I}$ is a distinguished family of orthogonal idempotents of A, then all right dg A-modules e_iA are homotopically projective.

Proof. It is a consequence of Remark 3.2 and [10, Theorem 3.1]. $\square$

Let us consider dg functors $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ between dg categories. By Examples 1.2 and 1.3, we then have dg functors $\mathcal{B}(F(?),?)\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B} \longrightarrow \mathrm{Dg}-K$ and $\mathcal{B}(?,G(?))\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B} \longrightarrow \mathrm{Dg}-K$.

Definition 7.7. In the situation of last paragraph, we say that the pair (F,G) is a dg *adjunction* or that F *is left* dg *adjoint to* G or that G *is right* dg *adjoint to* F when there is a natural isomorphism of dg functors $\eta\colon \mathcal{B}(F(?),?) \xrightarrow{\cong} \mathcal{A}(?,G(?))$. Due to Lemma 1.1 and Definition 1.4, this means that, for each pair of objects $(A,B) \in \mathcal{A}\times\mathcal{B}$, the map $\eta_{A,B}\colon \mathcal{B}(F(A),B) \longrightarrow \mathcal{A}(A,G(B))$ is an isomorphism in $\mathrm{Gr}-K$, natural on A and B, such that $\eta_{A,B}(d_{\mathcal{B}}(\beta)) = d_{\mathcal{A}}(\eta_{A,B}(\beta))$, for each homogeneous element $\beta \in \mathcal{B}(F(A),B)$.

Lemma 7.8. *Let A and B be* dg *algebras with enough idempotents. If $F\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ (resp. $F\colon (\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-B$) is a* dg *functor, then the induced functor $F = Z^0F\colon Z^0(\mathrm{Dg}-A) \cong \mathcal{C}(A) \longrightarrow \mathcal{C}(B) = Z^0(\mathrm{Dg}-B)$ (resp. $F := Z^0F\colon Z^0((\mathrm{Dg}-A)^{\mathrm{op}}) \cong \mathcal{C}(A)^{\mathrm{op}} \longrightarrow \mathcal{C}(B) = Z^0(\mathrm{Dg}-B)$) preserves conflations. If moreover F takes contractible* dg *modules to contractible* dg *modules, then the induced functor $F := H^0F\colon H^0(\mathrm{Dg}-A) \cong \mathcal{H}(A) \longrightarrow \mathcal{H}(B) = H^0(\mathrm{Dg}-B)$ (resp. $F := H^0F\colon H^0((\mathrm{Dg}-A)^{\mathrm{op}}) \cong \mathcal{H}(A)^{\mathrm{op}} \longrightarrow \mathcal{H}(B) = H^0(\mathrm{Dg}-B)$) is triangulated. When F is part of a* dg *adjunction, it takes contractible* dg *modules to contractible* dg *modules.*

Proof. We prove the covariant part of the lemma, the contravariant part being entirely similar. Since F is a dg functor the induced functor $Z^0F\colon Z^0(\mathrm{Dg}-A) = \mathcal{C}(A) \longrightarrow Z^0(\mathrm{Dg}-B) = \mathcal{C}(B)$ preserves exact sequences which split in in the underlying graded categories. That is, Z^0F takes conflations to conflations. As a consequence, H^0F takes triangles to triangles. The initial assertion then follows from [17, Lemma 2.27], bearing in mind that, by Lemma 7.2, we also have a natural isomorphism $H^0F \circ (?[1]) \cong (?[1]) \circ H^0F$. To end the proof, it will be enough to show that if (F, G) is a dg adjunction, then also (Z^0F, Z^0G) is an adjunction. Indeed, if this is proved and if (G, F) is a dg adjunction, then we will have that (Z^0G, Z^0F) is an adjunction. In any of the two situations, [17, Lemma 2.27] again gives that Z^0F preserves projective (=injective) objects with respect to the semi-split exact structures, which amounts to saying that F preserves contractible dg modules.

Let $\eta\colon \mathrm{HOM}_B(F(?), ?) \xrightarrow{\cong} \mathrm{HOM}_A(?, G(?))$ be a graded natural isomorphism which commutes with the differentials. Bearing in mind that, for each $M \in \mathrm{Dg} - A$ and $X \in \mathrm{Dg} - B$, we have $(Z^0F)(M) = F(M)$ and $(Z^0G)(X) = G(X)$, we then get an isomorphism of K-modules

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}(B)}((Z^0F)(M), X) \arrow[d, equal] & \mathrm{Hom}_{\mathcal{C}(A)}(M, (Z^0G)(X)) \arrow[d, equal] \\
Z^0(\mathrm{HOM}_B(F(M), X)) \arrow[r, "Z^0(\eta_{M,X})"] & Z^0(\mathrm{HOM}_A(M, G(X))),
\end{tikzcd}$$

which is natural on M and X since so is η. □

The following functors will be very important in the sequel.

Definition 7.9. Let A and B be dg algebras with enough idempotents and let $\Pi = \Pi_A\colon \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ and $\Upsilon = \Upsilon_A\colon \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ be the homotopically projective and the homotopically injective resolution functors, respectively.

1) If $F\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ is a dg functor which preserves contractible dg modules and we also put $H^0F = F\colon \mathcal{H}(A) \longrightarrow \mathcal{H}(B)$, then:
   (a) The composition $\mathbb{R}F\colon \mathcal{D}(A) \xrightarrow{\Upsilon} \mathcal{H}(A) \xrightarrow{F} \mathcal{H}(B) \xrightarrow{q_B} \mathcal{D}(B)$ is called the *(total) right derived functor of F.*
   (b) The composition $\mathbb{L}F\colon \mathcal{D}(A) \xrightarrow{\Pi} \mathcal{H}(A) \xrightarrow{F} \mathcal{H}(B) \xrightarrow{q_B} \mathcal{D}(B)$ is called the *(total) left derived functor of F.*
2) If $F\colon (\mathrm{Dg} - A)^{\mathrm{op}} \longrightarrow \mathrm{Dg} - B$ is a dg functor which preserves contractible dg modules, then the composition $\mathbb{R}F\colon \mathcal{D}(A)^{\mathrm{op}} \xrightarrow{\Pi^o} \mathcal{H}(A)^{\mathrm{op}} \xrightarrow{F} \mathcal{H}(B) \xrightarrow{q_B} \mathcal{D}(B)$ is called the *(total) right derived functor of F.*

Remark 7.10. If $F\colon (\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-B$ is a dg functor as in last definition, we can interpret it also as dg functor $F^o\colon \mathrm{Dg}-A \longrightarrow (\mathrm{Dg}-B)^{\mathrm{op}}$. We then have that $\mathbb{R}F = (\mathbb{L}F^o)^o$, where $\mathbb{L}F^o$ is the composition $\mathcal{D}(A) \xrightarrow{\Pi} \mathcal{H}(A) \xrightarrow{F^o} \mathcal{H}(B)^{\mathrm{op}} \xrightarrow{q_B^o} \mathcal{D}(B)^{\mathrm{op}}$. This is the reason for which we have not talked about left derived functors of contravariant dg functors.

Remark 7.11. If in any of the situations of last definition, the dg functor F also preserves acyclic dg modules, then the induced functor on the homotopy categories H^0F preserves quasi-isomorphisms. Then one gets a well-defined unique triangulated functor $F\colon \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ (resp. $F\colon \mathcal{D}(A)^{\mathrm{op}} \longrightarrow \mathcal{D}(B)$) such that $q_B \circ H^0F \cong F \circ q_A$ (resp. $q_B \circ H^0F \cong F \circ q_A^o$). It immediately follows that there are natural isomorphisms $\mathbb{L}F \cong F \cong \mathbb{R}F$.

If $F, G\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ are dg functors and $\tau\colon F \longrightarrow G$ is a homological natural transformation, then, for each right dg A-module M, we have that $\tau_M\colon F(M) \longrightarrow G(M)$ belongs to $Z^0(\mathrm{HOM}_B(F(M), G(M)) = \mathrm{Hom}_{\mathcal{C}(B)}(F(M), G(M))$. We then get a morphism, still denoted the same, $\tau_M\colon (H^0F)(M) = F(M) \longrightarrow G(M) = (H^0G)(M)$ in $\mathcal{H}(B)$. It is seen in a straightforward way that, when M varies, the τ_M give a natural transformation $H^0F \longrightarrow H^0G$. As a consequence we get induced natural transformations

$$q(\tau_{\Pi_A(?)})\colon q_B \circ H^0F \circ \Pi = \mathbb{L}F \longrightarrow \mathbb{L}G = q_B \circ H^0F \circ \Pi$$

and

$$q(\tau_{\Upsilon_A(?)})\colon q_B \circ H^0F \circ \Upsilon = \mathbb{R}F \longrightarrow \mathbb{R}G = q_B \circ H^0F \circ \Upsilon.$$

An analogous fact holds for when F and G are dg functors $(\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-B$.

Proposition 7.12. *Let A and B be* dg *algebras with enough idempotents. The following assertions hold:*

1) *If $F, G\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ are* dg *functors which take contractible* dg *modules to contractible* dg *modules and $\tau\colon F \longrightarrow G$ is a homological natural transformation of* dg *functors, then:*
   (a) *$q_B(\tau_{\Pi(?)})\colon \mathbb{L}F \longrightarrow \mathbb{L}G$ is a natural transformation of triangulated functors $\mathcal{D}(A) \longrightarrow \mathcal{D}(B)$.*
   (b) *$q_B(\tau_{\Upsilon(?)})\colon \mathbb{R}F \longrightarrow \mathbb{R}G$ is a natural transformation of triangulated functors $\mathcal{D}(A) \longrightarrow \mathcal{D}(B)$.*

2) *If* $F, G\colon (\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-B$ *are* dg *functors which take contractible* dg *modules to contractible* dg *modules and* $\tau\colon F \longrightarrow G$ *is a homological natural transformation of* dg *functors, then* $q_B(\tau_{\Pi^{\circ}_A()})\colon \mathbb{R}F \longrightarrow \mathbb{R}G$ *is natural transformation of triangulated functors* $\mathcal{D}(A)^{\mathrm{op}} \longrightarrow \mathcal{D}(B)$.

Abusing of notation, all these natural transformations of triangulated functors will be still denoted by τ. *Moreover, if in assertion 1.a (resp. 1.b, resp. 2), M is a right* dg *A-module such that* $\tau_{\Pi_A(M)}$ *(resp.* $\tau_{\Upsilon_A(M)}$, *resp.* $\tau_{\Pi^{\circ}_A(M)}$*) is a quasi-isomorphism (e.g. an isomorphism in* $\mathcal{H}(B)$ *or* $\mathrm{Dg}-B$*), then the evaluation of* $\tau\colon \mathbb{L}F \longrightarrow \mathbb{L}G$ *(resp.* $\tau\colon \mathbb{R}F \longrightarrow \mathbb{R}G$ *in 1.b) and 2.b) at M is an isomorphism in* $\mathcal{D}(B)$.

Proof. The proof in the three cases resemble each other very much. We just prove 1.b. The paragraph preceding this proposition shows that we have an induced natural transformation $q_B(\tau_{\Pi_A()})\colon \mathbb{L}F = q_B \circ H^0F \circ \Pi_A \longrightarrow q_B \circ H^0G \circ \Pi_A = \mathbb{L}G$. All we need to prove is that it is a natural transformation of triangulated functors, for which it is enough to check that the induced natural transformation $\tau\colon H^0F \longrightarrow H^0G$ is a natural transformation of triangulated functors. Indeed, since $\Pi_A\colon \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ and $q_B\colon \mathcal{H}(B) \longrightarrow \mathcal{D}(B)$ are triangulated functors, it will follow that $q_B(\tau_{\Pi(?)})\colon \mathbb{L}F = q_B \circ H^0F \circ \Pi_A \longrightarrow q_B \circ H^0G \circ \Pi_A = \mathbb{L}G$ is natural transformation of triangulated functors, as it is desired.

If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$ (*) is a triangle in $\mathcal{H}(A)$, then we may assume that it comes from a conflation $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ in $\mathcal{C}(A)$, where M is the cone of some morphism $\gamma[-1]\colon N[-1] \longrightarrow L$. If now $\xi_F\colon F \circ (?[1]) \cong (?[1]) \circ F$ and $\xi_G\colon G \circ (?[1]) \cong (?[1]) \circ G$ are the natural isomorphisms of Lemma 7.2, then the image of the triangle (*) by H^0F is

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N) \xrightarrow{\xi_{F,L} \circ F(\gamma)}$$

and the corresponding is true when replacing F by G. Due to the mentioned Lemma 7.2, we then have a commutative diagram in $\mathcal{H}(B)$ whose rows are triangles:

$$\begin{tikzcd}
F(L) \arrow[r, "F(\alpha)"] \arrow[d, "\tau_L"] & F(M) \arrow[r, "F(\beta)"] \arrow[d, "\tau_M"] & F(N) \arrow[r, "\xi_{F,L}\circ F(\gamma)"] \arrow[d, "\tau_N"] & F(L)[1] \arrow[d, "\tau_L[1]"] \\
G(L) \arrow[r, "G(\alpha)"] & G(M) \arrow[r, "G(\beta)"] & G(N) \arrow[r, "\xi_{G,L}\circ G(\gamma)"] & G(L)[1]
\end{tikzcd}$$

which shows that $q_B(\tau_{\Pi(?)})\colon H^0F \longrightarrow H^0G$ is a natural transformation of triangulated functors.

The last statement is a direct consequence of the definition of the triangulated transformation τ since the functor $q_?\colon \mathcal{H}(?) \longrightarrow \mathcal{D}(?)$ takes quasi-isomorphisms to isomorphisms. $\square$

Proposition 7.13. *Let A and B be* dg *algebras with enough idempotents. The following assertions hold:*

1) *If $(F\colon \mathrm{Dg}{-}A \longrightarrow \mathrm{Dg}{-}B, G\colon \mathrm{Dg}{-}B \longrightarrow \mathrm{Dg}{-}A)$ is a* dg *adjunction of* dg *functors, then $(\mathbb{L}F\colon \mathcal{D}(A) \longrightarrow \mathcal{D}(B), \mathbb{R}G\colon \mathcal{D}(B) \longrightarrow \mathcal{D}(A))$ is an adjunction of triangulated functors.*
2) *If $(F^o\colon \mathrm{Dg} - A \longrightarrow (\mathrm{Dg} - B)^{\mathrm{op}}, G\colon (\mathrm{Dg} - B)^{\mathrm{op}} \longrightarrow \mathrm{Dg} - A)$ is a* dg *adjunction of* dg *functors, then $((\mathbb{R}F)^o\colon \mathcal{D}(A) \longrightarrow \mathcal{D}(B)^{\mathrm{op}}, \mathbb{R}G\colon \mathcal{D}(B)^{\mathrm{op}} \longrightarrow \mathcal{D}(A))$ is an adjunction of triangulated functors.*

Proof. In the proof of Lemma 7.8 we have seen that, in the situation of assertion 1, one has that $(Z^0F\colon \mathcal{C}(A) \longrightarrow \mathcal{C}(B), Z^0G\colon \mathcal{C}(B) \longrightarrow \mathcal{C}(A))$ is an adjoint pair. A similar argument proves that, in the situation of assertion 2, one has that $(Z^0(F^o) = (Z^0F)^o\colon \mathcal{C}(A) \longrightarrow \mathcal{C}(B)^{\mathrm{op}}, Z^0G\colon \mathcal{C}(B)^{\mathrm{op}} \longrightarrow \mathcal{C}(A))$ is an adjoint pair.

With the obvious adaptation, [17, Lemma 2.27 and Proposition 2.28] and their proofs show that assertion 1 holds. As for assertion 2, note that [17, Lemma 2.27] also shows that $(H^0(F^o) = (H^0F)^o\colon \mathcal{H}(A) \longrightarrow \mathcal{H}(B)^{\mathrm{op}}, H^0G\colon \mathcal{H}(B)^{\mathrm{op}} \longrightarrow \mathcal{H}(A))$ is an adjoint pair of triangulated functors. Moreover, the adjoint pair $(\Pi_B\colon \mathcal{D}(B) \longrightarrow \mathcal{H}(B), q_B\colon \mathcal{H}(B) \longrightarrow \mathcal{D}(B))$ implies that the pair $(q_B^o\colon \mathcal{H}(B)^{\mathrm{op}} \longrightarrow \mathcal{D}(B)^{\mathrm{op}}, \Pi_B^o\colon \mathcal{D}(B)^{\mathrm{op}} \longrightarrow \mathcal{H}(B)^{\mathrm{op}})$ is also an adjoint pair. It then follows that the composition $(\mathbb{R}F)^o\colon \mathcal{D}(A) \xrightarrow{\Pi_A} \mathcal{H}(A) \xrightarrow{(H^0F)^o} \mathcal{H}(B)^{\mathrm{op}} \xrightarrow{q_B^o} \mathcal{D}(B)^{\mathrm{op}}$ (see Remark 7.10) is left adjoint to the composition $\mathcal{D}(A) \xleftarrow{q_A} \mathcal{H}(A) \xleftarrow{H^0G} \mathcal{H}(B)^{\mathrm{op}} \xleftarrow{\Pi_B^o} \mathcal{D}(B)^{\mathrm{op}}$, which is precisely $\mathbb{R}G$. $\square$

Proposition 7.14. *Let A, B and C be* dg *algebras with enough idempotents and denote by $\Pi_?\colon \mathcal{D}(?) \longrightarrow \mathcal{H}(?)$ and $\Upsilon_?\colon \mathcal{D}(?) \longrightarrow \mathcal{H}(?)$ the homotopically projective and homotopically injective resolution functors, for $? = A, B, C$. Suppose that all the* dg *functors appearing below preserve contractible* dg *modules. The following assertions hold:*

1) *Let $G\colon \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - B$ and $F\colon \mathrm{Dg} - B \longrightarrow \mathrm{Dg} - C$ be* dg *functors. Then:*
   (a) *There is a canonical natural transformation of triangulated functors $\rho\colon \mathbb{R}(F \circ G) \longrightarrow \mathbb{R}F \circ \mathbb{R}G$. When M is a right* dg

A-module such that $G(\Upsilon_A(M))$ is homotopically injective, then ρ_M is an isomorphism.

(b) *There is a canonical natural transformation of triangulated functors $\sigma\colon \mathbb{L}F \circ \mathbb{L}G \longrightarrow \mathbb{L}(F \circ G)$. When M is a right* dg *A-module such that $G(\Pi_A(M))$ is homotopically projective, then σ_M is an isomorphism.*

2) *If $G\colon (\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-B$ and $F\colon (\mathrm{Dg}-B)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-C$ are* dg *functors, then there is a canonical natural transformation $\tau\colon \mathbb{L}(F \circ G^o) \longrightarrow \mathbb{R}F \circ (\mathbb{R}G)^o$ of triangulated functors $\mathcal{D}(A) \longrightarrow \mathcal{D}(C)$. When M is a right* dg *A-module such that $G(\Pi_A(M))$ is homotopically projective, then τ_M is an isomorphism.*

3) *If $G\colon (\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-B$ and $F\colon \mathrm{Dg}-B \longrightarrow \mathrm{Dg}-C$ are* dg *functors, then there is a canonical natural transformation of triangulated functors $\omega\colon \mathbb{L}F \circ \mathbb{R}G \longrightarrow \mathbb{R}(F \circ G)$. When M is a right* dg *A-module such that $G(\Pi_A(M))$ is homotopically projective, ω_M is an isomorphism.*

4) *If $G\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ and $F\colon (\mathrm{Dg}-B)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-C$ are* dg *functors, then there is a canonical natural transformation of triangulated functors $\theta\colon \mathbb{R}(F \circ G^o) \longrightarrow \mathbb{R}F \circ (\mathbb{L}G)^o$. When M is a right* dg *A-module such that $G(\Pi_A(M))$ is homotopically projective, θ_M is an isomorphism.*

Proof. The arguments for the proofs are all very much alike and rely on the explicit definition of right and left derived functors in each case. We just provide the proof of assertions 1.a and 2, leaving the rest as an exercise to the reader.

1.a) We consider the unit $\lambda\colon 1_{\mathcal{H}(B)} \longrightarrow \Upsilon_B \circ q_B$ of the adjunction (q_B, Υ_B). We then get a canonical natural transformation of triangulated functors

$$\rho := (q_C \circ F)(\lambda_{(G\circ\Upsilon)(?)})\colon \mathbb{R}(F \circ G) = q_C \circ F \circ G \circ \Upsilon_A = \\ q_C \circ F \circ 1_{\mathcal{H}(B)} \circ G \circ \Upsilon_A \longrightarrow q_C \circ F \circ \Upsilon_B \circ q_B \circ G \circ \Upsilon_A = \mathbb{R}F \circ \mathbb{R}G,$$

where $F = H^0F\colon \mathcal{H}(B) \longrightarrow \mathcal{H}(C)$ and $G = H^0G\colon \mathcal{H}(A) \longrightarrow \mathcal{H}(B)$. If now $G(\Upsilon_A)(M)$ is homotopically injective, then $\lambda_{(G\circ\Upsilon_A)(M)}\colon (G \circ \Upsilon_A)(M) \xrightarrow{\cong} (\Upsilon_B \circ q_B \circ G \circ \Upsilon_A)(M)$ is an isomorphism, which implies that $\rho_M = (q_C \circ F)(\lambda_{(G\circ\Upsilon_A)(M)})$ is also an isomorphism.

2) The adjunction $(q_B^o\colon \mathcal{H}(B)^{\mathrm{op}} \longrightarrow \mathcal{D}(B)^{\mathrm{op}}, \Pi_B^o\colon \mathcal{D}(B)^{\mathrm{op}} \longrightarrow \mathcal{H}(B)^{\mathrm{op}})$ yields a unit $\mu\colon 1_{\mathcal{H}(B)^{\mathrm{op}}} \longrightarrow \Pi_B^o \circ q_B^o$. Then we get a natural transformation of triangulated functors

$$\sigma := (q_C \circ F)(\mu_{(G^o \circ \Pi_A)(?)}) \colon \mathbb{L}(F \circ G^o) = q_C \circ F \circ G^o \circ \Pi_A = \\ q_C \circ F \circ 1_{\mathcal{H}(B)^{\mathrm{op}}} \circ G^o \circ \Pi_A \longrightarrow q_C \circ F \circ \Pi_B^o \circ q_B^o \circ G^o \circ \Pi_A = \mathbb{R}F \circ (\mathbb{R}G)^o.$$

If now $G(\Pi_A(M))$ is homotopically projective, then $\mu_{(G^o \circ \Pi_A)(M)} \colon (G^o \circ \Pi_A)(M) \longrightarrow (\Pi_B^o \circ q_B^o \circ G^o \circ \Pi_A)(M)$ is an isomorphism in $\mathcal{H}(B)^{\mathrm{op}}$, which implies that $\sigma_M = (q_C \circ F)(\mu_{(G^o \circ \Pi_A)(M)})$ is an isomorphism. $\square$

Suppose now that A, B, C are dg algebras with enough idempotents and that $F \colon (\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C) \longrightarrow \mathrm{Dg} - B$ is a dg functor. We then have induced functors

$$Z^0F \colon Z^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C)) \longrightarrow Z^0(\mathrm{Dg} - B) = \mathcal{C}(B)$$

and

$$H^0F \colon H^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C)) \longrightarrow H^0(\mathrm{Dg} - B) = \mathcal{H}(B).$$

On the other hand, the objects of $\mathcal{C}(A) \otimes \mathcal{C}(B)$ are those of $Z^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C))$ (i.e. those of $(\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C)$). But if f is morphism in $\mathcal{C}(A)$ and g is a morphism in $\mathcal{C}(C)$, then, viewed as a morphism in $(\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C)$, we have that $f \otimes g$ is a 0-cycle and, hence, a morphism of $Z^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C))$. Indeed we have

$$d(f \otimes g) = d(f) \otimes g + (-1)^{|f|} f \otimes d(g) = 0,$$

because f and g are morphisms in $Z^0(\mathrm{Dg} - A) = \mathcal{C}(A)$ and $Z^0(\mathrm{Dg} - C) = \mathcal{C}(C)$, respectively. The assignments $(M, X) \rightsquigarrow (M, X)$ and $f \otimes g \rightsquigarrow f \otimes g$ give a functor $j \colon \mathcal{C}(A) \otimes \mathcal{C}(C) \longrightarrow Z^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C))$ and a composition

$$\mathcal{C}(A) \otimes \mathcal{C}(C) \xrightarrow{j} Z^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C)) \xrightarrow{Z^0F} \mathcal{C}(B).$$

Abusing the notation, we still denote by Z^0F this composition functor. Considering now f and g as above, suppose that either f or g is null-homotopic. We claim that $j(f \otimes g) = f \otimes g$ is a 0-boundary of $(\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C)$. Indeed if, say, $g = d(g')$ then

$$d(f \otimes g') = d(f) \otimes g' + (-1)^{|f|} f \otimes d(g') = f \otimes g$$

since $d(f) = 0$ and $|f| = 0$. A similar argument applies if we assume $f = d(f')$. This means that we have an induced functor $\underline{j} \colon \mathcal{H}(A) \otimes \mathcal{H}(C) \longrightarrow H^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C))$ and a corresponding composition

$$\mathcal{H}(A) \otimes \mathcal{H}(C) \xrightarrow{\underline{j}} H^0((\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C)) \xrightarrow{H^0F} \mathcal{H}(B),$$

which we shall still denote by H^0F.

A procedure similar to the one depicted in the previous paragraph can be undertaken with a dg functor $F\colon (\mathrm{Dg}-A)^{\mathrm{op}}\otimes(\mathrm{Dg}-C)\longrightarrow \mathrm{Dg}-B$, getting then functors $Z^0F\colon \mathcal{C}(A)^{\mathrm{op}}\otimes\mathcal{C}(C)\longrightarrow\mathcal{C}(B)$ and $H^0F\colon \mathcal{H}(A)^{\mathrm{op}}\otimes\mathcal{H}(C)\longrightarrow\mathcal{H}(B)$.

Proposition 7.15. *Let A,B,C be* dg *algebras with enough idempotents and let $F\colon (\mathrm{Dg}-A)\otimes(\mathrm{Dg}-C)\longrightarrow \mathrm{Dg}-B$ (resp. $F\colon (\mathrm{Dg}-A)^{\mathrm{op}}\otimes(\mathrm{Dg}-C)\longrightarrow \mathrm{Dg}-B$) be a* dg *functor. The following assertions hold:*

1) *The functor $F=Z^0F\colon \mathcal{C}(A)\otimes\mathcal{C}(C)\longrightarrow\mathcal{C}(B)$ (resp. $Z^0F\colon\mathcal{C}(A)^{\mathrm{op}}\otimes\mathcal{C}(C)\longrightarrow\mathcal{C}(B)$) preserves conflations on each variable.*
2) *If $F(P,X)$ and $F(M,Q)$ are contractible* dg *B-modules whenever P and Q are a contractible* dg *A-module and a contractible* dg *C-module, respectively, then the functor $F=H^0F\colon\mathcal{H}(A)\otimes\mathcal{H}(C)\longrightarrow\mathcal{H}(B)$ (resp. $F=H^0F\colon\mathcal{H}(A)^{\mathrm{op}}\otimes\mathcal{H}(C)\longrightarrow\mathcal{H}(B)$) is triangulated on both variables.*

Proof. Assertion 1 is a direct consequence of Lemma 7.8 bearing in mind Lemma 1.1. Moreover, if M is fixed, then the dg functor $F_M=F(M,?)\colon(\mathrm{Dg}-C)\longrightarrow(\mathrm{Dg}-B)$ takes contractible dg modules to contractible dg modules, which implies by Lemma 7.8 that $H^0F_M\colon\mathcal{H}(C)\longrightarrow\mathcal{H}(B)$ is a triangulated functor. But we clearly have $H^0F_M=H^0F(M,?)$, which says that $F=H^0F$ is triangulated on the second variable. A symmetric argument proves that it is triangulated on the first variable. $\square$

Our next goal is to see that, when a dg functor is part of a dg 'bifunctor' and certain conditions are satisfied, also its derived functor is part of a bifunctor which is triangulated on both variables.

Definition 7.16. Let A,B,C be dg algebras with enough idempotents.

1) If $F\colon(\mathrm{Dg}-A)\otimes(\mathrm{Dg}-C)\longrightarrow\mathrm{Dg}-B$ is a dg functor which preserves contractible dg modules in each variable, then we put
   (a) $\mathbb{L}F\colon\mathcal{D}(A)\otimes\mathcal{D}(C)\xrightarrow{\Pi_A\otimes\Pi_C}\mathcal{H}(A)\otimes\mathcal{H}(C)\xrightarrow{H^0F}\mathcal{H}(B)\xrightarrow{q_B}\mathcal{D}(B)$.
   (b) $\mathbb{R}F\colon\mathcal{D}(A)\otimes\mathcal{D}(C)\xrightarrow{\Upsilon_A\otimes\Upsilon_C}\mathcal{H}(A)\otimes\mathcal{H}(C)\xrightarrow{H^0F}\mathcal{H}(B)\xrightarrow{q_B}\mathcal{D}(B)$
2) If $F\colon(\mathrm{Dg}-A)^{\mathrm{op}}\otimes(\mathrm{Dg}-C)\longrightarrow\mathrm{Dg}-B$ be a dg functor which preserves contractible dg modules on each variable, then we put $\mathbb{R}F\colon\mathcal{D}(A)^{\mathrm{op}}\otimes\mathcal{D}(C)\xrightarrow{\Pi_A^o\otimes\Upsilon_C}\mathcal{H}(A)\otimes\mathcal{H}(C)\xrightarrow{H^0F}\mathcal{H}(B)\xrightarrow{q_B}\mathcal{D}(B)$.

By their definition all these functors are triangulated in each variable.

For each dg functor $F\colon(\mathrm{Dg}-A)\otimes(\mathrm{Dg}-C)\longrightarrow\mathrm{Dg}-B$ as in last definition, fixing an object M in $\mathrm{Dg}-A$ and X in $\mathrm{Dg}-C$, we get dg functors

$F_M = F(M, ?)\colon \mathrm{Dg} - C \longrightarrow \mathrm{Dg} - B$ and $F^X = F(?, X)\colon \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - B$. It is natural to ask whether we have natural isomorphisms $\mathbb{L}F(M, ?) \cong \mathbb{L}F_M$ and $\mathbb{L}F(?, X) \cong \mathbb{L}F^X$, and similarly for the right derived versions. For this, we have the following criterion:

Proposition 7.17. *Let A, B, C be* dg *algebras with enough idempotents. The following assertions hold:*

1) *Let $F\colon (\mathrm{Dg} - A) \otimes (\mathrm{Dg} - C) \longrightarrow \mathrm{Dg} - B$ be a* dg *functor. Then*
   (a) *if $F(?, Q)\colon \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - B$ preserves acyclic* dg *modules whenever Q is homotopically projective (resp. homotopically injective), then there is a natural isomorphism of triangulated functors $\mathbb{L}F(M, ?) \cong \mathbb{L}F_M\colon \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$ (resp. $\mathbb{R}F(M, ?) \cong \mathbb{R}F_M\colon \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$), for each right* dg *$A$-module M,*
   (b) *if $F(P, ?)\colon \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - B$ preserves acyclic* dg *modules whenever P is homotopically projective (resp. homotopically injective), then there is a natural isomorphism of triangulated functors $\mathbb{L}F(?, X) \cong \mathbb{L}F^X : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$, for each right* dg *$C$-module X.*
2) *Let $F\colon (\mathrm{Dg} - A)^{\mathrm{op}} \otimes (\mathrm{Dg} - C) \longrightarrow \mathrm{Dg} - B$ be a* dg *functor. Then*
   (a) *if $F(?, Q)\colon (\mathrm{Dg} - A)^{\mathrm{op}} \longrightarrow \mathrm{Dg} - B$ preserves acyclic* dg *modules whenever Q is homotopically injective, then there is a natural isomorphism of triangulated functors $\mathbb{R}F(M, ?) \cong \mathbb{R}F_M\colon \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$, for each right* dg *$A$-module M.*
   (b) *if $F(P, ?)\colon \mathrm{Dg} - C \longrightarrow \mathrm{Dg} - B$ preserves acyclic* dg *modules whenever P is homotopically projective, then there is a natural isomorphism of triangulated functors $\mathbb{R}F(?, X) \cong \mathbb{R}F^X : \mathcal{D}(A)^{\mathrm{op}} \longrightarrow \mathcal{D}(B)$ for each right* dg *C-module X.*

Proof. We will prove 1.b and 2.a, and leave to the reader the other ones whose proof follows entirely similar patterns. For 1.b, note that the action of $\mathbb{L}F(?, X)$ and $\mathbb{L}F^X$ on objects is given by

$$\mathbb{L}F(?, X)(M) = \mathbb{L}F(M, X) = F(\Pi_A(M), \Pi_C(X))$$

and

$$\mathbb{L}F^X(M) = F^X(\Pi_A(M)) = F(\Pi_A(M), X).$$

Moreover if $f\colon M \longrightarrow N$ is a morphism in $\mathcal{D}(A)$, then

$$\mathbb{L}F(?, X)(f) = \mathbb{L}F(f, 1_X) = F(\Pi_A(f), \Pi_C(1_X)) = F(\Pi_A(f), 1_{\Pi_C(X)}).$$

It follows that we can identify $\mathbb{L}F(?,X) = \mathbb{L}F^{\Pi_C(X)}$, where $F^{\Pi_C(X)} = F(?,\Pi_C(X))\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ is the 'left part' of F when the fixed second variable is $\Pi_C(X)$. We fix now the homotopically projective resolution map $\pi\colon \Pi_C(X) \longrightarrow X$. Note that π is a morphism in $\mathcal{H}(C)$, and hence the image of a morphism in $\mathcal{C}(C)$ by the canonical functor $\mathcal{C}(C) \longrightarrow \mathcal{H}(C)$. Fixing a lift, we can think of π as a morphism in $\mathcal{C}(C)$. We claim that $\pi_*\colon F^{\Pi_C(X)} = F(?,\Pi_C(X)) \longrightarrow F(?,X) = F^X$ is a homological natural transformation of dg functors $\mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$. Indeed note that, for a fixed M in $\mathrm{Dg}-A$, we have $(\pi_*)_M\colon F(M,\Pi_C(X)) \longrightarrow F(M,X)$ is the morphism $(\pi_*)_M = F(1_M,\pi)$. If now $f\colon M \longrightarrow N$ is a homogeneous morphism in $\mathrm{Dg}-A$, then

$$F^{\Pi_C(X)}(f) = F(f,1_{\Pi_C(X)}) = F(M,\Pi_C(X)) \longrightarrow F(N,\Pi_C(X))$$

while

$$F^X(f) = F(f,1_X)\colon F(M,X) \longrightarrow F(N,X).$$

We then have an equality

$$\begin{aligned} F^X(f)\circ(\pi_*)_M &= F(f,1_X)\circ F(1_M,\pi) \\ &= F(f,\pi) \\ &= (-1)^{|f||\pi|}F(1_N,s)\circ F(f,1_{\Pi_C(X)}) \\ &= (\pi_*)_N \circ F^{\Pi_C(X)}(f), \end{aligned}$$

using Lemma 1.1 and the fact that $|\pi| = 0$. It follows that π_* is a natural transformation of K-linear graded functors. In order to see that it is homological it remains to check that $(\pi_*)_M = F(1_M\otimes\pi) \in Z^0(\mathrm{HOM}_B(F(M,\Pi_C(X))),F(M,X))$, for all M in $\mathrm{Dg}-A$. But this is clear since $F(1_M\otimes\pi)$ is the image of $1_M\otimes\pi$ by the functor $Z^0F\colon \mathcal{C}(A)\otimes\mathcal{C}(C) \longrightarrow \mathcal{C}(B)$ (see Proposition 7.15).

Once we know that $\pi_*\colon F^{\Pi_C(X)} \longrightarrow F^X$ is a homological natural transformation of dg functors, Proposition 7.12 says that we have an induced natural transformation $\pi_*\colon \mathbb{L}F^{\Pi_C(X)} \longrightarrow \mathbb{L}F^X$ of triangulated functors $\mathcal{D}(A) \longrightarrow \mathcal{D}(B)$. But, when evaluating at an object M of $\mathcal{D}(A)$, we have that

$$\begin{aligned} &(\pi_*)_M\colon \mathbb{L}F^{\Pi_C(X)}(M) \\ &\qquad = F(\Pi_A(M),\Pi_C(X)) \longrightarrow F(\Pi_A(M),X) = \mathbb{L}F^X(M) \end{aligned}$$

is an isomorphism in $\mathcal{D}(B)$. Indeed, by hypothesis $F(\Pi_A(M),?)\colon \mathrm{Dg}-C \longrightarrow \mathrm{Dg}-B$ preserves acyclic dg modules, which implies that the induced triangulated functor $F(\Pi_A(M), ?)\colon H^0(\mathrm{Dg}-C) = \mathcal{H}(C) \longrightarrow \mathcal{H}(B) = H^0(\mathrm{Dg}-B)$ preserves quasi-isomorphisms. It follows that $(\pi_*)_M = F(1_{\Pi_A(M)}, \pi)\colon F(\Pi_A(M), \Pi_C(X)) \longrightarrow F(\Pi_A(M), X)$ is a quasi-isomorphism in $\mathcal{H}(B)$, which implies that (after applying $q_B\colon \mathcal{H}(B) \longrightarrow \mathcal{D}(B)$) it becomes an isomorphism in $\mathcal{D}(B)$. Then $(\pi_*)\colon \mathbb{L}F^{\Pi_C(X)} \longrightarrow \mathbb{L}F^X$ is a natural transformation of triangulated functor which is pointwise an isomorphism. Therefore it is a natural isomorphism. In particular $\mathbb{L}F(?, X) = \mathbb{L}F^{\Pi_C(X)}$ is naturally isomorphic to $\mathbb{L}F^X$ as triangulated functors.

The proof of 2.a follows an entirely similar pattern. We outline the argument, leaving the details to the reader. We have that $\mathbb{R}F(M, ?)(X) = F(\Pi_A(M), \Upsilon_A(X))$ and $\mathbb{R}F_M(X) = F_M(\Upsilon(X)) = F(M, \Upsilon(X))$. We can then identify $\mathbb{R}F(M, ?) = \mathbb{R}F_{\Pi_A(M)}$, where $F_{\Pi_A(M)} = F(\Pi_A(M), ?)\colon \mathrm{Dg}-C \longrightarrow \mathrm{Dg}-B$. If now $\pi\colon \Pi(M) \longrightarrow M$ is homotopically projective resolution, which we view as a morphism in $\mathcal{C}(A)$, then $\pi^*\colon F_M = F(M, ?) \longrightarrow F_{\Pi_A(M)} = F(\Pi_A(M), ?)$ is a homological natural transformation of dg functors $\mathrm{Dg}-C \longrightarrow \mathrm{Dg}-B$. The associated natural transformation of triangulated functor $\pi^*\colon \mathbb{R}F_M \longrightarrow \mathbb{R}F_{\Pi_A(M)}$, when evaluated at an object X of $\mathcal{D}(C)$, is $(\pi^*)_X = F(\pi, 1_{\Upsilon(X)})\colon \mathbb{R}F_M(X) = F(M, \Upsilon(X)) \longrightarrow F(\Pi_A(M), \Upsilon(X))$. This is a quasi-isomorphism of dg B-modules, and hence an isomorphism in $\mathcal{D}(B)$, for each $X \in \mathcal{D}(C)$. It follows that π^* is a natural isomorphism $\mathbb{R}F_M \xrightarrow{\cong} \mathbb{R}F_{\Pi_A(M)} = \mathbb{R}F(M, ?)$. □

8. The classical dg bifunctors

All throughout this section, we fix dg algebras with enough idempotents A, B and C and fix a distinguished family of orthogonal idempotents $(e_i)_{i\in I}$ in B, $(\epsilon_j)_{j\in J}$ in A and $(\nu_k)_{k\in\mathcal{K}}$ in C, all of which are homogeneous of degree zero and killed by the differential. If M a dg $C-A$-bimodule and X is a dg $B-A$-bimodule, the space of morphisms $\mathrm{HOM}_A(M, X)$ in $\mathrm{Dg}-A$ has a canonical structure of non-unitary graded $B-C$-bimodule, where the multiplication map is identified by the rule $(bfc)(m) = bf(cm)$, for all homogeneous elements $b \in B$, $f \in \mathrm{HOM}_A(M, X)$, $c \in C$ and $m \in M$. To avoid the non-unitary problem, we consider the largest unitary graded $B-C-$sub-bimodule

$$\overline{\mathrm{HOM}}_A(M, X) = B\,\mathrm{HOM}_A(M, X)C$$

of $\mathrm{HOM}_A(M, X)$. Note that, expressed in terms of the distinguished families of orthogonal idempotents, $\overline{\mathrm{HOM}}_A(M, X)$ consists of those $f \in$

$\mathrm{HOM}_A(M,X)$ such that Im(f) $\subseteq \oplus_{i\in I'}e_iX$, for some finite subset $I'\subseteq I$, and $f(\nu_kM)=0$, for all but finitely many $k\in\mathcal{K}$. We can say much more about the just defined graded $B-C$-bimodule

Lemma 8.1. *The differential* $d\colon \mathrm{HOM}_A(M,X)\longrightarrow \mathrm{HOM}_A(M,X)$ *satisfies Leibniz equality*

$$d(bfc)=d_B(b)fc+(-1)^{|b|}bd(f)c+(-1)^{|b|+|f|}bfd_C(c),$$

for all homogeneous elements $b\in B$, $f\in \mathrm{HOM}_A(M,X)$ *and* $c\in C$. *Moreover, it satisfies that* $d(\overline{\mathrm{HOM}}_A(M,X))\subseteq \overline{\mathrm{HOM}}_A(M,X)$*, so that, endowed with the restricted differential,* $\overline{\mathrm{HOM}}_A(M,X)$ *becomes a (unitary!)* dg $B-C$*-bimodule.*

Proof. We let act both members of the desired Leiniz equality on a homogeneous element $m\in M$. We then have

$$\begin{aligned}
d(bfc)(m) &= [d_X\circ(bfc)-(-1)^{|bfc|}(bfc)\circ d_M](m)\\
&= d_X(bf(cm))-(-1)^{|b|+|f|+|c|}bf(cd_M(m))\\
&= d_B(b)f(cm)+(-1)^{|b|}bd_X(f(cm))-(-1)^{|b|+|f|+|c|}bf(cd_M(m))\\
&= d_B(b)f(cm)+(-1)^{|b|}b(d_X\circ f-(-1)^{|f|}f\circ d_M)(cm)\\
&\quad+(-1)^{|b|+|f|}bf(d_M(cm)-(-1)^{|c|}cd_M(m))\\
&= d_B(b)f(cm)+(-1)^{|b|}bd(f)(cm)+(-1)^{|b|+|f|}bf(d_C(c)m)\\
&= [d_B(b)fc+(-1)^{|b|}bd(f)c+(-1)^{|b|+|f|}bfd_C(c)](m).
\end{aligned}$$

To prove the last statement, take a homogeneous element $f\in\overline{\mathrm{HOM}}_A(M,X)$. We have $d(f)=d_X\circ f-(-1)^{|f|}f\circ d_M$, which implies that $\mathrm{Im}(d(f))\subseteq d_X(Im(f))+Im(f)$. But if $I'\subseteq I$ is any finite subset such that $\mathrm{Im}(f)\subseteq\oplus_{i\in I'}e_iX$, then $\mathrm{Im}(d(f))\subseteq\oplus_{i\in I'}e_iX$. This is because $d_X(e_iX)\subseteq e_iX$ since $d_B(e_i)=0$ for all $i\in I'$. By analogous reason, we have that $d_M(\nu_kM)\subseteq\nu_kM$. This implies that if $\mathcal{K}'\subseteq\mathcal{K}$ is any finite subset such that $f(\nu_kM)=0$, for all $k\in\mathcal{K}\setminus\mathcal{K}'$, then we also have $d(f)(\nu_kM)=0$, for all $k\in\mathcal{K}\setminus\mathcal{K}'$. As a consequence, we get that $d(f)\in\overline{\mathrm{HOM}}_A(M,X)$. □

We can now prove

Proposition 8.2. *The assignment* $(M,X)\rightsquigarrow\overline{\mathrm{HOM}}_A(M,X)$ *is the definition on objects of a* dg *functor* $\overline{\mathrm{HOM}}_A(?,?)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}}\otimes(B-\mathrm{Dg}-A)\longrightarrow B-\mathrm{Dg}-C$.

Proof. We have obvious restriction of scalars functors $\rho\colon C - \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - A$ and $\rho'\colon B - \mathrm{Dg} - A \longrightarrow \mathrm{Dg} - A$, both of which are clearly dg functors since when M and N are dg $C - A-$bimodules, the differential $d\colon \mathrm{HOM}_R(M,N) \longrightarrow \mathrm{HOM}_R(M,N)$ is 'the same' when taking $R = A \otimes C^{\mathrm{op}}$ or when taking $R = A$. On the other hand, by Example 1.2, we have a canonical dg functor $\mathrm{HOM}_A(?,?)\colon (\mathrm{Dg}-A)^{\mathrm{op}} \otimes (\mathrm{Dg}-A) \longrightarrow \mathrm{Dg}-K$. We then get an induced dg functor

$$\begin{aligned}\mathrm{HOM}_A(?,?)\colon &(C-\mathrm{Dg}-A)^{\mathrm{op}} \otimes (B-\mathrm{Dg}-A)\\ &\xrightarrow{\rho^o\otimes\rho'} (\mathrm{Dg}-A)^{\mathrm{op}} \otimes (\mathrm{Dg}-A) \xrightarrow{\mathrm{Hom}_A(?,?)} \mathrm{Dg}-K.\end{aligned}$$

Recall that if $\alpha\colon N \longrightarrow M$ and $\varphi\colon X \longrightarrow Y$ are homogeneous morphisms in $C-\mathrm{Dg}-A$ and $B-\mathrm{Dg}-A$, respectively, then $\mathrm{HOM}_A(\alpha^o \otimes \varphi)\colon \mathrm{HOM}_A(M,X) \longrightarrow \mathrm{HOM}_A(N,Y)$ is the map defined by $\mathrm{HOM}_A(\alpha^o \otimes \varphi)(f) = (-1)^{(|\varphi|+|f|)|\alpha|}\varphi \circ f \circ \alpha$.

By definition, we have that $\overline{\mathrm{HOM}}_A(M,X)$ is a dg $B-C-$subbimodule of $\mathrm{HOM}_A(M,X)$ with the restricted differential. In order to have an induced graded functor $\overline{\mathrm{HOM}}_A(?,?)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}} \otimes (B-\mathrm{Dg}-A) \longrightarrow B-\mathrm{Dg}-C$, using Lemma 1.1, it is enough to prove the following two conditions:

a) For φ as above and each dg $C-A$-bimodule M, the map $\varphi_* := \mathrm{HOM}_A(1_M^o,\varphi)\colon \mathrm{HOM}_A(M,X) \longrightarrow \mathrm{HOM}_A(M,Y)$ is a morphisms of non-unitary $B-C-$bimodules such that $\varphi_*(\overline{\mathrm{HOM}}_A(M,X)) \subseteq \overline{\mathrm{HOM}}_A(M,Y)$.

b) For α as above and each dg $B-A$-bimodule X, the map $\alpha^* := \mathrm{HOM}_A(\alpha^0, 1_X)\colon \mathrm{HOM}_A(M,X) \longrightarrow \mathrm{HOM}_A(N,X)$ is a morphism of non-unitary graded $B-C-$bimodules such that $\alpha^*(\overline{\mathrm{HOM}}_A(M,X)) \subseteq \overline{\mathrm{HOM}}_A(N,X)$.

For condition a), we first check that φ_* is a morphism (of degree $|\varphi|$) of nonunitary graded left B-modules. According to the comments after Lemma 4.1, although applied to non-unitary graded left B-modules, we need to check that $\varphi_*(bf) = (-1)^{|\varphi||b|}b\varphi_*(f)$ or, equivalently, that $\varphi \circ (bf) = (-1)^{|\varphi||b|}b(\varphi \circ f)$, for any homogeneous element $f \in \mathrm{HOM}_A(M,X)$. By letting act both members of the desired equality on a homogeneous element $m \in M$, we get that

$$[\varphi \circ (bf)](m) = \varphi(bf(m)) = (-1)^{|\varphi||b|}b\varphi(f(m)) = [(-1)^{|\varphi||b|}b(\varphi \circ f)](m),$$

bearing in mind that φ is a morphism of graded $B-A-$bimodules and, hence, also a morphism of graded left B-modules. On the other hand, if

$c \in C$ is a homogeneous element, we have that $\varphi_*(fc) = \varphi \circ (fc)$ while $\varphi_*(f)c = (\varphi \circ f)c$. Both maps take $m \rightsquigarrow (\varphi \circ f)(cm)$, for each $m \in M$. Then φ_* is a morphism of nonunitary graded $B-C-$bimodules. Moreover, if $f \in \overline{\mathrm{HOM}}_A(M,X)$ and we fix finite subsets $I' \subset I$ and $F \subset \mathcal{K}$ such that $\mathrm{Im}(f) \subseteq \oplus_{i \in I'} e_i X$ and $f(\nu_k M) = 0$, for all $k \in \mathcal{K} \setminus F$, then we have that

$$\mathrm{Im}(\varphi_*(f)) = \mathrm{Im}(\varphi \circ f) \subseteq \varphi(\oplus_{i \in I'} e_i X) \subseteq \oplus_{i \in I'} e_i Y,$$

because φ is in particular a morphism left B-modules, and that $\varphi_*(f)(\nu_k M) = (\varphi \circ f)(\nu_k M) = 0$, for all $k \in \mathcal{K} \setminus F$. Therefore we have $\varphi_*(f) \in \overline{\mathrm{HOM}}_A(M,Y)$.

For condition b), we first prove that α^* is morphism of non-unitary graded left B-modules, which amounts to prove that $\alpha^*(bf) = (-1)^{|a||\alpha|} b\alpha^*(f)$, for any homogeneous element $f \in \mathrm{HOM}_A(M,X)$. On one hand, we have $\alpha^*(bf) = (-1)^{|\alpha|(|b|+|f|)}(bf) \circ \alpha$ while $(-1)^{|\alpha||b|} b\alpha^*(f) = (-1)^{|\alpha||b|}(-1)^{|\alpha||f|} b(f \circ \alpha)$. Since $(bf) \circ \alpha = b(f \circ \alpha)$ due to the definition of the multiplication map $B \otimes \mathrm{HOM}_A(N,X) \longrightarrow \mathrm{HOM}_A(N,X)$, we conclude that α^* is a morphism of degree $|\alpha|$ of graded left B-modules.

If $c \in C$ is a homogeneous element, then $\alpha^*(fc) = (-1)^{|\alpha|(|c|+|f|)}(fc) \circ \alpha$ and $\alpha^*(f)c = (-1)^{|\alpha||f|}(f \circ \alpha)c$. When we let these morphisms act on a homogeneous element $x \in N$, we get

$$[\alpha^*(fc)](x) = (-1)^{|\alpha|(|c|+|f|)}(fc)(\alpha(x)) = (-1)^{|\alpha|(|c|+|f|)} f(c\alpha(x))$$

while

$$[\alpha^*(f)c](x) = (-1)^{|\alpha||f|} f(\alpha(cx)) = (-1)^{|\alpha||f|}(-1)^{|\alpha||c|} f(c\alpha(x)),$$

bearing in mind that α is a morphism of graded left modules (see the comments after Lemma 4.1, applied to non-unitary left B-modules). It then follows that α^* is a homogeneous morphism of graded $B-C-$bimodules. On the other hand, if one considers finite subset $I' \subset I$ and $F \subset \mathcal{K}$ as for condition a), then $\mathrm{Im}(\alpha^*(f)) = \mathrm{Im}(f \circ \alpha) \subseteq \mathrm{Im}(f) \subseteq \oplus_{i \in I'} e_i X$ while $\alpha^*(f)(\nu_k N) = f(\alpha(\nu_k N)) \subseteq f(\nu_k M) = 0$, for all $k \in \mathcal{K} \setminus F$, bearing in mind that α is in particular a morphism of left C-modules.

To finish the proof, we just need to check that the induced map

$$\begin{aligned}
\overline{\mathrm{HOM}}_A(?,?)\colon\ & \mathrm{Hom}_{(C-\mathrm{Dg}-A)^{\mathrm{op}} \otimes (B-\mathrm{Dg}-A)}[(N,X),(M,Y)] \\
&= \mathrm{HOM}_{C-A}(M,N) \otimes \mathrm{HOM}_{B-A}(X,Y) \\
&\quad \longrightarrow \mathrm{HOM}_{B-C}(\overline{\mathrm{HOM}}_A(M,X), \overline{\mathrm{HOM}}_A(N,Y))
\end{aligned}$$

commutes with the differentials. But what we have done above shows that the map $\overline{\mathrm{HOM}}_A(?,?)$ is induced by the map

$$\begin{aligned}\mathrm{HOM}_A(?,?)\colon\ &\mathrm{Hom}_{(C-\mathrm{Dg}-A)^{\mathrm{op}}\otimes(B-\mathrm{Dg}-A)}[(N,X),(M,Y)]\\ &= \mathrm{HOM}_{C-A}(M,N)\otimes\mathrm{HOM}_{B-A}(X,Y)\\ &\longrightarrow \mathrm{HOM}_K(\mathrm{HOM}_A(M,X),\mathrm{HOM}_A(N,Y))\end{aligned} \tag{$*$}$$

given by the dg functor

$$\begin{aligned}\mathrm{HOM}_A(?,?)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}}\otimes(B-\mathrm{Dg}-A)\\ \stackrel{\rho^o\otimes\rho'}{\longrightarrow} (\mathrm{Dg}-A)^{\mathrm{op}}\otimes(\mathrm{Dg}-A)\stackrel{\mathrm{Hom}_A(?,?)}{\longrightarrow}\mathrm{Dg}-K.\end{aligned}$$

The differential in $\overline{\mathrm{HOM}}_A(P,Z)$ is the restriction of that of $\mathrm{HOM}_A(P,Z)$, for each $P\in C-\mathrm{Dg}-A$ and $Z\in B-\mathrm{Dg}-A$, and the differential on $\mathrm{HOM}_{B-C}(\mathrm{HOM}_A(M,X),\mathrm{HOM}_A(N,Y))$ is the restriction of the differential in $\mathrm{HOM}_K(\mathrm{HOM}_A(M,X),\mathrm{HOM}_A(N,Y))$. Therefore $\overline{\mathrm{HOM}}_A(?,?)$ commutes with the differentials due to the fact that the map (*) commutes with the differentials. $\square$

We want to emphasize a sort of 'dual' situation. Suppose now that X is again a dg $B-A$-bimodule and that W is a dg $B-C$-bimodule. Then the graded K-module $\mathrm{HOM}_{B^{\mathrm{op}}}(W,X)$ consisting of the morphisms $W\longrightarrow X$ in $B-\mathrm{Dg}$ should have a structure of non-unitary dg $C-A$-bimodule. Indeed, we can think of W and X as a dg $C^{\mathrm{op}}\otimes B^{\mathrm{op}}$-bimodule and a dg $A^{\mathrm{op}}-B^{\mathrm{op}}-$ bimodule, respectively. Then the first paragraph of this section says that $\mathrm{HOM}_{B^{\mathrm{op}}}(W,X)$ has a structure of non-unitary dg $A^{\mathrm{op}}-C^{\mathrm{op}}$-bimodule, which is equivalent to saying that it has a structure of non-unitary graded $C-A$-bimodule. Taking then $\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(W,X)=C\,\mathrm{HOM}_{B^{\mathrm{op}}}(W,X)A$, we get a (now unitary) dg $C-A$-bimodule. Our following result makes explicit this structure.

Corollary 8.3. *In the situation of last paragraph, the following assertions hold:*

1) *The structure of graded $C-A$-bimodule on $\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(W,X)$ is given by the rule $(cfa)(w)=(-1)^{(|c|+|a|)|w|+|c||f|}f(wc)a$, for all homogeneous elements $c\in C$, $f\in\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(W,X)$, $a\in A$ and $w\in W$.*
2) *$\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(W,X)$ consists of the $f\in\mathrm{HOM}_{B^{\mathrm{op}}}(W,X)$ such that $Im(f)\subset\oplus_{j\in J'}X\epsilon_j$, for some finite subset $J'\subseteq J$, and $f(W\nu_k)=0$ for all but finitely many $k\in\mathcal{K}$.*
3) *The assignment $(W,X)\rightsquigarrow\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(W,X)$ is the definition on objects of a* dg *functor $(B-\mathrm{Dg}-C)^{\mathrm{op}}\otimes(B-\mathrm{Dg}-A)\longrightarrow C-\mathrm{Dg}-A$.*

Proof. 1) Interpreting $\mathrm{HOM}_{B^{\mathrm{op}}}(W,X)$ as a non-unitary dg $A^{\mathrm{op}}-C^{\mathrm{op}}$-bimodule, the first paragraph of this section tells us that this structure is given by the rule $(a^o fc^o)(w)=a^o f(c^o w)$. But, by the identification of modules over a dg algebras as dg modules on the other side over the opposite dg algebra, we get that

$$\begin{aligned}(a^o fc^o)(w)&=a^o f(c^o w)=(-1)^{|c||w|}a^o f(wc)\\&=(-1)^{|c||w|}(-1)^{|f(wc)||a|}f(wc)a=(-1)^{|c||w|+|a|(|f|+|w|+|c|)}f(wc)a\end{aligned}$$

But, by analogous reason, we have an equality

$$\begin{aligned}(a^o fc^o)(w)&=(-1)^{(|a|+|f|)|c|}[c(a^o f)](w)\\&=(-1)^{(|a|+|f|)|c|}(-1)^{|a||f|}(cfa)(w)=(-1)^{|a||c|+|f||c|+|a||f|}(cfa)(w).\end{aligned}$$

Comparing these two expressions and cancelling signs appearing in both expressions, we get that $(-1)^{(|c|+|a|)|w|}f(wc)a=(-1)^{|f||c|}(cfa)(w)$, which gives the equality of assertion 1.

2) Considering the distinguished families of orthogonal idempotents $(\epsilon_j^o)_{j\in J}$ and $(\nu_k^0)_{k\in\mathcal{K}}$ in A^{op} and C^{op}, respectively, we know that $\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(W,X)=A^{\mathrm{op}}\,\mathrm{HOM}_{B^{\mathrm{op}}}(W,X)C^{\mathrm{op}}$ consists of those $f\in\mathrm{HOM}_{B^{\mathrm{op}}}(W,X)$ such that $\mathrm{Im}(f)\oplus_{j\in J'}\epsilon_j^o X$, for some finite subset $J'\subseteq J$, and $f(\nu_k^o W)=0$, for all but finitely many $k\in\mathcal{K}$. Bearing in mind that $\epsilon_j^o X=X\epsilon_j$ and $\nu_k^o W=W\nu_k$, for all $j\in J$ and $k\in\mathcal{K}$, the assertion follows.

3) is a direct consequence of Proposition 8.2. □

Let X be again a dg $B-A$-bimodule and let U be a dg $C-B$-bimodule. Then the dg K-module $U\otimes X:=U\otimes_K X$ has a canonical structure of dg $C-A$-bimodule by defining $c(u\otimes x)a=(cu)\otimes(xa)$. Clearly, this multiplication makes $U\otimes X$ into a graded $C-A$-bimodule. Moreover if $u\in U$, $x\in X$, $c\in C$ and $a\in A$ are homogeneous elements, then we have

$$\begin{aligned}d[c(u\otimes x)a]&=d(cu\otimes xa)\\&=d_U(cu)\otimes xa+(-1)^{|cu|}cu\otimes d_X(xa)\\&=d_U(cu)\otimes xa+(-1)^{|c|+|u|}cu\otimes(d_X(x)a+(-1)^{|x|}xd_A(a))\\&=d_U(cu)\otimes xa+(-1)^{|c|+|u|}cu\otimes d(x)a+(-1)^{|c|+|u|+|x|}cu\otimes xd_A(a)\\&=(d_C(c)u+(-1)^{|c|}cd_U(u))\otimes xa+(-1)^{|c|+|u|}cu\otimes d_X(x)a\\&\quad+(-1)^{|c|+|u|+|x|}cu\otimes xd_A(a)\end{aligned}$$

$$
\begin{aligned}
&= d_C(c)u \otimes xa + (-1)^{|c|}cd_U(u) \otimes xa + (-1)^{|c|+|u|}cu \otimes d_X(x)a \\
&\quad + (-1)^{|c|+|u|+|x|}cu \otimes xd_A(a) \\
&= d_C(c)u \otimes xa + (-1)^{|c|}c[d_U(u) \otimes x + (-1)^{|u|}u \otimes d_X(x)]a \\
&\quad + (-1)^{|c|+|u|+|x|}cu \otimes xd_A(a) \\
&= d_C(c)(u \otimes x)a + (-1)^{|c|}cd_{U\otimes X}(u \otimes x)a \\
&\quad + (-1)^{|c|+|u\otimes x|}c(u \otimes x)d_A(a)
\end{aligned}
$$

so that the differential of $U \otimes X$ satisfies Leibniz rule as a $C - A$-bimodule.

The K-submodule N of $U \otimes X$ generated by all differences $ub \otimes x - u \otimes bx$, where $u \in U$, $x \in X$ and $b \in B$ are homogeneous elements, is a graded $C - A-$subbimodule of $U \otimes X$. We will show that $d(N) \subseteq N$, which will imply that we get an induced graded map of degree $+1$,

$$d\colon U \otimes_B X := \frac{U \otimes X}{N} \longrightarrow \frac{U \otimes X}{N} = U \otimes_B X,$$

making $U \otimes_B X$ into a dg $C - A$-bimodule. Indeed, we leave to the reader checking the following equality, for all homogeneous elements $u \in U$, $x \in X$ and $b \in B$:

$$
\begin{aligned}
&d(ub \otimes x - u \otimes bx) \\
&\qquad = d_U(u)b \otimes x - d_U(u) \otimes bx + (-1)^{|u|}(ud_B(b) \otimes x - u \otimes d_B(b)x) \\
&\qquad\quad + (-1)^{|u|+|v|}(ub \otimes d_X(x) - u \otimes bd_X(x)).
\end{aligned}
$$

This shows that $d(ub \otimes x - u \otimes bx) \in N$ and, hence, that $d(N) \subseteq N$ as desired.

Proposition 8.4. *Let A, B and C be* dg *algebras with enough idempotents. The assignment $(U, X) \rightsquigarrow U \otimes_B X$ is the definition on objects of a* dg *functor*

$$?\otimes_B ?\colon\ (C - \mathrm{Dg} - B) \otimes (B - \mathrm{Dg} - A) \longrightarrow C - \mathrm{Dg} - A.$$

Proof. For simplicity, we denote by T the dg functor that we want to define, so that $T(U, X) = U \otimes_A X$. If now $\alpha\colon U \longrightarrow V$ and $\varphi\colon X \longrightarrow Y$ are homogeneous morphisms in $C - \mathrm{Dg} - B$ and $B - \mathrm{Dg} - A$, respectively, we define $T(\alpha \otimes \varphi)\colon U \otimes_B X \longrightarrow V \otimes_B Y$ by the rule $T(\alpha \otimes \varphi)(u \otimes x) = (-1)^{|\varphi||u|}\alpha(u) \otimes \varphi(x)$, for all homogeneous elements $u \in U$ and $x \in X$.

We first prove that $T(\alpha\otimes\varphi)$ is well-defined. If $b\in B$ is any homogeneous element, then we have

$$\begin{aligned}T(\alpha\otimes\varphi)(ub\otimes x)&=(-1)^{|\varphi|(|u|+|b|)}\alpha(ub)\otimes\varphi(x)\\&=(-1)^{|\varphi|(|u|+|b|)}\alpha(u)b\otimes\varphi(x)=(-1)^{|\varphi||u|}(-1)^{|\varphi||b|}\alpha(u)\otimes b\varphi(x)\\&=(-1)^{|\varphi||u|}\alpha(u)\otimes\varphi(bx)=T(\alpha\otimes\varphi)(u\otimes bx)\end{aligned}$$

using the facts that α is a morphism of graded right B-modules and φ is a morphism of graded left B-modules. Therefore $T(\alpha\otimes\varphi)$ is a well-defined morphism in $\mathrm{GR}-K$, and we clearly have $|T(\alpha\otimes\varphi)|=|\alpha|+|\varphi|$. It is very easy to see that $T(\alpha\otimes\varphi)$ is morphism in $\mathrm{GR}-A$. On the other hand, if $c\in C$ is a homogeneous element, then we have

$$\begin{aligned}T(\alpha\otimes\varphi)[c(u\otimes x)]&=T(\alpha\otimes\varphi)(cu\otimes x)\\&=(-1)^{|\varphi|(|c|+|u|)}\alpha(cu)\otimes\varphi(x)\\&=(-1)^{|\varphi|(|c|+|u|)}(-1)^{|\alpha||c|}c\alpha(u)\otimes\varphi(x)\\&=(-1)^{(|\alpha|+|\varphi|)|c|}(-1)^{|\varphi||u|}c(\alpha(u)\otimes\varphi(x))\\&=(-1)^{|T(\alpha\otimes\varphi)||c|}[cT(\alpha\otimes\varphi)](u\otimes x),\end{aligned}$$

bearing in mind that α is a morphism of graded left C-modules. It follows that $T(\alpha\otimes\varphi)[c(u\otimes x)]=(-1)^{|T(\alpha\otimes\varphi)||c|}[cT(\alpha\otimes\varphi)](u\otimes x)$, so that $T(\alpha\otimes\varphi)$ is also a morphism in $C-\mathrm{GR}$, and hence a morphism in $C-\mathrm{GR}-A$ (see the comments after Lemma 4.1).

We now check Conditions 2(a)–2(c) of Lemma 1.1:

Condition 2(c). Note that we have $T(\alpha\otimes 1_Z)=\alpha\otimes 1_Z$, for each dg $B-A$-bimodule Z, while $T(1_W\otimes\varphi)(w\otimes x)=(-1)^{|\varphi||w|}w\otimes\varphi(x)$, for each dg $C-B$-module W and all homogeneous elements $w\in W$ and $x\in X$. We then have

$$\begin{aligned}[T(\alpha\otimes 1_Y)\circ T(1_M\otimes\varphi)](u\otimes x)&=(-1)^{|\varphi||u|}T(\alpha\otimes 1_Y)(u\otimes\varphi(x))\\&=(-1)^{|\varphi||u|}\alpha(u)\otimes\varphi(x)=T(\alpha\otimes\varphi)(u\otimes x)\\&=(-1)^{|\varphi||u|}\alpha(u)\otimes\varphi(x)=(-1)^{|\varphi||\alpha|}T(1_V\otimes\varphi)(\alpha(u)\otimes x)\\&=(-1)^{|\varphi||\alpha|}[T(1_V\otimes\varphi)\circ T(\alpha\otimes 1_X)](u\otimes x)\end{aligned}$$

for all homogeneous elements $u\in U$ and $x\in X$. Therefore condition 2.c of the mentioned lemma is satisfied.

Condition 2(a). If U is a fixed dg $C-B$-bimodule and we consider the assignments $T_U\colon B-\mathrm{Dg}-A\longrightarrow C-\mathrm{Dg}-A$ given by $T_U(X)=U\otimes_B X$ on objects and $T_U(\varphi) = T(1_U \otimes \varphi)$ on morphisms, we need to check that T_U is a dg functor. We have $T_U(1_X) = T(1_U \otimes 1_X)\colon u \otimes x \rightsquigarrow (-1)^{|1_X||u|} u \otimes x = u \otimes x$, so that $T_U(1_X) = 1_{T_U(X)}$. Moreover, if $\varphi\colon X \longrightarrow Y$ and $\psi\colon Y \longrightarrow Z$ are homogeneous morphisms in $B - \mathrm{Dg} - A$, then we have

$$\begin{aligned}
T_U(\psi \circ \varphi)(u \otimes x) &= T(1_U \otimes (\psi \circ \varphi))(u \otimes x) \\
&= (-1)^{(|\varphi|+|\psi|)|u|} u \otimes (\psi \circ \varphi)(x) = (-1)^{|\varphi||u|}(-1)^{|\psi||u|} u \otimes (\psi \circ \varphi)(x) \\
&= (-1)^{|\varphi||u|} T(1_U \otimes \psi)(u \otimes \varphi(x)) = [T(1_U \otimes \psi) \circ T(1_U \otimes \varphi)](u \otimes x) \\
&= [T_U(\psi) \circ T_U(\varphi)](u \otimes x)
\end{aligned}$$

It then follows that T_U is a graded functor. We need to see that it commutes with the differentials, which means that the diagram

$$\begin{tikzcd}
\mathrm{HOM}_{B-A}(X,Y) \arrow[r, "d"] \arrow[d, "T_U"] & \mathrm{HOM}_{B-A}(X,Y) \arrow[d, "T_U"] \\
\mathrm{HOM}_{C-A}(U \otimes_B X, U \otimes_B Y) \arrow[r, "\delta"] & \mathrm{HOM}_{C-A}(U \otimes_B X, U \otimes_B Y)
\end{tikzcd}$$

commutes, where d and δ are the differentials on Hom spaces of $B - \mathrm{Dg} - A$ and $C - \mathrm{Dg} - A$, respectively. We fix any homogeneous element $\varphi \in \mathrm{HOM}_{B-A}(X,Y)$ and shall prove that $(\delta \circ T_U)(\varphi) = (T_U \circ d)(\varphi)$. Letting act the two members of the desired equality on $u \otimes x$, where $u \in U$ and $x \in X$ are homogeneous elements, we get:

$$\begin{aligned}
[(\delta \circ T_U)(\varphi)](u \otimes x) &= [d_{U \otimes_B Y} \circ T_U(\varphi) - (-1)^{|\varphi|} T_U(\varphi) \circ d_{U \otimes_B X}](u \otimes x) \\
&= (-1)^{|\varphi||u|} d_{U \otimes_B Y}(u \otimes \varphi(x)) - (-1)^{|\varphi|} T_U(\varphi)(d_U(u) \otimes x \\
&\quad + (-1)^{|u|} u \otimes d_X(x)) \\
&= (-1)^{|\varphi||u|}[d_U(u) \otimes \varphi(x) + (-1)^{|u|} u \otimes d_Y(\varphi(x))] \\
&\quad - (-1)^{|\varphi|}[(-1)^{|\varphi|(|u|+1)} d_U(u) \otimes \varphi(x) \\
&\quad + (-1)^{|u|}(-1)^{|\varphi||u|} u \otimes \varphi(d_X(x))] \\
&= (-1)^{(|\varphi|+1)|u|} u \otimes d_Y(\varphi(x)) - (-1)^{(|\varphi|+1)|u|}(-1)^{|\varphi|} u \otimes \varphi(d_X(x)) \\
&= (-1)^{(|\varphi|+1)|u|} u \otimes d(\varphi)(x) = T(1_U \otimes d(\varphi))(u \otimes x) \\
&= [(T_U \circ d)(\varphi)](u \otimes x).
\end{aligned}$$

Condition 2.(b) Let us fix a dg $B - A$-bimodule X and consider the assignments $T^X = ? \otimes_B X\colon (C - \mathrm{Dg} - B) \longrightarrow C - \mathrm{Dg} - A$ given on objects by $U \rightsquigarrow U \otimes_B X$ and on morphisms by $\alpha \rightsquigarrow T(\alpha \otimes 1_X) =$

$\alpha \otimes 1_X$. It is straightforward to see that $T^X(\beta \circ \alpha) = T^X(\beta) \circ T^X(\alpha)$, whenever α and β are composable morphisms in $C - \mathrm{Dg} - B$, and that $T^X(1_U) = 1_{T^X(U)}$, so that we have a graded functor $C - \mathrm{Dg} - B \longrightarrow C - \mathrm{Dg} - A$. It remains to see that T^X commutes with the differentials. For that, we fix arbitrary dg $C - B$-bimodules U and V and denote by $d\colon \mathrm{Hom}_B(U,V) \longrightarrow \mathrm{Hom}_B(U,V)$ and $\delta\colon \mathrm{Hom}_A(U \otimes_B X, V \otimes_B X) \longrightarrow \mathrm{Hom}_A(U \otimes_B X, V \otimes_B X)$ the respective differentials on Hom spaces. We need to check that $(? \otimes_B X)(d(\alpha)) = \delta[(? \otimes_B X)(\alpha)]$. That is, we need to check that $d(\alpha) \otimes 1_X = \delta(\alpha \otimes 1_X)$, for any homogeneous element $\alpha \in \mathrm{Hom}_B(U,V)$. But if $u \in U$ and $x \in X$ are homogeneous elements, then we have an equality

$$\begin{aligned}
&\delta(\alpha \otimes 1_X)(u \otimes x)\\
&\quad = [d_{V\otimes_B X} \circ (\alpha \otimes 1_X) - (-1)^{|\alpha\otimes 1_X|}(\alpha \otimes 1_X) \circ d_{U\otimes_B X}](u \otimes x)\\
&\quad = d_{V\otimes_B X}(\alpha(u) \otimes x) - (-1)^{|\alpha|}(\alpha \otimes 1_X)[d_U(u) \otimes x\\
&\qquad + (-1)^{|u|} u \otimes d_X(x)]\\
&\quad = d_V(\alpha(u)) \otimes x + (-1)^{|\alpha(u)|}\alpha(u) \otimes d_X(x) - (-1)^{|\alpha|}\alpha(d_U(u)) \otimes x\\
&\qquad - (-1)^{|\alpha|+|u|}\alpha(u) \otimes d_X(x)\\
&\quad = d_V(\alpha(u)) \otimes x - (-1)^{|\alpha|}\alpha(d_U(u)) \otimes x\\
&\quad = (d_V \circ \alpha - (-1)^{|\alpha|}\alpha \circ d_U)(u) \otimes x = d(\alpha)(u) \otimes x\\
&\quad = (d(\alpha) \otimes 1_X)(u \otimes x). \qquad \square
\end{aligned}$$

9. The classical dg adjunctions

In this section we show that the classical tensor-Hom adjunction and the adjunction between contravariant Hom functors for module categories over rings with unit can be extended to the dg setting.

Theorem 9.1. *Let A, B and C be* dg *algebras with enough idempotents and let X be a* dg *$B - A-$bimodules. The pair*

$$\begin{aligned}
(? \otimes_B X \colon C - \mathrm{Dg} - B \longrightarrow C - \mathrm{Dg} - A,\\
\overline{\mathrm{HOM}}_A(X, ?) \colon C - \mathrm{Dg} - A \longrightarrow C - \mathrm{Dg} - B)
\end{aligned}$$

is a dg *adjunction. As a consequence, we have an adjunction*

$$\begin{aligned}
(? \otimes^{\mathbb{L}}_B X \colon \mathcal{D}(B \otimes C^{\mathrm{op}}) \longrightarrow \mathcal{D}(A \otimes C^{\mathrm{op}}),\\
\mathbb{R}\,\mathrm{Hom}_A(X, ?) \colon \mathcal{D}(A \otimes C^{\mathrm{op}}) \longrightarrow \mathcal{D}(B \otimes C^{\mathrm{op}}))
\end{aligned}$$

of triangulated functors, where $? \otimes_B^{\mathbb{L}} := \mathbb{L}(? \otimes_B X)$ *and* $\mathbb{R}\operatorname{Hom}_A(X, ?) := \mathbb{R}(\overline{\mathrm{HOM}}_A(X, ?))$.

Proof. As usual, we fix distinguished families of orthogonal idempotents $(\epsilon_j)_{j\in J}$, $(e_i)_{i\in I}$ and $(\nu_k)_{k\in\mathcal{K}}$ in A, B and C, respectively. Whenever U and M are objects in $C-\mathrm{Dg}-B$ and $C-\mathrm{Dg}-A$, respectively, we define

$$\eta_{U,M}\colon \mathrm{HOM}_{C-A}(U \otimes_B X, M) \longrightarrow \mathrm{HOM}_{C-B}(U, \overline{\mathrm{HOM}}_A(X, M))$$

by the rule $[\eta_{U,M}(f)(u)](x) = f(u \otimes x)$, for all homogeneous elements $f \in \mathrm{HOM}_{C-A}(U \otimes_B X, M)$, $u \in U$ and $x \in X$. We need to check that η is well-defined. We start by checking that $\eta(f)(u) \in \overline{\mathrm{HOM}}_A(X, M)$. If $a \in A$ is any homogeneous element, then we have

$$[\eta(f)(u)](xa) = f(u \otimes xa) = f(u \otimes x)a = [\eta(f)(u)](x)a,$$

so that $\eta(f)(u)$ is a homogeneous element of $\mathrm{HOM}_A(X, M)$. Moreover if $i \in I$ is such that $ue_i = 0$, then we have that $\eta(f)(u)(e_iX) = f(u\otimes e_iX) = 0$, because $f(u\otimes e_ix) = f(ue_i\otimes x) = 0$. It then follows that $\eta(f)(u)$ vanishes on all but finitely many e_iX. On the other hand, we know that there is a finite subset $F \subset \mathcal{K}$ such that $\nu_k u = 0$, for all $k \in \mathcal{K} \setminus F$. It then follows that

$$[(\nu_k\eta(f))(u)](x) = \nu_k f(u \otimes x) = f(\nu_k u \otimes x) = 0,$$

for all $k \in \mathcal{K} \setminus F$. It follows that $\operatorname{Im}(\eta(f)(u)) \subseteq \sum_{k\in F} \nu_k M$. Therefore we get that $\eta(f)(u) \in \overline{\mathrm{HOM}}_A(X, M)$.

We next check that $\eta(f)\colon U \longrightarrow \overline{\mathrm{HOM}}_A(X, M)$ is a morphism of dg $C-B$-bimodules. We need to check that

$$\eta(f)(cub) = (-1)^{|c||\eta(f)|}c(\eta(f)(u))b = (-1)^{|c||f|}c(\eta(f)(u))b,$$

for all homogeneous elements $c \in C$, $u \in U$ and $b \in B$ (see the comments after Lemma 4.1). Indeed we have

$$\begin{aligned}[\eta(f)(cub)](x) &= f(cub \otimes x) = (-1)^{|c||f|}cf(u \otimes bx)\\ &= (-1)^{|c||f|}c[\eta(f)(u)](bx) = (-1)^{|c||f|}[c(\eta(f)(u))b](x),\end{aligned}$$

due to the definition of the structure of $C-B$-bimodule on $\overline{\mathrm{HOM}}_A(X, M)$. It follows that $\eta = \eta_{U,M}$ is well-defined.

For the naturality of η, recall that $\mathrm{HOM}_A(?, M)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow \mathrm{Dg}-K$ takes a homogeneous morphism $\alpha\colon N \longrightarrow N'$ in $C-\mathrm{Dg}-A$ to the map

$$\alpha^* = \mathrm{HOM}_A(\alpha^o, M)\colon \mathrm{HOM}_A(N', M) \longrightarrow \mathrm{HOM}_A(N, M)$$

given by $\alpha^*(\beta) = (-1)^{|\alpha||\beta|}\beta \circ \alpha$ (see the proof of Proposition 8.2). A similar fact is true for $\mathrm{HOM}_{C-B}(?, W)\colon (C - \mathrm{Dg} - B)^{\mathrm{op}} \longrightarrow \mathrm{Dg} - K$, for any dg $C - B$-bimodule W. With this in mind, if $\varphi\colon U \longrightarrow V$ is a homogeneous morphism in $C - \mathrm{Dg} - B$, then we have that

$$[\varphi^* \circ \eta_{V,M}](g) = (-1)^{|\eta(g)||\varphi|}\eta_{V,M}(g) \circ \varphi = (-1)^{|g||\varphi|}\eta_{V,M}(g) \circ \varphi,$$

for each homogeneous element $g \in \mathrm{HOM}_{C-A}(V \otimes_B X, M)$. On the other hand, we have

$$[\eta_{U,M} \circ (\varphi \otimes 1_X)^*](g) = (-1)^{|g||\varphi|}\eta_{U,M}(g \circ (\varphi \otimes 1_X)).$$

Taking homogeneous elements $u \in U$ and $x \in X$, we then have

$$\begin{aligned}
&[(\varphi^* \circ \eta_{V,M})(g)](u)(x) = (-1)^{|g||\varphi|}[\eta_{V,M}(g) \circ \varphi](u)(x) \\
&\quad = (-1)^{|g||\varphi|}g(\varphi(u) \otimes x) = (-1)^{|g||\varphi|}[g \circ (\varphi \otimes 1_X)](u \otimes x) \\
&\quad = (-1)^{|g||\varphi|}\eta_{U,M}(g \circ (\varphi \otimes 1_X))(u)(x) = [(\eta_{U,M} \circ (\varphi \otimes 1_X)^*)(g)](u)(x).
\end{aligned}$$

This shows that $\varphi^* \circ \eta_{V,M} = \eta_{U,M} \circ (\varphi \otimes 1_X)^*$, which proves the naturality of η on the variable U. The naturality on the variable M is shown as in the classical (ungraded) context.

It remains to prove that η commutes with the differentials. For this, we denote by $d\colon \mathrm{HOM}_{C-A}(U \otimes_B X, M) \longrightarrow \mathrm{HOM}_{C-A}(U \otimes_B X, M)$ and $\delta\colon \mathrm{HOM}_{C-B}(U, \overline{\mathrm{HOM}}_A(X, M)) \longrightarrow \mathrm{HOM}_{C-B}(U, \overline{\mathrm{HOM}}_A(X, M))$ the respective differentials on Hom spaces in the dg categories $C - \mathrm{Dg} - A$ and $C - \mathrm{Dg} - B$, respectively. We need to prove that $\delta(\eta(f)) = \eta(d(f))$, for each homogeneous element $f \in \mathrm{HOM}_{C-A}(U \otimes_B X, M)$. If $u \in U$ and $x \in X$ are homogeneous elements, then we have:

$$\begin{aligned}
&[\delta(\eta(f))](u)(x) = [d_{\overline{\mathrm{HOM}}_A(X,M)} \circ \eta(f) - (-1)^{|\eta(f)|}\eta(f) \circ d_U](u)(x) \\
&\quad = [d_{\overline{\mathrm{HOM}}_A(X,M)}(\eta(f)(u)) - (-1)^{|f|}\eta(f)(d_U(u))](x) \\
&\quad = [d_M \circ \eta(f)(u) - (-1)^{|u|+|f|}\eta(f)(u) \circ d_X - (-1)^{|f|}\eta(f)(d_U(u))](x) \\
&\quad = d_M(f(u \otimes x)) - (-1)^{|u|+|f|}f(u \otimes d_X(x)) - (-1)^{|f|}f(d_U(u) \otimes x) \\
&\quad = d_M(f(u \otimes x)) - (-1)^{|f|}f(d_U(u) \otimes x + (-1)^{|u|}u \otimes d_X(x)) \\
&\quad = d_M(f(u \otimes x)) - (-1)^{|f|}f(d_{U \otimes_B X}(u \otimes x)) \\
&\quad = (d_M \circ f - f \circ d_{U \otimes_B X})(u \otimes x) = d(f)(u \otimes x) = \eta(d(f))(u)(x).
\end{aligned}$$

Therefore we have $\delta(\eta(f)) = \eta(d(f))$, as desired. □

We now consider a particular case of the last adjunction. Let $\iota\colon B \longrightarrow A$ be a homomorphism of dg algebras with enough idempotents. All throughout the rest of the paper, we assume that such a homomorphism makes A into a (unitary!) $B-B$-bimodule (equivalently, that $A = \iota(B)A\iota(B)$). This means that if $(e_i)_{i\in I}$ is any distinguished family of orthogonal idempotents of B then, after deleting those $\iota(e_i)$ which are zero, the family $(\iota(e_i)i \in I)$ is a distinguished family of orthogonal idempotents of A. Note that we have an obvious *restriction of scalars functor* $\iota_*\colon C - \mathrm{Dg} - A \longrightarrow C - \mathrm{Dg} - B$, which is clearly a dg functor that preserves acyclic and contractible dg modules. In particular, we have $\mathbb{R}\iota_* = \iota_*$ (see Remark 7.11). We can apply the last proposition to the bimodule $X = {}_BA_A$. But note the following:

Lemma 9.2. *In the situation of preceding paragraph, consider the* dg *functor*
$$\overline{\mathrm{HOM}}_A(A,?)\colon C - \mathrm{Dg} - A \longrightarrow C - \mathrm{Dg} - B.$$
There is a natural isomorphism of dg *functors* $\iota_* \cong \overline{\mathrm{HOM}}_A(A,?)$. *As a consequence, there is a natural isomorphism of triangulated functors*
$$\mathbb{R}\iota_* = \iota_* \cong \mathbb{R}\operatorname{Hom}_A(A,?)\colon \mathcal{D}(A \otimes C^{\mathrm{op}}) \xrightarrow{\cong} \mathcal{D}(B \otimes C^{\mathrm{op}}).$$

Proof. Recall that if M is a dg $C-A$-bimodule, then $\overline{\mathrm{HOM}}_A(A,M)$ consists of the morphisms $f\colon A \longrightarrow M$ in $\mathrm{Dg} - A$ such that $f(e_iA) = 0$, for all but finitely many $i \in I$, and $\mathrm{Im}(f) \subseteq \bigoplus_{k\in F} \nu_k M$, for some finite subset $F \subset \mathcal{K}$. If $m \in M$ is any homogeneous element and we consider the homogeneous morphism $\lambda_m\colon A \longrightarrow M$ in $\mathrm{GR} - A$ given by $\lambda_m(a) = ma$, then $\lambda_m \in \overline{\mathrm{HOM}}_A(A,M)$. Indeed since $me_i = 0$, for all but finitely many $i \in I$, we get that also $\lambda_m(e_iA) = me_iA = 0$, for all but finitely many $i \in I$. On the other hand, since there is a finite subset $F \subset \mathcal{K}$ such that $\nu_k m = 0$, for all $k \in \mathcal{K} \setminus F$, we get that $\mathrm{Im}(\lambda_m) = mA \subseteq \oplus_{k\in F}\nu_k M$.

The induced map $\lambda_M\colon M \longrightarrow \overline{\mathrm{HOM}}_A(A,M)$ is clearly a morphism in $C - \mathrm{GR} - B$. Defining $\Psi\colon \overline{\mathrm{HOM}}_A(A,M) \longrightarrow M$ by the rule $\Psi(f) = \sum_i f(e_i)$, we get an inverse for λ in $C - \mathrm{GR} - B$. Then, for each M in $C - \mathrm{Dg} - A$, we have a morphism of degree zero $\lambda_M\colon \iota_*(M) \longrightarrow \overline{\mathrm{HOM}}_A(A,M)$ in $C - \mathrm{Dg} - B$. To check that, when M varies, we get a bijective natural transformation of dg functors $\lambda\colon \iota_* \longrightarrow \overline{\mathrm{HOM}}_A(A,?)$ is easy and left to the reader. In order to see that we have a natural isomorphism of dg functors we just need to check that λ is a homological natural transformation, which amounts to check that if $d\colon \mathrm{HOM}_{C-B}(M, \overline{\mathrm{HOM}}_A(A,M)) \longrightarrow \mathrm{HOM}_{C-B}(M, \overline{\mathrm{HOM}}_A(A,M))$ is the differential on Hom spaces of the dg category $C-\mathrm{Dg}-B$, then $d(\lambda_M)=0$. To see this, for each homogeneous element $m \in M$, we have:

$$\begin{aligned}[d(\lambda_M)](m) &= [d_{\overline{\mathrm{HOM}}_A(A,M)} \circ \lambda_M - (-1)^{|\lambda_M|}\lambda_M \circ d_M](m) \\ &= d_{\overline{\mathrm{HOM}}_A(A,M)}(\lambda_M(m)) - \lambda_M(d_M(m)) \\ &= d_M \circ \lambda_M(m) - (-1)^{|m|}\lambda_M(m) \circ d_A - \lambda_M(d_M(m)).\end{aligned}$$

When applying both members of this equality to a homogeneous element $a \in A$, we get that

$$\{[d(\lambda_M)](m)\}(a) = d_M(ma) - (-1)^{|m|} m d_A(a) - d_M(m)a = 0.$$

The corresponding natural isomorphism of triangulated functors follows from Proposition 7.12. □

This justifies the following terminology:

Definition 9.3. If $\iota\colon B \longrightarrow A$ is a homomorphism of dg algebras with enough idempotents as above, then the dg functor $? \otimes_B A\colon C-\mathrm{Dg}-B \longrightarrow C-\mathrm{Dg}-A$ is called the *extension of scalars functor* associated to ι. It is denoted by $\iota^*\colon C-\mathrm{Dg}-B \longrightarrow C-\mathrm{Dg}-A$.

As an immediate consequence of Theorem 9.1 and Lemma 9.2, we get:

Corollary 9.4. *Let $\iota\colon B \longrightarrow A$ a homomorphism of* dg *algebras as above. The pair $(\iota^*\colon C-\mathrm{Dg}-B \longrightarrow C-\mathrm{Dg}-A, \iota_*\colon C-\mathrm{Dg}-A \longrightarrow C-\mathrm{Dg}-B)$ is a* dg *adjunction. Therefore we have an adjoint pair of triangulated functors $(\mathbb{L}\iota^*\colon \mathcal{D}(B\otimes C^{\mathrm{op}}) \longrightarrow \mathcal{D}(A\otimes C^{\mathrm{op}}), \mathbb{R}\iota_* = \iota_*\colon \mathcal{D}(A\otimes C^{\mathrm{op}}) \longrightarrow \mathcal{D}(B\otimes C^{\mathrm{op}}))$.*

We move now to study a less known adjunction.

Theorem 9.5. *Let A and B be* dg *algebras with enough idempotents and let X be a* dg *$B-A$-bimodule. The pair*

$$\begin{aligned}(\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(?,X)^o\colon B-\mathrm{Dg}-C \longrightarrow (C-\mathrm{Dg}-A)^{\mathrm{op}}, \\ \overline{\mathrm{HOM}}_A(?,X)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow B-\mathrm{Dg}-C)\end{aligned}$$

is a dg *adjunction. In particular, the pair*

$$\begin{aligned}(\mathbb{R}\,\mathrm{Hom}_{B^{\mathrm{op}}}(?,X)^o\colon \mathcal{D}(C\otimes B^{\mathrm{op}}) \longrightarrow \mathcal{D}(A\otimes C^{\mathrm{op}})^{\mathrm{op}}, \\ \mathbb{R}\,\mathrm{Hom}_A(?,X)\colon \mathcal{D}(A\otimes C^{\mathrm{op}})^{\mathrm{op}} \longrightarrow \mathcal{D}(C\otimes B^{\mathrm{op}}))\end{aligned}$$

is an adjoint pair of triangulated functors, where $\mathbb{R}\,\mathrm{Hom}(?,X) := \mathbb{R}(\overline{\mathrm{HOM}}(?,X))$ in both cases.

Proof. All throughout the proof, we fix distinguished families of orthogonal idempotents $(e_i)_{i\in I}$, $(\epsilon_j)_{j\in J}$ and $(\nu_k)_{k\in\mathcal{K}}$ in B, A and C, respectively. Let U be a dg $B-C$-bimodule and M be a dg $C-A$-bimodule. By the initial paragraph of Section 8 and by Corollary 8.3, we have:

a) $\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X)$ consists of the $f\in \mathrm{HOM}_{B^{\mathrm{op}}}(U,X)$ such that $\mathrm{Im}(f)\subseteq \oplus_{j\in J'}X\epsilon_j$, for some finite subset $J'\subset J$, and $f(U\nu_k)=0$ for all but finitely many $k\in\mathcal{K}$.

b) $\overline{\mathrm{HOM}}_{A}(M,X)$ consists of the $g\in \mathrm{HOM}_{A}(M,X)$ such that $\mathrm{Im}(g)\subseteq \oplus_{i\in I'}e_iX$, for some finite subset $I'\subset I$, and $g(\nu_kM)=0$ for all but finitely many $k\in\mathcal{K}$.

We define a K-linear map

$$\begin{array}{ccc}
\mathrm{Hom}_{(C-\mathrm{Dg}-A)^{\mathrm{op}}}(\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X),M) & & \mathrm{Hom}_{B-\mathrm{Dg}-C}(U,\overline{\mathrm{HOM}}_{A}(M,X)) \\
\| & & \| \\
\mathrm{HOM}_{C-A}(M,\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X)) & \xrightarrow{\xi=\xi_{U,M}} & \mathrm{HOM}_{B-C}(U,\overline{\mathrm{HOM}}_{A}(M,X))
\end{array}$$

by the rule $[\xi(f)(u)](m)=(-1)^{|u||m|}f(m)(u)$, for all homogeneous elements $u\in U$, $m\in M$, and $f\in \mathrm{HOM}_{C-A}(M,\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X))$. We first check that if f and u are fixed, then the assignment $m\rightsquigarrow[\xi(f)(u)](m)=(-1)^{|u||m|}f(m)(u)$ gives a homogeneous morphism $M\longrightarrow X$ in $\mathrm{GR}-A$. Indeed we have

$$[\xi(f)(u)](ma)=(-1)^{|u|(|m|+|a|)}f(ma)(u)=(-1)^{|u|(|m|+|a|)}[f(m)a](u).$$

But, by the structure of right dg A-module on $\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X)$ (see Corollary 8.3), we see $[f(m)a](u)=(-1)^{|a||u|}f(m)(u)a$, hence $[\xi(f)(u)](ma)=(-1)^{|u||m|}f(m)(u)a$. On the other hand, we get $[\xi(f)(u)](m)a=(-1)^{|u||m|}f(m)(u)a$. This shows that $\xi(f)(u)$ is a homogeneous morphism $M\longrightarrow X$ in $\mathrm{GR}-A$.

In order to check that ξ is well-defined, we also need to see that the just defined morphism $\xi(f)(u)$ is really in $\overline{\mathrm{HOM}}_{A}(M,X)$ (see point b) above). We have $u=\sum_{i\in F_u}e_iu$, for some finite subset $F_u\subseteq I$, and then $[f(m)](u)=\sum_{i\in F_u}e_if(m)(u)\in\bigoplus_{i\in F_u}e_iX$, for all $m\in M$, using the fact that $f(m)$ is a morphism of graded left B-modules. That is, we have that $\mathrm{Im}(\xi(f)(u))\subseteq\oplus_{i\in F_u}e_iX$. On the other hand, we have a finite subset $\mathcal{K}'\subset\mathcal{K}$ such that $u\nu_k=0$, for all $k\in\mathcal{K}\setminus\mathcal{K}'$. Bearing in mind the explicit definition of the $C-A$-bimodule structure on $\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X)$ (see Corollary 8.3) and the fact that f is a morphism of $C-A-$bimodules, we get that

$$\begin{aligned}
[\xi(f)(u)](\nu_km)&=(-1)^{|u||\nu_km|}f(\nu_km)(u)\\
&=(-1)^{|u||m|}(\nu_kf(m))(u)=(-1)^{|u||m|}(-1)^{|\nu_k||u|+|\nu_k||f(m)|}f(m)(u\nu_k),
\end{aligned}$$

which implies that $[\xi(f)(u)](\nu_k m) = 0$, for all $k \in \mathcal{K} \setminus \mathcal{K}'$, independently of m. It follows that $\xi(f)(u)](\nu_k M) = 0$ for almost all $k \in \mathcal{K}$, so that $\xi(f)(u) \in \overline{\mathrm{HOM}}_A(M, X)$.

As a final step to check that ξ is well-defined, we will see that $\xi(f)$ is really a morphism $U \longrightarrow \overline{\mathrm{HOM}}_A(M, X)$ in $B - \mathrm{GR} - C$. That is, we need to check the equalities $\xi(f)(bu) = (-1)^{|f||b|} b\xi(f)(u)$ (see the comments after Lemma 4.1) and $\xi(f)(uc) = \xi(f)(u)c$, for all homogeneous elements $b \in B$, $u \in U$ and $c \in C$. We apply both members of the first desired equality to a homogeneous element $m \in M$ and get:

$$
\begin{aligned}
&[\xi(f)(bu)](m) \\
&\qquad = (-1)^{(|b|+|u|)|m|} f(m)(bu) = (-1)^{(|b|+|u|)|m|} (-1)^{|f(m)||b|} bf(m)(u) \\
&\qquad = (-1)^{(|b|+|u|)|m|} (-1)^{((||f|+m|)|b|} bf(m)(u) = (-1)^{|u||m|+|f||b|} bf(m)(u) \\
&\qquad = (-1)^{|f||b|} (-1)^{|u||m|} bf(m)(u) = (-1)^{|f||b|} b[\xi(f)(u)(m)] \\
&\qquad = (-1)^{|f||b|} [b\xi(f)(u)](m),
\end{aligned}
$$

using that $f(m)$ is a morphism in $B - \mathrm{GR}$ of degree $|f| + |m|$. On the other hand, we have

$$
[\xi(f)(uc)](m) = (-1)^{|uc||m|} f(m)(uc) = (-1)^{|u||m|+|c||m|} f(m)(uc)
$$

while, using Corollary 8.3 and the comments after Lemma 4.1, we also get

$$
\begin{aligned}
&[\xi(f)(u)c](m) = \xi(f)(u)(cm) \\
&\qquad = (-1)^{|u||cm|} f(cm)(u) = (-1)^{|u||cm|} (-1)^{|c||f|} [cf(m)](u) \\
&\qquad = (-1)^{|u||cm|+|c||f|} (-1)^{|c||u|+|c||f(m)|} f(m)(uc) \\
&\qquad = (-1)^{|u||m|+|c||m|} f(m)(uc).
\end{aligned}
$$

We now prove the naturality of ξ on both variables. Let $\alpha \colon M \longrightarrow N$ be a homogeneous morphism in $C - \mathrm{Dg} - A$. With the obvious meaning of the vertical arrows, we need to prove that the following diagram is commutative:

$$
\begin{tikzcd}
\mathrm{Hom}_{C-A}(M, \overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U, X)) \arrow[r, "\xi_{U,M}"] & \mathrm{Hom}_{B-C}(U, \overline{\mathrm{HOM}}_A(M, X)) \\
\mathrm{Hom}_{C-A}(N, \overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U, X)) \arrow[r, "\xi_{U,N}"] \arrow[u, "\alpha^*"] & \mathrm{Hom}_{B-C}(U, \overline{\mathrm{HOM}}_A(N, X)) \arrow[u, "\overline{\mathrm{HOM}}_A(\alpha, X)_*"']
\end{tikzcd}
$$

For this, we take any homogeneous morphism $g\colon N \longrightarrow \overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U, X)$ in $C - \mathrm{Dg} - A$. We then have that $(\xi_{U,M} \circ \alpha^*)(g) = (-1)^{|\alpha||g|}\xi_{U,M}(g \circ \alpha)$ and therefore

$$\begin{aligned}
&[(\overline{\mathrm{HOM}}_A(\alpha, X)_* \circ \xi_{U,N})(g)(u)](m)\\
&\quad = [(\overline{\mathrm{HOM}}_A(\alpha, X) \circ \xi(g))(u)](m) = (-1)^{|\alpha||\xi(g)(u)|}[\xi(g)(u) \circ \alpha](m)\\
&\quad = (-1)^{|\alpha|(|g|+|u|)}\xi(g)(u)(\alpha(m)) = (-1)^{|\alpha|(|g|+|u|)}(-1)^{|u||\alpha(m)|}g(\alpha(m))(u)\\
&\quad = (-1)^{|\alpha|(|g|+|u|)}(-1)^{|u|(|\alpha|+|m|)}(g \circ \alpha)(m)(u)\\
&\quad = (-1)^{|\alpha||g|+|u||m|}(g \circ \alpha)(m)(u) = (-1)^{|\alpha||g|}\xi_{U,M}(g \circ \alpha)(u)(m)\\
&\quad = [(\xi_{U,M} \circ \alpha^*)(g)(u)](m),
\end{aligned}$$

which proves the naturality of ξ on the variable M.

Let now $\varphi\colon U \longrightarrow V$ be a homogeneous morphism in $B - \mathrm{Dg} - C$. For the naturality of ξ on the 'variable' U, we need to check that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{C-A}(M, \overline{HOM}_{B^{\mathrm{op}}}(U, X)) \arrow[r, "\xi_{U,M}"] & \mathrm{Hom}_{B-C}(U, \overline{HOM}_A(M, X)) \\
\mathrm{Hom}_{C-A}(M, \overline{HOM}_{B^{\mathrm{op}}}(V, X)) \arrow[r, "\xi_{V,M}"] \arrow[u, "\overline{HOM}_{B^{\mathrm{op}}}(\varphi, X)_*"'] & \mathrm{Hom}_{B-C}(V, \overline{HOM}_A(M, X)) \arrow[u, "\varphi^*"]
\end{tikzcd}$$

Let $f\colon M \longrightarrow \overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(V, X)$ be a homogeneous morphism in $C-\mathrm{Dg}-A$. Then we have that $[\xi_{U,M} \circ \overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(\varphi, X)_*](f) = \xi_{U,M}(\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(\varphi, X) \circ f)$, while $(\varphi^* \circ \xi_{V,M})(f) = (-1)^{|f||\varphi|}\xi_{V,M}(f) \circ \varphi$. If now $u \in U$ and $m \in M$ are homogeneous elements, then we have equalities:

$$\begin{aligned}
&[\xi_{U,M}(\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(\varphi, X) \circ f)(u)](m)\\
&\quad = (-1)^{|u||m|}[(\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(\varphi, X) \circ f)(m)](u)\\
&\quad = (-1)^{|u||m|}(-1)^{|\varphi||f(m)|}[f(m) \circ \varphi](u)\\
&\quad = (-1)^{|u||m|+|\varphi||f|+|\varphi||m|}f(m)(\varphi(u))\\
&\quad = (-1)^{|f||\varphi|}(-1)^{|\varphi(u)||m|}f(m)(\varphi(u))\\
&\quad = (-1)^{|f||\varphi|}[\xi_{V,M}(f)(\varphi(u))](m)\\
&\quad = (-1)^{|f||\varphi|}[(\xi_{V,M}(f) \circ \varphi)(u)](m)
\end{aligned}$$

which proves the naturality of ξ on the 'variable' U.

Next we check that ξ commutes with the differentials. That is, we need to prove that, for each dg $C-A$-bimodule M and each dg $B-C$-bimodule U, the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{HOM}_{C-A}(M,\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X)) \arrow[r, "d"] \arrow[d, "\xi_{U,M}"] & \mathrm{HOM}_{C-A}(M,\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X)) \arrow[d, "\xi_{U,M}"] \\
\mathrm{HOM}_{B-C}(U,\overline{\mathrm{HOM}}_{A}(M,X)) \arrow[r, "\delta"] & \mathrm{HOM}_{B-C}(U,\overline{\mathrm{HOM}}_{A}(M,X)),
\end{tikzcd}$$

Here d and δ are the differentials on Hom spaces in $C-\mathrm{Dg}-A$ and $B-\mathrm{Dg}-C$, respectively. Put $\xi=\xi_{U,M}$ for simplicity. Then, for all homogeneous elements $f\in \mathrm{HOM}_{C-A}(M,\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X))$, $u\in U$ and $m\in M$, we have

$$\begin{aligned}
[(\xi\circ d)(f)(u)](m) &= [\xi(d(f))(u)][m]=(-1)^{|u||m|}d(f)(m)(u)\\
&=(-1)^{|u||m|}[(d_{\overline{\mathrm{HOM}}(U,X)}\circ f-(-1)^{|f|}f\circ d_M)(m)](u)\\
&=(-1)^{|u||m|}[d_X\circ f(m)-(-1)^{|f(m)|}f(m)\circ d_U\\
&\quad -(-1)^{|f|}f(d_M(m))](u)\\
&=(-1)^{|u||m|}[d_{\overline{\mathrm{HOM}}(U,X)}(f(m))-(-1)^{|f|}f(d_M(m))](u)\\
&=(-1)^{|u||m|}[d_X(f(m)(u))-(-1)^{|f|}f(d_M(m))(u)\\
&\quad -(-1)^{|f|+|m|}f(m)(d_U(u))]\\
&=(-1)^{|u||m|}d_X(f(m)(u))-(-1)^{|f|+|u|}(-1)^{|u|(|m|+1)}f(d_M(m))(u)\\
&\quad -(-1)^{|f|}(-1)^{(|u|+1)|m|}f(m)(d_U(u))\\
&=[d_X\circ\xi(f)(u)-(-1)^{|\xi(f)(u)|}\xi(f)(u)\circ d_M\\
&\quad -(-1)^{|f|}\xi(f)(d_U(u))](m)\\
&=[d_{\overline{\mathrm{HOM}}(M,X)}(\xi(f)(u))-(-1)^{|f|}\xi(f)(d_U(u))](m)\\
&=[(\delta\circ\xi)(f)(u)][m].
\end{aligned}$$

where we used that $(\delta\circ\xi)(f)=d_{\overline{\mathrm{HOM}}(M,X)}\circ\xi(f)-(-1)^{|\xi(f)|}\xi(f)\circ d_U$ in the last equality.

Finally, in order to prove the bijective condition of ξ, note that exchanging the roles of U and X and of A and B^{op}, one has a well-defined K-linear map of degree zero $\xi'=\xi'_{U,M}\colon \mathrm{HOM}_{B-C}(U,\overline{\mathrm{HOM}}_A(M,X))\longrightarrow \mathrm{HOM}_{C-A}(M,\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(U,X))$, given by the rule

$$[\xi'(g)(m)](u)=(-1)^{|u||m|}g(u)(m).$$

Clearly $\xi'_{M,U}$ is inverse to $\xi_{M,U}$. □

We will now show that, under appropriate assumptions, the derived functors of covariant and contravariant $\overline{\mathrm{HOM}}$ are part of a bifunctor which is triangulated on both variables. We need the following auxiliary result.

Lemma 9.6. *Let A, B and C be* dg *algebras with enough idempotents and let P and Q be* dg *$C-A-$bimodules such that P is homotopically projective and Q is homotopically injective as right* dg *A-modules. Then the following assertions hold:*

1) *The functor $\overline{\mathrm{HOM}}_A(P,?)\colon B-\mathrm{Dg}-A \longrightarrow B-\mathrm{Dg}-C$ preserves acyclic* dg *bimodules.*
2) *The functor $\overline{\mathrm{HOM}}_A(?,Q)\colon (B-\mathrm{Dg}-A)^{\mathrm{op}} \longrightarrow C-\mathrm{Dg}-B$ preserves acyclic* dg *bimodules.*

Proof. The proofs of the two assertions are rather similar. We only prove 1). Let X be an acyclic dg $B-A$-bimodule. We know that the non-unitary dg $B-C$-bimodule $\mathrm{HOM}_A(P,X)$ is acyclic since P_A is homotopically projective. By definition, we have that $\overline{\mathrm{HOM}}_A(P,X) = B\,\mathrm{HOM}_A(P,X)C$ and the differential on this (unitary) dg $B-C$-bimodule is the restriction of the differential of $\mathrm{HOM}_A(P,X)$. Let now $f \in Z^n(\overline{\mathrm{HOM}}_A(P,X))$ be any n-cycle. By the acyclicity of $\mathrm{HOM}_A(P,X)$, we have a $g \in \mathrm{HOM}_A(P,X)^{n-1}$ such that $f = d(g)$, where d is the differential of $\mathrm{HOM}_A(P,X)$. If $(e_i)_{i\in I}$ and $(\nu_k)_{k\in\mathcal{K}}$ are distinguished families of orthogonal idempotents of B and C, respectively, then there are finite subset $I' \subset I$ and $\mathcal{K}' \subset \mathcal{K}$ such that $f = \sum_{i\in I',k\in\mathcal{K}'} e_i f \nu_k$. Taking $g' = \sum_{i\in I',k\in\mathcal{K}'} e_i g \nu_k$ and using Leibniz rule for the non-unitary dg $B-C$-bimodule $\mathrm{HOM}_A(P,X)$ (see Lemma 8.1), we get an element $g' \in \overline{\mathrm{HOM}}_A(P,X)^{n-1}$ such that

$$d(g') = d(\sum_{i\in I',k\in\mathcal{K}'} e_i g \nu_k) = \sum_{i\in I',k\in\mathcal{K}'} e_i d(g) \nu_k = \sum_{i\in I',k\in\mathcal{K}'} e_i f \nu_k = f. \quad \square$$

We say that an algebra with enough idempotents A is *K-projective* (resp. *K-flat*) when it is projective (resp. flat) as a K-module.

Corollary 9.7. *Let A, B and C be* dg *algebras with enough idempotents. The* dg *functor $\overline{\mathrm{HOM}}_A(?,?)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}} \otimes (B-\mathrm{Dg}-A) \longrightarrow B-\mathrm{Dg}-C$ preserves contractibility on each variable. If $\mathbb{R}\,\mathrm{HOM}_A(?,?) := \mathbb{R}(\overline{\mathrm{HOM}}(?,?))\colon \mathcal{D}(A\otimes C^{\mathrm{op}})^{\mathrm{op}} \otimes \mathcal{D}(A\otimes B^{\mathrm{op}}) \longrightarrow \mathcal{D}(C\otimes B^{\mathrm{op}})$ is the associated bi-triangulated functor (see Definition 7.16), then the following assertions hold:*

1) *If C is K-projective or X is a homotopically injective* dg $B-A$*-bimodule, then there is a natural isomorphism of triangulated functors*

$$\begin{aligned}\mathbb{R}\,\mathrm{HOM}_A(?,X) &\cong \mathbb{R}\,\mathrm{Hom}_A(?,X)\\ &:= \mathbb{R}(\overline{\mathrm{HOM}}_A(?,X))\colon \mathcal{D}(A\otimes C^{\mathrm{op}})^{\mathrm{op}} \longrightarrow \mathcal{D}(C\otimes B^{\mathrm{op}}).\end{aligned}$$

2) *If either B is K-flat or M is a homotopically projective* dg $C-A$*-bimodule, then there is a natural isomorphism of triangulated functors*

$$\begin{aligned}\mathbb{R}\,\mathrm{HOM}_A(M,?) &\cong \mathbb{R}\,\mathrm{Hom}_A(M,?)\\ &:= \mathbb{R}(\overline{\mathrm{HOM}}_A(M,?))\colon \mathcal{D}(A\otimes B^{\mathrm{op}}) \longrightarrow \mathcal{D}(C\otimes B^{\mathrm{op}}).\end{aligned}$$

In particular, if C is K-projective (e.g. if $C=K$) one has an isomorphism $\mathbb{R}\,\mathrm{Hom}_A(M,?)(X)\cong\mathbb{R}\,\mathrm{Hom}_A(?,X)(M)$ in $\mathcal{D}(C\otimes B^{\mathrm{op}})$, for all dg $B-A-$*bimodules X and all homotopically projective* dg $C-A$*-bimodules M.*

Proof. By Theorems 9.1 and 9.5, we know that, for fixed M and X in $C-\mathrm{Dg}-A$ and $B-\mathrm{Dg}-A$, the dg functors $\overline{\mathrm{HOM}}_A(M,?)\colon B-\mathrm{Dg}-A\longrightarrow B-\mathrm{Dg}-C$ and $\overline{\mathrm{HOM}}_A(?,X)\colon (C-\mathrm{Dg}-A)^{\mathrm{op}}\longrightarrow B-\mathrm{Dg}-C$ are part of a dg adjunction. By Lemma 7.8, both of them preserve contractible dg modules, which shows the first statement of the corollary.

The last statement is a direct consequence of assertions 1 and 2. Note that when C is K-projective (resp. B is K-flat), the restriction of scalars functor $C-\mathrm{Dg}-A\longrightarrow\mathrm{Dg}-A$ (resp. $B-\mathrm{Dg}-A\longrightarrow\mathrm{Dg}-A$) preserves homotopically projective (resp. homotopically injective) dg modules (this is well-known in the context of dg modules over small dg categories, but the reader can easily adapt the proof of [17, Lemma 3.6] to get a direct proof by her/himself). Then assertion 1, when C is K-projective, is a direct consequence of Proposition 7.17(2.b) and Lemma 9.6. Similarly, assertion 2 for K-flat B follows from Proposition 7.17(2.a) and Lemma 9.6.

To check what remains of assertions 1 and 2, we just prove what remains of assertion 2 since the argument for assertion 1 is entirely dual. Recall from the proof of Proposition 7.17 that we have a natural isomorphism of triangulated functors $\mathcal{D}(A\otimes B^{\mathrm{op}})\longrightarrow\mathcal{D}(C\otimes B^{\mathrm{op}})$

$$\mathbb{R}\,\mathrm{HOM}_A(M,?)\cong\mathbb{R}(\overline{\mathrm{HOM}}_A(\Pi_{C-A}(M),?))=\mathbb{R}\,\mathrm{Hom}_A(\Pi_{C-A}(M),?).$$

Then $\mathbb{R}\,\mathrm{HOM}_A(M,?)$ is the composition

$$\begin{aligned}&\mathcal{D}(A\otimes B^{\mathrm{op}})\xrightarrow{\Upsilon_{B-A}}\mathcal{H}(A\otimes B^{\mathrm{op}})\\ &\qquad\xrightarrow{\overline{\mathrm{HOM}}_A(\Pi_{C-A}(M),?)}\mathcal{H}(C\otimes B^{\mathrm{op}})\xrightarrow{q_{C\otimes B^{\mathrm{op}}}}\mathcal{D}(C\otimes B^{\mathrm{op}}),\end{aligned}$$

where Υ_{B-A} (resp. Π_{C-A}) is the homotopically injective (resp. homotopically projective) resolution functor for dg $B-A-$bimodules (resp. dg $C-A-$bimodules). But, when M is homotopically projective, the homotopically projective resolution $\pi\colon \Pi_{C-A}(M) \longrightarrow M$ is an isomorphism in $\mathcal{H}(A \otimes C^{\mathrm{op}})$. Considering the bi-triangulated functor

$$\overline{\mathrm{HOM}}_A(?,?)\colon \mathcal{H}(A \otimes C^{\mathrm{op}})^{\mathrm{op}} \otimes \mathcal{H}(A \otimes B^{\mathrm{op}}) \longrightarrow \mathcal{H}(C \otimes B^{\mathrm{op}})$$

(see Proposition 7.15), we deduce that π induces a natural isomorphism of triangulated functors

$$\pi^*\colon \overline{\mathrm{HOM}}_A(M,?) \xrightarrow{\cong} \overline{\mathrm{HOM}}_A(\Pi_{C-A}(M),?)\colon \mathcal{H}(A\otimes B^{\mathrm{op}}) \longrightarrow \mathcal{H}(C\otimes B^{\mathrm{op}}).$$

Then $\mathbb{R}\,\mathrm{HOM}_A(M,?)$ is naturally isomorphic to the composition

$$\mathcal{D}(A\otimes B^{\mathrm{op}}) \xrightarrow{\Upsilon_{B-A}} \mathcal{H}(A\otimes B^{\mathrm{op}}) \xrightarrow{\overline{\mathrm{HOM}}_A(M,?)} \mathcal{H}(C\otimes B^{\mathrm{op}}) \xrightarrow{q_{C\otimes B^{\mathrm{op}}}} \mathcal{D}(C\otimes B^{\mathrm{op}}),$$

which is precisely $\mathbb{R}\,\mathrm{Hom}_A(M,?)\colon \mathcal{D}(A \otimes B^{\mathrm{op}}) \longrightarrow \mathcal{D}(C \otimes B^{\mathrm{op}})$. □

10. Dualities for perfect complexes

In this final section, we shall consider the adjunction of Theorem 9.5 when $C = K$. That is, we consider the adjunction

$$(\mathbb{R}\,\mathrm{Hom}_{B^{\mathrm{op}}}(?,X)^o\colon \mathcal{D}(B^{\mathrm{op}}) \longrightarrow \mathcal{D}(A)^{\mathrm{op}},$$
$$\mathbb{R}\,\mathrm{Hom}_A(?,X)\colon \mathcal{D}(A)^{\mathrm{op}} \longrightarrow \mathcal{D}(B^{\mathrm{op}})).$$

Remark 10.1. We denote the unit of this adjunction by

$$\lambda\colon 1_{\mathcal{D}(B^{\mathrm{op}})} \longrightarrow \mathbb{R}\,\mathrm{Hom}_A(?,X) \circ \mathbb{R}\,\mathrm{Hom}_{B^{\mathrm{op}}}(?,X)^o.$$

Note that the counit $\rho^o\colon \mathbb{R}\,\mathrm{Hom}_{B^{\mathrm{op}}}(?,X)^o \circ \mathbb{R}\,\mathrm{Hom}_A(?,X) \longrightarrow 1_{\mathcal{D}(A)^{\mathrm{op}}}$, when evaluated at any right dg A-module, is a morphisms in $\mathcal{D}(A)^{\mathrm{op}}$. We then change this perspective, and see it as natural transformation $\rho\colon 1_{\mathcal{D}(A)} \longrightarrow \mathbb{R}\,\mathrm{Hom}_{B^{\mathrm{op}}}(?,X) \circ \mathbb{R}\,\mathrm{Hom}_A(?,X)^o$.

Recall that if $(F\colon \mathcal{C} \longrightarrow \mathcal{D}, G\colon \mathcal{D} \longrightarrow \mathcal{C})$ is an adjoint pair of arbitrary categories $\mathcal{C}$ and $\mathcal{D}$, an object $C \in \mathrm{Ob}(\mathcal{C})$ (resp. $D \in \mathrm{Ob}(\mathcal{D})$) is called *reflexive* (resp *coreflexive*) with respect to the given adjunction when the evaluation of the unit at C (resp. the evaluation of the counit at D) is an isomorphism. The following fact is well-known:

Lemma 10.2. *In the situation of the previous paragraph, the functors F and G define by restriction mutually quasi-inverse equivalences of categories between the full subcategories of reflexive and coreflexive objects.*

Coming back to the situation of Theorem 9.5, with $C = K$, the following definition comes then naturally.

Definition 10.3. A left dg B-module U will be called *homologically X-reflexive* when the unit map

$$\lambda_U \colon U \longrightarrow \mathbb{R}\operatorname{Hom}_A(\mathbb{R}\operatorname{Hom}_{B^{\mathrm{op}}}(U, X), X)$$

is an isomorphism. A right dg A-module M is called *homologically X-coreflexive* when the counit map

$$\rho_M \colon M \longrightarrow \mathbb{R}\operatorname{Hom}_{B^{\mathrm{op}}}(\mathbb{R}\operatorname{Hom}_A(M, X), X)$$

is an isomorphism. Fixing again distinguished families of orthogonal idempotents $(e_i)_{i\in I}$ and $(\epsilon_j)_{j\in J}$ in B and A, respectively, we shall say that the dg $B-A$-bimodule X is *left (resp. right) homologically faithfully balanced* (see [18]) when each Be_i (resp. $\epsilon_j A$) is homologically X-reflexive (resp. homologically X-coreflexive). We will say that X is *homologically faithfully balanced* when it is left and right homologically faithfully balanced.

In the sequel we denote by $\mathrm{per}(A)$ (resp. $\mathrm{per}(A^{\mathrm{op}})$) the (thick) subcategory of $\mathcal{D}(A)$ (resp. $\mathcal{D}(A^{\mathrm{op}})$) formed by the compact objects. It will be called the *perfect right (resp. left) derived category of A*.

Recall that if $\mathcal{C}$ and $\mathcal{D}$ are triangulated categories, then a *triangulated duality* or a *duality of triangulated categories* $\mathcal{C} \stackrel{\cong^o}{\longrightarrow} \mathcal{D}$ is an equivalence of triangulated categories categories $\mathcal{C}^{\mathrm{op}} \stackrel{\cong}{\longrightarrow} \mathcal{D}$. As the following proposition shows, the definitions 10.3 are independent of the considered distinguished families of idempotents.

Proposition 10.4. *Let A and B be* dg *algebras with enough idempotents, on which we fix distinguished families of orthogonal idempotents $(\epsilon_j)_{j\in J}$ and $(e_i)_{i\in I}$, respectively, and let X be a* dg *$B-A$-bimodule. The following assertions hold:*

1) *X is left homologically faithfully balanced if, and only if, all objects of $\mathrm{per}(B^{\mathrm{op}})$ are homologically X-reflexive. Then $\mathbb{R}\operatorname{Hom}_{B^{\mathrm{op}}}(?, X)$ and $\mathbb{R}\operatorname{Hom}_A(?, X)$ define quasi-inverse dualities of triangulated categories $\mathrm{per}(B^{\mathrm{op}}) \stackrel{\cong^o}{\longleftrightarrow} thick_{\mathcal{D}(A)}(e_i X : i \in I)$.*

2) *X is right homologically faithfully balanced if, and only if, all objects of* $\mathrm{per}(A)$ *are homologically X-coreflexive. Then* $\mathbb{R}\mathrm{Hom}_A(?,X)$ *and* $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(?,X)$ *define quasi-inverse dualities of triangulated categories* $\mathrm{per}(A) \overset{\cong^o}{\longleftrightarrow} \mathit{thick}_{\mathcal{D}(B^{\mathrm{op}})}(X\epsilon_j : i \in I)$.

3) *If $A = B$ then the regular* dg *bimodule $X = A$ is homologically faithfully balanced. In particular* $\mathbb{R}\mathrm{Hom}_A(?,A)$ *and* $\mathbb{R}\mathrm{Hom}_{A^{\mathrm{op}}}(?,A)$ *define quasi-inverse dualities* $\mathrm{per}(A) \overset{\cong^o}{\longrightarrow} \mathrm{per}(A^{\mathrm{op}})$.

Proof. The classes $\mathrm{Ref}(X) = \{U \in \mathcal{D}(B^{\mathrm{op}}) : \lambda_U \text{ is an isomorphism}\}$ and $\mathrm{Coref}(X) = \{M \in \mathcal{D}(A) : \rho_M \text{ is an isomorphism}\}$ of X-reflexive and X-coreflexive objects are thick subcategories of $\mathcal{D}(B^{\mathrm{op}})$ and $\mathcal{D}(A)$, respectively (see the last three lines of the introduction). On the other hand, note that $\mathrm{per}(B^{\mathrm{op}}) = \mathrm{thick}_{\mathcal{D}(B^{\mathrm{op}})}(Be_i : i \in I)$ and that $\mathrm{per}(A) = \mathrm{thick}_{\mathcal{D}(A)}(\epsilon_j A : j \in J)$ (see Theorem 3.1, Remark 3.2 and [10, Section 5])

1) When X is left faithfully balanced, we then have that $\mathrm{per}(B^{\mathrm{op}}) \subseteq \mathrm{Ref}(X)$. Using now Lemma 10.2, we conclude that $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(?,X)$ and $\mathbb{R}\mathrm{Hom}_A(?,X)$ define quasi-inverse dualities between $\mathrm{per}(B^{\mathrm{op}})$ and the image of $\mathrm{per}(B^{\mathrm{op}})$ by $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(?,X)$. This image is precisely $\mathrm{thick}_{\mathcal{D}(A)}(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Be_i,X) : i \in I)$. In order to prove assertion 1, it remains to check that there is an isomorphism $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Be_i,X) \cong e_iX$ in $\mathcal{D}(A)$, for each $i \in I$. To see that, note that, due to the homotopically projective condition of Be_i (see Example 7.6), if $\Pi\colon \mathcal{D}(B^{\mathrm{op}}) \longrightarrow \mathcal{H}(B^{\mathrm{op}})$ is the homotopically projective resolution functor, then $\Pi(Be_i) \cong Be_i$ in $\mathcal{H}(B^{\mathrm{op}})$. We then have that $\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Be_i,X) = \overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(Be_i,X)$. But the map $\Psi\colon \overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(Be_i,X) \longrightarrow e_iX$, given by $\Psi(f) = f(e_i)$ is an isomorphism of right dg A-modules. Indeed, by Corollary 8.3, we have that $\Psi(fa) = (fa)(e_i) = (-1)^{|a||e_i|}f(e_i)a = f(e_i)a = \Psi(f)a$ since $|e_i| = 0$, which immediately implies that Ψ is an isomorphism in $\mathrm{GR}-A$ and $\mathrm{Gr}-A$. On the other hand, if δ is the differential of $\overline{\mathrm{HOM}}_{B^{\mathrm{op}}}(Be_i,X)$ and $d_{e_iX} = (d_X)_{|e_iX}$ is the differential of e_iX, then we have $(d_{e_iX} \circ \Psi)(f) = d_{e_iX}(f(e_i)) = d_X(f(e_i))$ while we have

$$\begin{aligned}(\Psi \circ \delta)(f) &= \Psi(\delta(f)) = \Psi(d_X \circ f - (-1)^{|f|} f \circ d_{Be_i}) \\ &= [d_X \circ f - (-1)^{|f|} f \circ d_{Be_i}](e_i) = d_X(f(e_i))\end{aligned}$$

since $d_{Be_i}(e_i) = 0$ because the differential of B vanishes on e_i. Therefore Ψ is an isomorphism of right dg A-modules, thus ending the proof of assertion 1.

2) Assertion 2 is proved as assertion 1 by exchanging the roles of A and B^{op}. Due to the fact that $\epsilon_j A$ is a homotopically projective right dg

A-module, in a way analogous to that of the previous paragraph, one checks that the map $\Phi\colon \mathbb{R}\operatorname{Hom}_A(\epsilon_j A, X) = \overline{\mathrm{HOM}}_A(\epsilon_j A, X) \longrightarrow X\epsilon_j$, given by $\Phi(g) = g(\epsilon_j)$ is an isomorphism of left dg B-modules.

3) For simplicity, put

$$F = \overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(?, A)\colon (A - \mathrm{Dg})^{\mathrm{op}} \longrightarrow \mathrm{Dg} - A$$

and

$$G = \overline{\mathrm{HOM}}_A(?, A)\colon (\mathrm{Dg} - A)^{\mathrm{op}} \longrightarrow A - \mathrm{Dg}.$$

We then have $G \circ F^o\colon A - \mathrm{Dg} \longrightarrow A - \mathrm{Dg}$ and want to get information about the unit $\lambda\colon 1_{\mathcal{D}(A^{\mathrm{op}})} \longrightarrow \mathbb{R}G \circ \mathbb{L}(F^o) = \mathbb{R}G \circ (\mathbb{R}F)^o$ (see Remark 7.10). For this, we consider the unit $\widetilde{\lambda}\colon 1_{A-\mathrm{Dg}} \longrightarrow G \circ F^o$ of the adjunction (F^o, G). By Proposition 7.12, we have an induced natural transformation of triangulated functors $\widetilde{\lambda}\colon 1_{\mathcal{D}(A^{\mathrm{op}})} \longrightarrow \mathbb{L}(G \circ F^o)$ and, by Proposition 7.14(2), we get another natural transformation of triangulated functors $\delta\colon \mathbb{L}(G \circ F^o) \longrightarrow \mathbb{R}G \circ (\mathbb{R}F)^o$. It is not hard to see that λ is the composition $1_{\mathcal{D}(A^{\mathrm{op}})} \xrightarrow{\widetilde{\lambda}} \mathbb{L}(G \circ F^o) \xrightarrow{\delta} \mathbb{R}G \circ (\mathbb{R}F)^o$. If $j \in J$ is arbitrary, then $\Pi_A(A\epsilon_j) \cong A\epsilon_j$ in $\mathcal{H}(A^{\mathrm{op}})$ since $A\epsilon_j$ is homotopically projective. Moreover, by the proof of assertion 1 (with A and ϵ_j instead of B and e_i), we know that $G^o(A\epsilon_j) = \overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(A\epsilon_j, A) \cong \epsilon_j A$ in $\mathrm{Dg} - A$. It then follows from Proposition 7.14(2) that $\delta_{A\epsilon_j}$ is an isomorphism. Moreover, by Proposition 7.12, we know that if

$$\widetilde{\lambda}_{A\epsilon_j}\colon A\epsilon_j \longrightarrow (G \circ F^o)(A\epsilon_j) = \overline{\mathrm{HOM}}_A(\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(A\epsilon_j, A), A)$$

is a quasi-isomorphism (e.g. an isomorphism in $A - \mathrm{Dg}$), then also

$$\widetilde{\lambda}_{A\epsilon_j}\colon A\epsilon_j \longrightarrow \mathbb{L}(G \circ F^o)$$

is an isomorphism, and this will imply that

$$\lambda_{A\epsilon_j} = \delta_{A\epsilon_j} \circ \widetilde{\lambda_{A\epsilon_j}}\colon A\epsilon_j \longrightarrow [\mathbb{R}G \circ (\mathbb{R}F)^o](A\epsilon_j)$$

is an isomorphism in $\mathcal{D}(A^{\mathrm{op}})$ and, hence, that $X = {}_AA_A$ is left homologically faithfully balanced.

We are led to give an explicit definition of

$$\widetilde{\lambda}_U\colon U \longrightarrow \overline{\mathrm{HOM}}_A(\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(U, A), A),$$

for any left dg A-module. If we consider the natural isomorphism

$$\begin{array}{ccc} \operatorname{Hom}_{(\mathrm{Dg}-A)^{\mathrm{op}}}(\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(U, A), M) & & \operatorname{Hom}_{A-\mathrm{Dg}}(U, \overline{\mathrm{HOM}}_A(M, A)) \\ \| & & \| \\ \mathrm{HOM}_A(M, \overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(U, A)) & \xrightarrow{\xi = \xi_{U,M}} & \mathrm{HOM}_{A^{\mathrm{op}}}(U, \overline{\mathrm{HOM}}_A(M, A)) \end{array}$$

(see the proof of Theorem 9.5), the standard theory of adjunction gives that
$\widetilde{\lambda}_U = \xi_{U,\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(U,A)}(1_{\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(U,A)})$. We then get that

$$\widetilde{\lambda}_U(u)(\alpha) = \xi(1_{\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(U,A)})(u)(\alpha) = (-1)^{|\alpha||u|}\alpha(u),$$

for all homogeneous elements $u \in U$ and $\alpha \in \overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(U,A)$. Abusing the notation and denoting by $(?)^*$ both functors $\overline{\mathrm{HOM}}_A(?,A)$ and $\overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(?,A)$, we get a natural transformation $\widetilde{\lambda}\colon 1_{A-\mathrm{Dg}} \longrightarrow (?)^{**}$, where $\widetilde{\lambda}_U\colon U \longrightarrow U^{**}$ is given by the rule $\widetilde{\lambda}_U(\alpha)(u) = (-1)^{|\alpha||u|}\alpha(u)$.

Consider the case $U = A\epsilon_j$ and let us consider now the isomorphisms $\Psi\colon (A\epsilon_j)^* = \overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(A\epsilon_j, A) \longrightarrow \epsilon_j A$ and $\Phi\colon (\epsilon_j A)^* = \overline{\mathrm{HOM}}_A(\epsilon_j A, A) \longrightarrow A\epsilon_j$ given in the proofs of Assertions 1 and 2, when $X = A$. The composition

$$A\epsilon_j \xrightarrow{\Phi^{-1}} (\epsilon_j A)^* \xrightarrow{\Psi^*} (A\epsilon_j)^{**}$$

is then an isomorphism of left dg A-modules. Note that $\Phi^{-1}(u) = f_u$, where $f_u(a) = ua$ for all $a \in \epsilon_j A$. We then have that

$$(\Psi^* \circ \Phi^{-1})(u) = \Psi^*(f_u) = (-1)^{|\Psi||f_u|} f_u \circ \Psi = f_u \circ \Psi,$$

using the action of the functor $(?)^* = \overline{\mathrm{HOM}}_{A^{\mathrm{op}}}(?,A)$ on homogeneous morphisms (see the proof of Proposition 8.2) and the fact that $|\Psi| = 0$. Taking into account the comments after Lemma 4.1, we then have that

$$\begin{aligned}[(\Psi^* \circ \Phi^{-1})(u)](\alpha) &= (f_u \circ \Psi)(\alpha) = f_u(\alpha(\epsilon_j)) = u\alpha(\epsilon_j) \\ &= (-1)^{|u||\alpha|}\alpha(u\epsilon_j) = (-1)^{|u||\alpha|}\alpha(u),\end{aligned}$$

for each $\alpha \in (A\epsilon_j)^* = \overline{\mathrm{HOM}}_A(A\epsilon_j, A)$. It follows that $\widetilde{\lambda}_{A\epsilon_j} = \Psi^* \circ \Phi^{-1}$, and hence that $\lambda_{A\epsilon_j}$ is an isomorphism, for all $j \in J$.

By a left-right symmetric argument, one checks that ${}_A A_A$ is also right homologically faithfully balanced. The part of assertion 3 concerning duality follows from assertions 1 and 2 and from the first paragraph of this proof. □

We end with a result that has proved very useful in [20]:

Proposition 10.5. *Let $\iota\colon A \longrightarrow B$ be a homomorphism of* dg *algebras with enough idempotents such that $B = \iota(A)B\iota(A)$, and let us consider the* dg *functors:*

$$F\colon (\mathrm{Dg} - A)^{\mathrm{op}} \xrightarrow{\overline{\mathrm{HOM}}_A(?,A)} A - \mathrm{Dg} \xrightarrow{\iota^*} B - \mathrm{Dg}$$

and

$$G\colon (\mathrm{Dg}-A)^{\mathrm{op}} \stackrel{\iota^{*o}}{\longrightarrow} (\mathrm{Dg}-B)^{\mathrm{op}} \stackrel{\overline{\mathrm{HOM}}_B(?,B)}{\longrightarrow} B-\mathrm{Dg},$$

where we denote by ι^* *both extension of scalars functors* $B\otimes_A ?\colon A-\mathrm{Dg} \longrightarrow B-\mathrm{Dg}$ *and* $?\otimes B\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$. *There is homological natural transformation of* dg *functors* $\eta\colon F \longrightarrow G$ *whose triangulated version, when evaluated at compact objects, gives a natural isomorphism*

$$\eta\colon [(B\otimes_A^{\mathbb{L}}?)\circ\mathbb{R}\operatorname{Hom}_A(?,A)]_{|per(A)^{\mathrm{op}}} \stackrel{\cong}{\longrightarrow} [\mathbb{R}\operatorname{Hom}_B(?,B)\circ(?\otimes_A^{\mathbb{L}}B)]_{\mathrm{per}(A)^{\mathrm{op}}}$$

of triangulated functors $\mathrm{per}(A)^{\mathrm{op}} \longrightarrow \mathcal{D}(B)$.

Proof. All throughout the proof we fix a distinguished family of orthogonal idempotents $(e_i)_{i\in}$ in A. Note that, after deleting the terms which are zero, $(\iota(e_i))_{i\in I}$ is also a distinguished family of orthogonal idempotents in B. For each right dg A-module M, we define

$$\eta_M\colon F(M) = B\otimes_A \overline{\mathrm{HOM}}_A(M,A) \longrightarrow \overline{\mathrm{HOM}}_B(M\otimes_A B,B) = G(M)$$

by the rule $\eta_M(b\otimes f)(m\otimes b') = b\iota(f(m))b'$, for all homogeneous elements $b,b'\in B$, $f\in\overline{\mathrm{HOM}}_A(M,A)$ and $m\in M$. We first check that $\eta := \eta_M$ is well-defined. Note that if $a\in A$ is a homogeneous element and b,b',f,m are homogeneous elements above, then we have

$$\eta(b\iota(a)\otimes f)(m\otimes b') = b\iota(a)\iota(f(m))b'$$

while

$$\eta(b\otimes af)(m\otimes b') = b\iota((af)(m))b' = b\iota(af(m))b' = b\iota(a)\iota(f(m))b',$$

bearing in mind that the structure of right and left A-module on B is given by $a\cdot b\cdot a' = \iota(a)b\iota(a')$. Moreover, if $b_1,b_2\in B$ are homogeneous elements, then we have that

$$\eta(b\otimes f)((m\otimes b_1)b_2) = \eta(b\otimes f)(m\otimes(b_1b_2)) = b\iota(f(m))(b_1b_2)$$

and

$$\eta(b\otimes f)(m\otimes b_1)b_2 = (b\iota(f(m))b_1)b_2,$$

so that $\eta(b\otimes f)$ is a homogeneous morphism $M\otimes_A B \longrightarrow B$ in $\mathrm{Dg}-B$. On the other hand, if $b=\sum_{i\in F} e_ib$, for a finite subset $F\subset I$, we clearly have that $\mathrm{Im}(\eta(b\otimes f))\subseteq \oplus_{i\in F}e_iB$, thus showing that $\eta(b\otimes f)\in\overline{\mathrm{HOM}}_B(M\otimes B,B)$ (see the initial paragraph of Section 8). Therefore $\eta=\eta_M$ is a morphism $F(M) = B \otimes_A \overline{\mathrm{HOM}}_A(M,A) \longrightarrow \overline{\mathrm{HOM}}_B(M \otimes_A B, B)$ in $\mathrm{Gr}-K$. In order to check that it is actually a morphism in $B-\mathrm{Gr}$, we need to check that $\eta(b_1(b\otimes f)) = (-1)^{|b_1||\eta|}b_1\eta(b\otimes f) = b_1\eta(b\otimes f)$. But this is clear since

$$\eta(b_1(b\otimes f))(m\otimes b') = \eta((b_1b)\otimes f)(m\otimes b') = (b_1b)\iota(f(m))b'$$

while

$$[b_1\eta(b\otimes f)](m\otimes b') = b_1\eta(b\otimes f)(m\otimes b') = b_1(b\iota(f(m))b'),$$

for each homogeneous element $b_1 \in B$.

We now prove the naturality of η. Recall from the proof of Proposition 8.4 that the action of $B\otimes_A ?\colon A-\mathrm{Dg} \longrightarrow B-\mathrm{Dg}$ on homogeneous morphisms is given by the rule $(B\otimes ?)(\alpha)\colon b\otimes x \rightsquigarrow (-1)^{|\alpha||b|}b\otimes\alpha(x)$, for all homogeneous elements $\alpha \in \mathrm{HOM}_{B^{\mathrm{op}}}(X,Y)$, $x\in X$ and $b \in B$. Let $\varphi\colon M \longrightarrow N$ be a homogeneous morphism in $\mathrm{Dg}-A$. Then

$$F(\varphi) = [(B\otimes_A ?)\circ \overline{\mathrm{HOM}}_A(?,A)](\varphi) = (B\otimes_A ?)(\varphi^*).$$

This is a morphism

$$F(N) = B\otimes_A \overline{\mathrm{HOM}}_A(N,A) \longrightarrow B\otimes_A \overline{\mathrm{HOM}}_A(M,A) = F(M)$$

which takes $b\otimes g \rightsquigarrow (-1)^{|\varphi^*||b|}b\otimes\varphi^*(g) = (-1)^{|\varphi||b|}b\otimes\varphi^*(g)$, for all homogeneous elements $b\in B$ and $g \in \overline{\mathrm{HOM}}_A(M,A)$. By the definition of φ^* (see the proof of Proposition 8.2), we then get that

$$F(\varphi)(b\otimes g) = (-1)^{|\varphi||b|}(-1)^{|\varphi||g|}b\otimes(g\circ\varphi) = (-1)^{|\varphi|(|g|+|b|)}b\otimes(g\circ\varphi)$$

On the other hand, we have that

$$G(\varphi) = [\overline{\mathrm{HOM}}_B(?,B)\circ(?\otimes_A B)](\varphi) = (\varphi\otimes 1_B)^*.$$

This is a morphism

$$G(N) = \overline{\mathrm{HOM}}_B(N\otimes_A B, B) \longrightarrow \overline{\mathrm{HOM}}_B(M\otimes_A B, B) = G(M)$$

which takes $u \rightsquigarrow (\varphi\otimes 1_B)^*(u) = (-1)^{|\varphi||u|}u\circ(\varphi\otimes 1_B)$, for each homogeneous element $u \in \overline{\mathrm{HOM}}_B(N\otimes_A B, B)$. We then have the following equalities, for all homogeneous elements $b\in B$ and $g \in \overline{\mathrm{HOM}}_A(N,A)$:

$$[G(\varphi)\circ\eta_N](b\otimes g) = G(\varphi)(\eta_N(b\otimes g)) = (-1)^{|\varphi|(|b|+|g|)}\eta_N(b\otimes g)\circ(\varphi\otimes 1_B)$$

and

$$[\eta_M \circ F(\varphi)](b \otimes g) = (-1)^{|\varphi|(|b|+|g|)}\eta_M(b \otimes (g \circ \varphi)).$$

But, for all homogeneous elements $m \in M$ and $b' \in B$, we also have

$$\begin{aligned}[\eta_N(b \otimes g) \circ (\varphi \otimes 1_B)](m \otimes b') &= \eta_N(b \otimes g)(\varphi(m) \otimes b') \\ &= b\iota((g \circ \varphi)(m)))b' = \eta_M(b \otimes (g \circ \varphi))(m \otimes b').\end{aligned}$$

It follows that $G(\varphi) \circ \eta_N = \eta_M \circ F(\varphi)$, so that η is a natural transformation of dg functors.

We next prove that hat η is homological, i.e., that $d_{G(M)} \circ \eta_M - \eta_M \circ d_{F(M)} = 0$. We denote by $d = d_{F(M))}\colon B \otimes_A \overline{\mathrm{HOM}}_A(M, A) \longrightarrow B \otimes_A \overline{\mathrm{HOM}}_A(M, A)$ and $\delta = d_{G(M)}\colon \overline{\mathrm{HOM}}_B(M \otimes_A B, B) \longrightarrow \overline{\mathrm{HOM}}_B(M \otimes_A B, B)$ the respective differentials. We need to check that $\delta(\eta(b \otimes f)) = \eta(d(b \otimes f))$, for all homogeneous elements $b \in B$ and $f \in \overline{\mathrm{HOM}}_A(M, A)$. For this, we shall apply both members of this desired equality to a tensor $m \otimes b'$, where $m \in M$ and $b' \in B$ are homogeneous elements. We then have:

$$\begin{aligned}&[\delta(\eta(b \otimes f))](m \otimes b') \\ &\quad = [d_B \circ \eta(b \otimes f)) - (-1)^{|b|+|f|}\eta(b \otimes f)) \circ d_{M \otimes_A B}](m \otimes b') \\ &\quad = d_B(b\iota(f(m))b') - (-1)^{|b|+|f|}\eta(b \otimes f))(d_M(m) \otimes b' \\ &\qquad + (-1)^{|m|}m \otimes d_B(b')) \\ &\quad = d_B(b)\iota(f(m))b' + (-1)^{|b|}bd_B(\iota(f(m))b') - (-1)^{|b|+|f|}[b\iota(f(d_M(m)))b' \\ &\qquad + (-1)^{|m|}b\iota(f(m))d_B(b')] \\ &\quad = d_B(b)\iota(f(m))b' + (-1)^{|b|}b[d_B(\iota(f(m)))b' + (-1)^{|f|+|m|}\iota(f(m))d_B(b')] \\ &\qquad - (-1)^{|b|+|f|}b\iota(f(d_M(m)))b' - (-1)^{|b|+|f|+|m|}b\iota(f(m))d_B(b') \\ &\quad = d_B(b)\iota(f(m))b' + (-1)^{|b|}bd_B(\iota(f(m)))b' \\ &\qquad - (-1)^{|b|+|f|}b\iota(f(d_M(m)))b' \qquad (*)\end{aligned}$$

while we also have

$$\begin{aligned}\eta(d(b \otimes f))(m \otimes b') &= [\eta(d_B(b) \otimes f) + (-1)^{|b|}\eta(b \otimes d_H(f)))](m \otimes b') \\ &= d_B(b)\iota(f(m))b' + (-1)^{|b|}b\iota(d_H(f)(m))b' = d_B(b)\iota(f(m))b' \\ &\quad + (-1)^{|b|}b\iota((d_A \circ f)(m) - (-1)^{|f|}(f \circ d_M)(m))b' \qquad (**)\end{aligned}$$

The expression $(*)$ and $(**)$ are equal because $\iota\colon A \longrightarrow B$ is a morphism of dg algebras with enough idempotents and, hence, one has that $\iota(d_A(a)) = d_B(\iota(a))$, for all $a \in A$.

For the final assertion, we start by pointing out that $\iota^*\colon \mathrm{Dg}-A \longrightarrow \mathrm{Dg}-B$ (resp. $\iota^*\colon A-\mathrm{Dg} \longrightarrow B-\mathrm{Dg}$) preserves homotopically projective dg modules. Indeed if $P \in \mathrm{Dg}-A$ is homotopically projective and $Y \in \mathrm{Dg}-B$ is acyclic, then dg adjunction gives an isomorphism $\mathrm{HOM}_B(\iota^*(P), Y) \cong \mathrm{HOM}_A(P, \iota_*(Y))$ in $\mathrm{Dg}-K$. Since ι_* preserves acyclic dg modules, we conclude that the last dg K-module is acyclic and, hence, $\iota^*(P)$ is homotopically projective.

By Proposition 7.12, we have an induced natural transformation of triangulated functors $\eta\colon \mathbb{R}(\iota^* \circ \overline{\mathrm{HOM}}_A(?, A)) \longrightarrow \mathbb{R}(\overline{\mathrm{HOM}}_B(?, B) \circ \iota^{*o})$. On the other hand, by Proposition 7.14, we have natural transformations of triangulated functors $u\colon \mathbb{L}\iota^* \circ \mathbb{R}\,\mathrm{Hom}_A(?, A) \longrightarrow \mathbb{R}(\iota^* \circ \overline{\mathrm{HOM}}_A(?, A))$ and $v\colon \mathbb{R}(\overline{\mathrm{HOM}}_B(?, B) \circ \iota^{*o}) \longrightarrow \mathbb{R}\,\mathrm{Hom}_B(?, B) \circ (\mathbb{L}\iota^*)^o$. The composition

$$\begin{aligned}\mathbb{L}\iota^* \circ \mathbb{R}\,\mathrm{Hom}_A(?, A) &\xrightarrow{u} \mathbb{R}(\iota^* \circ \overline{\mathrm{HOM}}_A(?, A))\\ &\xrightarrow{\eta} \mathbb{R}(\overline{\mathrm{HOM}}_B(?, B) \circ \iota^{*o}) \xrightarrow{v} \mathbb{R}\,\mathrm{Hom}_B(?, B) \circ (\mathbb{L}\iota^*)^o\end{aligned}$$

is then the desired natural transformation, which we want to prove that is an isomorphism when evaluated at any $M \in \mathrm{per}(A)$. Since $\mathrm{per}(A) = \mathrm{thick}_{\mathcal{D}(A)}(e_iA : i \in I)$ it is enough to prove that $(v \circ \eta \circ u)_{e_iA} = v_{e_iA} \circ \eta_{e_iA} \circ u_{e_iA}$ is an isomorphism, for all $i \in I$.

Proposition 7.14(4) together with the previous to the last paragraph tell us that v is a natural isomorphism. On the other hand, $\Pi_A(e_iA) \cong e_iA$ in $\mathcal{H}(A)$ since e_iA is homotopically projective, for all $i \in I$. But $\overline{\mathrm{HOM}}_A(e_iA, A) \cong Ae_i$ and, by the previous to the last paragraph again, also $[\iota^* \circ \overline{\mathrm{HOM}}_A(?, A)](e_iA)$ is homotopically projective. Proposition 7.14(3) then gives that u_{e_iA} is an isomoprhism and Proposition 7.12 implies that, in order to check that η_{e_iA} is an isomorphism in $\mathcal{D}(B^{\mathrm{op}})$ and hence end the proof, it is enough to prove that

$$\eta_{e_iA}\colon F(e_iA) = B \otimes_A \overline{\mathrm{HOM}}_A(e_iA, A) \longrightarrow \overline{\mathrm{HOM}}_B(e_iA \otimes B, B) = G(e_iA)$$

is an isomorphism of left dg B-modules.

We proceed to prove this fact. Recall from the proof of Proposition 10.4 that we have isomorphisms $\Phi_A\colon \overline{\mathrm{HOM}}_A(e_iA, A) \xrightarrow{\cong} Ae_i$ and $\Phi_B\colon \overline{\mathrm{HOM}}_B(e_iB, B) \xrightarrow{\cong} Be_i$, in $A-\mathrm{Dg}$ and $B-\mathrm{Dg}$, respectively, mapping $f \rightsquigarrow f(e_i)$ in both cases. Note that $\Phi_B^{-1}\colon Be_i \longrightarrow \overline{\mathrm{HOM}}_B(e_iB, B)$ is given by the rule $\Phi_B^{-1}(b)(e_i) = b$ or, equivalently, by the rule $\Phi_B^{-1}(b)(b') = bb'$, for all homogeneous elements $b \in Be_i$ and $b' \in e_iB$. Note also that since Φ_A has degree zero, the induced isomorphism $(B\otimes ?)(\Phi_A)\colon B \otimes_A \overline{\mathrm{HOM}}_A(e_i, A) \xrightarrow{\cong} B \otimes_A Ae_i$ takes $b \otimes f \rightsquigarrow b \otimes f(e_i)$, so that $(B\otimes_A ?)(\Phi_A) = 1_B \otimes \Phi_A$. We now claim that

$$\eta_{e_iA}\colon B\otimes_A \overline{\mathrm{HOM}}_A(e_iA,A)\longrightarrow \overline{\mathrm{HOM}}_B(e_iA\otimes_A B,B)$$

can be decomposed as the composition of morphisms in $B-\mathrm{Dg}$

$$\begin{aligned} &B\otimes_A \overline{\mathrm{HOM}}_A(e_iA,A) \overset{1_B\otimes\Phi_A}{\longrightarrow} B\otimes_A Ae_i \overset{\mu'}{\longrightarrow} Be_i \\ &\qquad \overset{\Phi_B^{-1}}{\longrightarrow} \overline{\mathrm{HOM}}_B(e_iB,B) \overset{\mu^*}{\longrightarrow} \overline{\mathrm{HOM}}_B(e_iA\otimes_A B,B), \end{aligned}$$

where $\mu'\colon B\otimes_A Ae_i\longrightarrow Be_i$ and $\mu\colon e_iA\otimes B\longrightarrow B$ are the multiplication maps, $b\otimes a \rightsquigarrow b\iota(a)$ and $a\otimes b\rightsquigarrow \iota(a)b$. Indeed we have:

$$\begin{aligned} &(\mu^*\circ\Phi_B^{-1}\circ\mu'\circ(1_B\otimes\Phi))(b\otimes f) = (\mu^*\circ\Phi_B^{-1}\circ\mu')(b\otimes f(e_i)) \\ &\quad = (\mu^*\circ\Phi_B^{-1})(b\iota(f(e_i))) = \Phi_B^{-1}(b\iota(f(e_i)))\circ\mu \end{aligned}$$

since μ has zero degree. When we take homogeneous elements $x\in e_iA$ and $b'\in B$ and make act the last morphism on $x\otimes b'$, we get

$$\begin{aligned} &(\mu^*\circ\Phi_B^{-1}\circ\mu'\circ(1_B\otimes\Phi))(b\otimes f)(x\otimes b') = \Phi_B^{-1}(b\iota(f(e_i)))(\iota(x)b') \\ &\quad = b\iota(f(e_i))\iota(x)b' = b\iota(f(e_ix))b' = b\iota(f(x))b' = \eta_{e_iA}(b\otimes f)(x\otimes b'), \end{aligned}$$

using the fact that $\iota\colon A\longrightarrow B$ is an algebra homomorphism and $f\colon e_iA\longrightarrow A$ is a morphism of right A-modules.

The proof is hence reduced to check that $\mu'\colon B\otimes_A Ae_i\longrightarrow Be_i$ and $\mu\colon e_iA\otimes B\longrightarrow e_iB$ are isomorphisms in $B-\mathrm{Dg}$ and $\mathrm{Dg}-B$, respectively. But this is clear. Their inverses map $b\rightsquigarrow b\otimes e_i$ and $b\rightsquigarrow e_i\otimes b$, respectively. □

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