Algebra and Discrete Mathematics
Number 4. **(2003).** pp. 21 – 32

RESEARCH ARTICLE

Post-critically finite self-similar groups

E. Bondarenko, V. Nekrashevych

Communicated by V. V. Kirichenko

Dedicated to R. I. Grigorchuk on the occasion of his 50th birthday

Abstract. We describe in terms of automata theory the automatic actions with post-critically finite limit space. We prove that these actions are precisely the actions by bounded automata and that any self-similar action by bounded automata is contracting.

1. Introduction

The aim of this paper is to show a connection between two notions, which have appeared in rather different fields of mathematics. One is the notion of a post-critically finite self-similar set (other related terms are: "nested fractal" or "finitely ramified fractal"). It appeared during the study of harmonic functions and Brownian motion on fractals. The class of post-critically finite fractals is a convenient setup for such studies. See the papers [1, 10, 3, 17] for the definition of a post-critically finite self-similar sets and for applications of this notion to harmonic analysis on fractals.

The second notion appeared during the study of groups generated by finite automata (or, equivalently, groups acting on rooted trees). Many interesting examples of such groups where found (like the Grigorchuk group [7], groups defined by Aleshin [13], Sushchansky [18], Gupta-Sidki groups [14] and many others), and these particular examples where generalized to different classes of groups acting on rooted trees: branch groups [4], self-similar (state closed) groups [15, 5], GGS-groups [2], AT groups [8, 9], spinal groups [12].

The second author acknowledges the support of Swiss National Science Foundation and Alexander von Huboldt Foundation

S. Sidki has defined in his work [16] a series of subgroups of the group of finite automata, using the notions of activity growth and circuit structure. In particular, he has defined the notion of a bounded automaton. The set of all automorphisms of the regular rooted tree, which are defined by bounded automata is a group. It is interesting that most of the known interesting examples of groups acting on rooted trees (in particular, all the above mentioned examples) are subgroups of the group of bounded automata. Also every finitely automatic GGS-group, AT-group or spinal group is a subgroup of the group of bounded automata.

We prove in our paper that a self-similar (state closed) group is a subgroup of the group of all bounded automata if and only if its *limit space* is a post-critically finite self-similar space. The limit space of a self-similar group was defined in [11] (see also [15]). This establishes the mentioned above connection between the harmonic analysis on fractals and group theory.

The structure of the paper is the following. Section "Self-similar groups" is a review of the basic definitions of the theory of self-similar groups of automata. We define the notions of self-similar groups, automata, Moore diagrams, contracting groups, nucleus of a contracting group and establish notations.

Third section "Limit spaces" gives the definition and the basic properties of the limit space of a contracting self-similar group as a quotient of the space of infinite sequences. We also discuss the notion of *tiles* of a limit space (the images of the cylindric sets of the space of sequences).

The main results of the section "Post-critically finite limit space" are Corollary 4.2, giving a criterion when the limit space of a self-similar group action is post-critically finite and Proposition 4.3, stating that a post-critically finite limit space is 1-dimensional.

The last section "Automata with bounded cyclic structure" is the main part of the article. We prove Theorem 5.3, which says that every self-similar subgroup of the group $\mathcal{B}$ of bounded automata is contracting and that a contracting group has a post-critically finite limit space if and only if it is a subgroup of $\mathcal{B}$.

2. Self-similar groups

We review in this section the basic definitions and theorems concerning self-similar groups. For a more detailed account, see [15].

Let X be a finite set, which will be called *alphabet*. By X^* we denote the set of all finite words $x_1x_2\ldots x_n$ over the alphabet X, including the empty word $\emptyset$.

Definition 2.1. *A faithful action of a group G on the set X^* is* self-similar *(or* state closed*) if for every $g \in G$ and for every $x \in X$ there exist $h \in G$ and $y \in X$ such that*

$$g(xw) = yh(w)$$

for all $w \in X^*$.

We will write formally

$$g \cdot x = y \cdot h, \tag{1}$$

if for every $w \in X^*$ we have $g(xw) = yh(w)$. If one identifies every letter $x \in X$ with the map $w \mapsto xw : X^* \longrightarrow X^*$, then equation (1) will become a correct equality of two transformations.

The notion of a self-similar action is closely related with the notion of an *automaton.*

Definition 2.2. *An* automaton *$\mathcal{A}$ over the alphabet X is a tuple $\langle Q, \pi, \lambda\rangle$, where Q is a set (the set of* internal states *of the automaton), and $\pi : Q \times X \longrightarrow Q$ and $\lambda : Q \times X \longrightarrow Q$ are maps (the* transition *and the* output functions*, respectively).*

An automaton is finite *if its set of states Q is finite. A subset $Q' \subset Q$ is called* sub-automaton *if for all $q \in Q'$ and $x \in X$ we have $\pi(q, x) \in Q'$. If Q' is a sub-automaton, then we identify it with the automaton $\langle Q', \pi|_{Q'\times X}, \lambda|_{Q'\times X}\rangle$.*

For every state $q \in Q$ and $x \in X$ we also write formally

$$q \cdot x = y \cdot p, \tag{2}$$

where $y = \lambda(q, x)$ and $p = \pi(q, x)$.

We will also often use in our paper another notation for the functions π and λ:

$$\pi(q, x) = q|_x, \qquad \lambda(q, x) = q(x).$$

The transition and output functions are naturally extended to functions $\pi : Q \times X^* \longrightarrow Q$ and $\lambda : Q \times X^* \longrightarrow X^*$ by the formulae:

$$\pi(q, xv) = \pi\left(\pi(q, x), v\right), \qquad \lambda(q, xv) = \lambda(q, x)\lambda\left(\pi(q, x), v\right),$$

or, in the other notation:

$$q|_{xv} = q|_x|_v, \qquad q(xv) = q(x)q|_x(v).$$

We also put $q|_\emptyset = q$, $q(\emptyset) = \emptyset$.

Hence we get for every state q a map $v \mapsto q(v)$, defining the *action of the state* q on the words. It is easy to see that we have

$$q_1 q_2|_v = q_1|_{q_2(v)} g_2|_v, \qquad q(vw) = q(v) q|_v(w)$$

for all $q, q_1, q_2 \in Q$ and $v, w \in X^*$. Here $q_1 q_2$ is the product of transformations q_1 and q_2, i.e., $q_1 q_2(w) = q_1 (q_2(w))$.

The above definitions imply the following description of self-similar actions in terms of automata theory.

Proposition 2.1. *A faithful action of a group G on the set X^* is self-similar if and only if there exists an automaton with the set of states G such that the action of the states of the automaton on X^* coincides with the original action of G.*

The automaton from Proposition 2.1 is called *complete automaton* of the action.

It is convenient to represent automata by their *Moore diagrams.* If $\mathcal{A} = \langle Q, \pi, \lambda \rangle$ is an automaton, then its Moore diagram is a directed graph with the set of vertices Q in which we have for every pair $x \in X, q \in Q$ an arrow from $q \in Q$ into $\pi(q, x)$ labelled by the pair of letters $(x; \lambda(q, x))$.

Let $q \in Q$ be a state and let $v \in X^*$ be a word. In order to find the image $q(v)$ of the word v under the action of the state q one needs to find a path in the Moore diagram, which starts at the state q with the consecutive labels of the form $(x_1; y_1), (x_2; y_2), \ldots (x_n; y_n)$, where $x_1 x_2 \ldots x_n = v$, then $q(v) = y_1 y_2 \ldots y_n$.

Definition 2.3. *We say that an automaton $\mathcal{A} = \langle Q, \pi, \lambda \rangle$* has finite nucleus *if there exists its finite sub-automaton $\mathcal{N} \subset Q$ such that for every $q \in \mathcal{A}$ there exists $n \in \mathbb{N}$ such that $q|_v \in \mathcal{N}$ for all $v \in X^*$ such that $|v| \geq n$.*

A self-similar action of a group G on X^* is said to be contracting *if its full automaton has a finite nucleus.*

In general, if $\mathcal{A}$ is an automaton, then its *nucleus* is the set

$$\mathcal{N} = \bigcup_{q \in Q} \bigcap_{n \in \mathbb{N}} \{q|_v \ : \ v \in X^*, |v| \geq n\}.$$

For more on contracting actions, see the papers [15, 6].

3. Limit spaces

One of important properties of contracting actions is there strong relation to Dynamical Systems, exhibited in the following notion of *limit space.*

Denote by $X^{-\omega}$ the set of all infinite to the left sequences of the form $\ldots x_2x_1$, where x_i are letters of the alphabet X. We introduce on the set $X^{-\omega}$ the topology of the infinite power of the discrete set X. Then the space $X^{-\omega}$ is a compact totally disconnected metrizable topological space without isolated points. Thus it is homeomorphic to the Cantor space.

Definition 3.1. *Let (G, X^*) be a contracting group action over the alphabet X. We say that two points $\ldots x_2x_1, \ldots y_2y_1 \in X^{-\omega}$ are* asymptotically equivalent *(with respect to the action of the group G) if there exists a bounded sequence $\{g_k\}_{k\geq 1}$ of group elements such that for every $k \in \mathbb{N}$ we have*

$$g_k(x_k \ldots x_1) = y_k \ldots y_1.$$

Here a sequence $\{g_k\}_{k\geq 1}$ is said to be *bounded* if the set of its values is finite.

It is easy to see that the defined relation is an equivalence. The quotient of the space $X^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of the action and is denoted $\mathcal{J}_G$.

We have the following properties of the limit space (see [11]).

Theorem 3.1. *The asymptotic equivalence relation is closed and has finite equivalence classes. The limit space $\mathcal{J}_G$ is metrizable and finite-dimensional. The shift $\sigma : \ldots x_2x_1 \mapsto \ldots x_3x_2$ induces a continuous surjective map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$.*

We will also use the following description of the asymptotic equivalence relation (for the proof see [11]).

Proposition 3.2. *Two sequences $\ldots x_2x_1, \ldots ay_2y_1 \in X^{-\omega}$ are asymptotically equivalent if and only if there exists a sequence $g_1, g_2, \ldots$ of elements of the nucleus such that $g_i \cdot x_i = y_i \cdot g_{i-1}$, i.e., if there exists a left-infinite path $\ldots e_2, e_1$ in the Moore diagram of the nucleus such that the edge e_i is labelled by $(x_i; y_i)$.*

A *left-infinite path* in a directed graph is a sequence $\ldots e_2, e_1$ of its arrows such that beginning of e_i is equal to the end of e_{i+1}. The end of the last edge e_1 is called the *end* of the left-infinite path.

The dynamical system $(\mathcal{J}_G, \mathsf{s})$ has a special Markov partition coming from the its presentation as a shift-invariant quotient.

Definition 3.2. *For every finite word $v \in X^*$ the respective* tile $\mathcal{T}_v$ *is the image of the cylindrical set $X^{-\omega}v$ in the limit space $\mathcal{J}_G$. We say that $\mathcal{T}_v$ is a tile of the level number $|v|$.*

We have the following obvious properties of the tiles.

1. Every tile $\mathcal{T}_v$ is a compact set.

2. $\mathsf{s}(\mathcal{T}_{vx}) = \mathcal{T}_v$.

3. $\mathcal{T}_v = \cup_{x \in X} \mathcal{T}_{xv}$.

In particular, the image of a tile $\mathcal{T}_v$ of nth level under the shift map s is a union of d tiles $\mathcal{T}_u$ of the nth level, i.e., that the tiles of one level for a Markov partition of the dynamical system $(\mathcal{J}_G, \mathcal{T})$.

Actually, a usual definition of a Markov partition requires that two tiles do not overlap, i.e., that they do not have common interior points. We have the following criterion (for a proof see also [11]).

We say that a self-similar action satisfies the *open set condition* if for every $g \in G$ there exists $v \in X^*$ such that $g|_v = 1$.

Theorem 3.3. *If a contracting action of a group G on X^* satisfies the open set condition then for every $n \geq 0$ and for every $v \in X^n$ the boundary of the tile $\mathcal{T}_v$ is equal to the set*

$$\partial \mathcal{T}_v = \bigcup_{u \in X^n, u \neq v} \mathcal{T}_u \cap \mathcal{T}_v,$$

and the tiles of one level have disjoint interiors.

If the action does not satisfy the open set condition, then there exists $n \in \mathbb{N}$ and a tile of nth level, which is covered by other tiles of nth level.

4. Post-critically finite limit spaces

Following [1], we adopt the following definition.

Definition 4.1. *We say that a contracting action (G, X^*), has a* post-critically finite (p.c.f.) limit space *if intersection of every two different tiles of one level is finite.*

We obtain directly from Theorem 3.3 that a contracting action has a p.c.f. limit space if and only if it satisfies the open set condition and the boundary of every tile is finite.

The following is an easy corollary of Theorem 3.3 and Proposition 3.2.

Proposition 4.1. *The image of a sequence $\ldots x_{n+1}x_n \ldots x_1 \in X^{-\omega}$ belongs to the boundary of the tile $\mathcal{T}_{x_n \ldots x_1}$ if and only if there exists a sequence $\{g_k\}$ of elements of the nucleus such that $g_{k+1} \cdot x_{k+1} = x_k \cdot g_k$ and $g_n(x_n \ldots x_1) \neq x_n \ldots x_1$.*

This gives us an alternative way of defining p.c.f. limit spaces.

Corollary 4.2. *A contracting action (G, X^*) has a p.c.f. limit space if and only if there exists only a finite number of left-infinite paths in the Moore diagram of its nucleus which end in a non-trivial state.*

Proof. We say that a sequence $\ldots x_2 x_1 \in X^{-\omega}$ is read on a left-infinite path $\ldots e_2 e_1$, if the label of the edge e_i is $(x_i; y_i)$ for some $y_i \in X$. If the path $\ldots e_2 e_1$ passes through the states $\ldots g_2 g_1 g_0$ (here g_i is the beginning and g_{i-1} is the end of the edge e_i), then the state g_{n-1} is uniquely defined by g_n and x_n, since $g_{n-1} = g_n|_{x_n}$. Consequently, any given sequence $\ldots x_2 x_1$ is read not more than on $|\mathcal{N}|$ left-infinite paths of the nucleus $\mathcal{N}$. In particular, every asymptotic equivalence class on $X^{-\omega}$ has not more than $|\mathcal{N}|$ elements.

For every non-trivial state $g \in \mathcal{N}$ denote by B_g be the set of sequences, which are read on the left-infinite paths of the nucleus, which end in g.

Suppose that there is infinitely many left-infinite paths in the nucleus ending in a nontrivial state. Then there exists a state $g \in \mathcal{N} \setminus \{1\}$ for which the set B_g is infinite.

Since the state g is non-trivial, there exists a word $v \in X^*$ such that $g(v) \neq v$. Then for every $\ldots x_2 x_1 \in B_g$, there exists a sequence $\ldots y_2 y_1$ such that $\ldots x_2 x_1 v$ is asymptotically equivalent to $\ldots y_2 y_1 g(v)$. Hence, every point of $\mathcal{J}_G$ represented by a sequence from $B_g v$ belongs both to $\mathcal{T}_v$ and to $\mathcal{T}_{g(v)}$. This show that the intersection $\mathcal{T}_v \cap \mathcal{T}_{g(v)}$ is infinite, since the asymptotic equivalence classes are finite.

On the other hand, Proposition 4.1 shows, that if the sequence $\ldots x_2 x_1 v$, represents a point of the intersection $\mathcal{T}_v \cap \mathcal{T}_u$ for $u \in X^{|v|}, u \neq v$, then the sequence $\ldots x_2 x_1$ is read on some path of the nucleus, which ends in a non-trivial state. Therefore, if the intersection $\mathcal{T}_v \cap \mathcal{T}_u$ is infinite, then the set of left-infinite paths in the nucleus is infinite. □

Proposition 4.3. *If the limit space of a contracting action is post-critically finite, then its topological dimension is ≤ 1.*

Proof. We have to prove that every point $\zeta \in \mathcal{J}_G$ has a basis of neighborhoods with 0-dimensional boundaries.

Let $T_n(\zeta)$ be the union of the tiles of nth level, containing ζ. It is easy to see that $\{T_n(\zeta) \,:\, n \in \mathbb{N}\}$ is a base of neighborhoods of ζ. □

5. Automata with bounded cyclic structure

We take Corollary 4.2 as a justification of the following definition.

Definition 5.1. *A self-similar contracting group is said to be* post-critically finite *(p.c.f. for short) if there exists only a finite number of inverse paths in the nucleus ending at a non-trivial state.*

A more precise description of the structure of the nucleus of a p.c.f. group is given in the next proposition.

Recall, that an automatic transformation q of X^* is said to be *finitary* (see [19]) if there exists $n \in \mathbb{N}$ such that $q|_v = 1$ for all $v \in X^n$ (then $q(x_1 \dots x_m) = q(x_1 \dots x_n)x_{n+1} \dots x_m$). The minimal number n is called *depth* of q.

The set of all finitary automatic transformations of X^* is a locally finite group. If G is a finite subgroup of the group of finitary transformations, then the *depth* of G is the greatest depth of its elements.

If we have a subset A of the vertex set of a graph Γ, then we consider it to be a subgraph of Γ, taking all the edges, which start and end at the vertices of A.

We say that a directed graph is a *simple cycle* if its vertices $g_1, g_2, \dots, g_n$ and edges $e_1, e_2, \dots, e_n$ can be indexed so that e_i starts at g_i and ends at g_{i+1} (here all g_i and all e_i are pairwise different and $g_{n+1} = g_1$).

Proposition 5.1. *Let $\mathcal{N}$ be the nucleus of a p.c.f. group, and let $\mathcal{N}_0$ be the subgraph of finitary elements of $\mathcal{N}$ and $\mathcal{N}_1 = \mathcal{N} \setminus \mathcal{N}_0$. Then $\mathcal{N}_1$ is a disjoint union of simple cycles.*

Proof. The set $\mathcal{N}_0$ is obviously a sub-automaton, i.e., for every $g \in \mathcal{N}_0$ and $x \in X$ we have $g|_x \in \mathcal{N}_0$. It follows then from the definition of a nucleus that every vertex of the graph $\mathcal{N}_1$ has an incoming arrow. This means that every vertex of the graph $\mathcal{N}_1$ is an end of a left-infinite path. On the other hand, there exists for every $g \in \mathcal{N}_1$ at least one $x \in X$ such that $g|_x \in \mathcal{N}_1$, since all elements of $\mathcal{N}_1$ are not finitary. Thus, every vertex of $\mathcal{N}_1$ has an outgoing arrow and is a beginning of a right-infinite path.

Let g be an arbitrary vertex of the graph $\mathcal{N}_1$. We have a left-infinite path γ_- ending in g. Suppose that we have a (pre-)periodic right-infinite path γ_+ starting at g, i.e., a path of the form $\gamma_+ = qppp \dots = qp^\omega$, were q is a finite path, p is a finite simple cycle and the set of edges of the paths p and q are disjoint. Note that there always exists a (pre-)periodic path beginning at g.

If q is not empty, then we get an infinite set of different left-infinite paths in the graph $\mathcal{N}_1$: $\{\gamma_- qp^n\}_{n \in \mathbb{N}}$, what contradicts to the post-critical finiteness of the action.

Hence the pre-period q is empty. In particular, every element of $\mathcal{N}_1$ belongs to a finite cycle, i.e., for every $g \in \mathcal{N}_1$ there exists $v \in X^*$ such that $g|_v = g$.

Suppose now that there exist two different letters $x, y \in X$ such that $g|_x$ and $g|_y$ belong to $\mathcal{N}_1$. The element $g|_x$ belongs to a finite cycle p_x in $\mathcal{N}_1$. The cycle p_x must contain the element g, otherwise we get a strictly pre-periodic path starting at g. Similarly, there exists a cycle p_y , which contains g and $g|_y$. The cycles p_x and p_y are different and intersect in the vertex g. Hence we get an infinite set of left-infinite paths in $\mathcal{N}_1$ of the form $\ldots p_3p_2p_1$, where p_i are either p_x or p_y (seen as paths starting at g) in an arbitrary way.

Hence, for every $g \in \mathcal{N}_1$ there exists only one letter $x \in X$ such that $g|_x \in \mathcal{N}_1$. This (together with the condition that every vertex of $\mathcal{N}_1$ has an incoming edge) implies that $\mathcal{N}_1$ is a disjoint union of simple cycles. □

The following notion was defined and studied by Said Sidki in [16].

Definition 5.2. *We say that an automatic transformation q is* bounded *if the sequence $\theta(k, q)$ is bounded, where $\theta(k, q)$ is the number of words $v \in X^k$ such that $q|_v$ acts non-trivially on the first level X^1 of the tree X^*.*

The following proposition is proved in [16] (Corollary 14).

Proposition 5.2. *An automatic transformation is bounded if and only if it is defined by a finite automaton in which every two non-trivial cycles are disjoint and not connected by a directed path.*

Here a cycle is *trivial* if its only vertex is the trivial state. In particular, every finitary transformation is bounded, since it has no non-trivial cycles.

Theorem 5.3. *The set $\mathcal{B}$ of all bounded automorphisms of the tree X^* is a group.*

A finitely generated self-similar automorphism group G of the tree X^* has a p.c.f. limit space if and only if it is a subgroup of $\mathcal{B}$. In particular, every finitely generated self-similar subgroup of $\mathcal{B}$ is contracting.

Proof. The fact that $\mathcal{B}$ is a group, is proved in [16]. We have also proved that the nucleus of every p.c.f. group G is a subset of $\mathcal{B}$. This implies that G is a subgroup of $\mathcal{B}$.

In the other direction, suppose that we have a self-similar finitely generated subgroup $G \leq \mathcal{B}$. Then G is generated by a finite automaton S whose all non-trivial cycles are disjoint. Let S_0 be the subautomaton of all finitary tranformations, and let $S_2 = S \setminus S_0$. Then all non-trivial cycles belong to S_2. Let S_1 be the union of all these cycles.

Let $g \in S_1$ and $v \in X^*$ be arbitrary. Then either $g|_v$ belongs to the same cycle as g, or $g|_v \notin S_1$, since no two different cycles of S_1 can be connected by a directed path. If $g|_v \notin S_1$, then all states $g|_{vu}$ of $g|_v$ do not belong to S_1. But this is possible only when $g|_v \in S_0$. Therefore, there exists $m \in \mathbb{N}$ such that for every $g \in S$ and every $v \in X^m$ either $g|_v \in S_0$, or $g|_v \in S_1$. Then the group $G_1 = \langle G|_{X^m} \rangle$ is also self-similar and is generated by a subset of the set $S_0 \cup S_1$. The group G is contracting if and only if G_1 is contracting. Their nuclei will coincide. Therefore, if we prove our theorem for G_1, then it will follow for G, so we assume that $S_2 = S_1$.

Let n_1 be the least common multiple of the lengths of cycles in S_1. Then for every $u \in X^{n_1}$ and $s \in S_1$ we have either $s|_u \in S_0$ or $s|_u = s$. Moreover, it follows from the conditions of the theorem that the word $u \in X^{n_1}$ such that $s|_u = s$ is unique for every $s \in S_1$.

Let $\mathcal{N}_1$ be the set of all elements $h \in G \setminus 1$ such that there exists one word $u(h) \in X^{n_1}$ such that $h|_{u(h)} = h$ and for all the other words $u \in X^{n_1}$ the restriction $h|_u$ belongs to $\langle S_0 \rangle$. It is easy to see that the set $\mathcal{N}_1$ is finite (every its element h is uniquely defined by the permutation it induces on X^{n_1} and its restrictions in the words $u \in X^{n_1}$, note also that the group $\langle S_0 \rangle$ is finite).

Let us denote by $l_1(g)$ the minimal number of elements of $S_1 \cup S_1^{-1}$ in a decomposition of g into a product of elements of $S \cup S^{-1}$.

Let us prove that there exists for every $g \in G$ a number k such that for every $v \in X^{n_1 k}$ the restriction $g|_v$ belongs to $\mathcal{N}_1 \cup \{S_0\}$. We will prove this by induction on $l_1(g)$.

If $l_1(g) = 1$, then $g = h_1 s h_2$, where $h_1, h_2 \in \langle S_0 \rangle$ and $s \in S_1$. The elements h_1, h_2 are finitary, thus there exists k such that for every $v \in X^{n_1 k}$ the restriction $h_i|_v$ is trivial. Then we have $h_1 s h_2|_v = s|_{h_2(v)}$, thus $g|_v$ is either equal to $s \in \mathcal{N}_1$ or belongs to $S_0 \cup S_0^{-1}$. Thus the claim is proved for the case $l_1(g) = 1$.

Suppose that the claim is proved for all elements $g \in G$ such that $l_1(g) < m$. Let $g = s_1 s_2 \ldots s_k$, where $s_i \in S \cup S^{-1}$. For every $u \in X^{n_1}$ the restriction $s_i|_u$ is equal either to s_i or belongs to S_0. Consequently, either $g|_u = g$ for one u and $g|_u \in \langle S_0 \rangle$ for all the other $u \in X^{n_1}$, or $l_1(g|_u) < l_1(g)$ for every $u \in X^{n_1}$. In the first case we have $g \in \mathcal{N}_1$ and in the second we apply the induction hypothesis, and the claim is proved.

Consequently, the group G is contracting with the nucleus equal to a subset of the set $\{g|_v \ : \ g \in \mathcal{N}_1, v \in X^*, |v| < n_1\}$. Note that any restriction $g|_v$ of an element of $\mathcal{N}_1$ either belongs to $\mathcal{N}_1$ or is finitary.

Let us prove that the limit space of the group G is p.c.f.. Suppose that we have a left-infinite path in the nucleus of the group. Let

$$\ldots h_3, h_2, h_1$$

be the elements of the nucleus $\mathcal{N}$ it passes through and let the letters

$\ldots, x_3, x_2, x_1$ be the letters labeling its edges. In other words, we have

$$h_n = h_{n+1}|_{x_n}$$

for every $n \geq 1$.

The number of possibilities for h_n is finite, thus it follows from the arguments above that every element h_i belongs to $\mathcal{N}_1 \cup \langle S_0 \rangle$. The elements of $\langle S_0 \rangle$ can belong only to the ending of the sequence h_i of the length not greater than the depth of the group $\langle S_0 \rangle$. The rest of the sequences h_i and x_i is periodic with period n_1. Hence, there exists only a finite number of possibilities for such a sequence, and the limit space of the group is p.c.f. □

Corollary 5.4. *The word problem is solvable in polynomial time for every finitely generated subgroup of $\mathcal{B}$.*

Proof. The word problem in every finitely generated contracting group is solvable in polynomial time (see [6]). □

References

[1] Jun Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 2001.

[2] G. Baumslag. *Topics in combinatorial group theory.* Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[3] Tom Lindstrøm. Brownian motion on nested fractals. *Mem. Am. Math. Soc.*, 420:128 p., 1990.

[4] Rostislav I. Grigorchuk. Just infinite branch groups. In Aner Shalev, Marcus P. F. du Sautoy, and Dan Segal, editors, *New horizons in pro-p groups*, volume 184 of *Progress in Mathematics*, pages 121–179. Birkhäuser Verlag, Basel, etc., 2000.

[5] Said N. Sidki. *Regular Trees and their Automorphisms*, volume 56 of *Monografias de Matematica*. IMPA, Rio de Janeiro, 1998.

[6] Volodymyr V. Nekrashevych. Virtual endomorphisms of groups. *Algebra and Discrete Mathematics*, 1(1):96–136, 2002.

[7] Rostislav I. Grigorchuk. On Burnside's problem on periodic groups. *Funtional Anal. Appl.*, 14(1):41–43, 1980.

[8] Yuriĭ I. Merzlyakov. Infinite finitely generated periodic groups. *Dokl. Akad. Nauk SSSR*, 268(4):803–805, 1983.

[9] A. V. Rozhkov. *Finiteness conditions in automorphism groups of trees.* PhD thesis, Cheliabinsk, 1996. Habilitation thesis.

[10] Jun Kigami. Laplacians on self-similar sets — analysis on fractals. *Transl., Ser. 2, Am. Math. Soc. 161, 75-93 (1994); translation from Sugaku 44, No.1, 13-28 (1992)*, 1992.

[11] Volodymyr V. Nekrashevych. Limit spaces of self-similar group actions. preprint, Geneva University, available at `http://www.unige.ch/math/biblio/preprint/2002/limit.ps`, 2002.

[12] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunik̀. Branch groups. preprint.

[13] S. V. Aleshin. Finite automata and the Burnside problem for periodic groups. *Mat. Zametki*, 11:319–328, 1972. (in Russian).

[14] Narain D. Gupta and Said N. Sidki. On the Burnside problem for periodic groups. *Math. Z.*, 182:385–388, 1983.

[15] L. Bartholdi, R. Grigorchuk, and V. Nekrashevych. From fractal groups to fractal sets. In Peter Grabner and Wolfgang Woess, editors, *Fractals in Graz 2001. Analysis – Dynamics – Geometry – Stochastics*, pages 25–118. Birkhäuser Verlag. Basel. Boston. Berlin., 2003.

[16] Said N. Sidki. Automorphisms of one-rooted trees: growth, circuit structure and acyclicity. *J. of Mathematical Sciences (New York)*, 100(1):1925–1943, 2000.

[17] C. Sabot. Existence and uniqueness of diffusions of finitely ramified self-similar fractals. *Ann. Sci. Éc. Norm. Supér., IV. Sér.*, 30(5):605–673, 1997.

[18] Vitaliĭ I. Sushchansky. Periodic permutation p-groups and the unrestricted Burnside problem. *DAN SSSR.*, 247(3):557–562, 1979. (in Russian).

[19] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and Vitaliĭ I. Sushchanskii. Automata, dynamical systems and groups. *Proceedings of the Steklov Institute of Mathematics*, 231:128–203, 2000.

Contact information

V. Nekrashevych

Kyiv Taras Shevchenko University
Mechanics and Mathematics Faculty
Algebra Department
Volodymyrska, 60
Kyiv, Ukraine, 01033
E-Mail: `nazaruk@ukrpack.net`