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Dynamics of finite groups acting on the boundary of homogenous rooted tree

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Abstract. Criterion of embedding of finite groups into the automorphism groups of a homogenous rooted tree of a spherical index n is formulated. The sets of natural numbers which are the lengths of all orbits of finite groups acting on the boundary of tree are described.

1. Introduction

Let X be a finite set such that $|X| = n$. We put $\overline{X}_m = X \times \cdots \times X$ (m times) for $m \in \mathbb{N}$ and $\overline{X}_0 = \{\emptyset\}$. The elements of these sets we call *vertices* and vertex $\emptyset$ we call *root*. Now we organize the vertices as follows: the vertex $(x_1, x_2, \ldots, x_{m-1}, x_m) \in \overline{X}_m$ we connect with the vertex $(x_1, x_2, \ldots, x_{m-1})$ for $m \in \mathbb{N} \setminus \{1\}$ and all vertices $x_1 \in X_1$ we connect with the root. In this way we obtain the graph T_X which is a *homogenous rooted tree of the spherical index n*. Now we denote by ∂T_X the boundary of the tree T_X, that is $\partial T_X = X^\omega$. We denote by G_X the automorphisms group of the tree T_X. Obviously the group G_X operates on ∂T_X.

Theorem 1. *A finite group G has a faithful representation by automorphisms of the tree T_X if and only if G has a subnormal series*

$$G = G_1 \rhd G_2 \rhd \cdots \rhd G_{k+1} = \{1\}$$

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such that for every i, $1 \leq i \leq k$, *the quotient* G_i/G_{i+1} *can be faithful represented by permutations of the set* X.

Recall that *orbits* of an action of a group G on a set A are classes of the equivalence relation $\sim_G$ defined by the condition

$$x \sim_G y \Leftrightarrow \exists g \in G : x^g = y; x, y \in A.$$

The *length* of an orbit is its cardinality. If the group G is finite, then the cardinality of every its orbit is a divisor of its order. By the symbol $Orb(G, A)$ we denote the set of all orbit lengths of the group G on the set A. For a finite group G the set $Orb(G, A)$ is obviously finite. A positive integer is called $n-$*number* if for any its prime divisor p the inequality $p \leq n$ holds true. The set of all $n-$numbers will be denoted by E_n.

Theorem 2.

1) *A positive integer number* k *belongs to the set* $Orb(G, \partial T_X)$ *for some finite subgroup* $G < G_X$ *if and only if* k *is a* $n-$*number.*

2) *For any finite subset* $D \subset E_n$ *there exists a finite subgroup* $G < G_X$ *such that* $Orb(G, \partial T_X) = D$.

Theorems 1, 2 can be generalized to the case of spherically homogenous rooted trees (for definitions see [5]).

2. Preliminaries

Here we state the well-known facts about the group G_X.

Lemma 1. *For any* X *the group* G_X *is isomorphic to the infinite wreath power of symmetric groups* S_n, $|X| = n$, *that is*

$$G_X \simeq \wr_{i=1}^{\infty} S_n^{(i)}, \quad S_n^{(i)} = S_n.$$

Proof see, for example, in [4].
The definition of a finitely or infinitely iterated wreath product we can find in [2],[3]. According to [3] every element u of the wreath product $\wr_{i=1}^{\infty} S_n^{(i)}$ is defined by infinite tuple of the type

$$u = [u_1, u_2(x_1), u_3(x_1, x_2), \ldots],$$

where $u_1 \in S_n$, $u_i(x_1, \dots, x_{i-1}) \in S_n^{\overline{X}_{i-1}}$ for $i > 1$. Following [3] we call such a tuple a *tableau*. The action of the tableau u on a sequence $\overline{x} \in X^\omega$, is defined by the equality

$$\overline{x}^u = (x_1, x_2, x_3, \dots)^u = \left(x_1^{u_1}, x_2^{u_2(x_1)}, x_3^{u_3(x_1,x_2)}, \dots\right).$$

We denote by $G_{X,m}$ the subgroup of G_X which contains all automorphisms $u \in G_X$ of the type

$$[u_1, u_2(x_1), \dots, u_m(x_1, \dots, x_{m-1}), \varepsilon, \varepsilon, \dots].$$

It is clear that $G_{X,1} \leqslant G_{X,2} \leqslant \dots$. Let $FG_X = \bigcup_{m=1}^{\infty} G_{X,m}$.

Lemma 2. *The subgroup FG_X is a locally finite π-group, where π is the set of prime divisors of n.*

Proof. For every m the group $G_{X,m}$ is isomorphic to the wreath product $\wr_{i=1}^{m} S_n^{(i)}$. Since the symmetric group S_n is π-group, is both $G_{X,m}$ and FG_X are π-groups. Obviously for any m the group $G_{X,m}$ is finite and $FG_X = \bigcup_{m=1}^{\infty} G_{X,m}$ is locally finite. □

We use also two statements about wreath product of permutation groups.

Lemma 3. *Let (V_i, X_i) be a subgroup of a permutation group (U_i, X_i) for $i = 1, 2, \dots, k$. Then the wreath product $\wr_{i=1}^{k} U_i$ contains a subgroup isomorphic to $\wr_{i=1}^{k} V_i$.*

Proof. According [3] each element of wreath product $\wr_{i=1}^{k} U_i$ can be presented by a tableau of the type

$$[u_1, u_2(x_1), \dots, u_k(x_1, \dots, x_{k-1})],$$

where $u_1 \in U_1$, $u_i(x_1 \dots, k_{i-1}) \in U_i^{X_1 \times \dots \times X_{i-1}}$ for $2 \le i \le k$.
The set of tableaus

$$[v_1, v_2(x_1), \dots, v_k(x_1, \dots, x_{k-1})]$$

such that $v_1 \in V_1$, $v_i(x_1, \dots, x_{i-1}) \in V_i^{X_1 \times \dots \times X_{i-1}}$, $2 \le i \le k$ forms a subgroup of the wreath product $\wr_{i=1}^{k} U_i$ which is isomorphic to $\wr_{i=1}^{k} V_i$. □

The following statement is well known Kaloujnine-Krasner's theorem for finitely iterated wreath products [2].

Lemma 4. *Let a group G have a subnormal series $G = G_1 \rhd G_2 \rhd \cdots \rhd G_{k+1} = \{1\}$, with the quotients being $G_i/G_{i+1} = H_i$ $(i = 1, \ldots, k)$. If H_i can be faithful represented by permutation of the set X for all $i = 1, 2, \ldots, k$, then the group G can be embedded into the wreath product $\wr_{i=1}^{k} H_i$ of permutation groups $(H_1, X), (H_2, X), \ldots, (H_k, X)$.*

3. Proof of Theorem 1

1) Let G have a subnormal series $G = G_1 \rhd G_2 \rhd \cdots \rhd G_{k+1} = \{1\}$ such that for every $i = 1, \ldots, k$, the quotient group $G_i/G_{i+1} = H_i$ can be faithful represented by permutations of the set X. By Kaloujnine-Krasner's theorem the group G is isomorphically embedded into the wreath product $\wr_{i=1}^{k} H_i$ of permutation groups $(H_1, X), (H_2, X), \ldots, (H_k, X)$. By lemma 3 the wreath product $\wr_{i=1}^{k} H_i$ is isomorphic to a subgroup of the wreath power $\wr_{i=1}^{k} S_n^{(i)}$ of symmetric groups of degrees n. Hence, we have the sequence of embeddings

$$G \hookrightarrow \wr_{i=1}^{k} H_i \hookrightarrow \wr_{i=1}^{k} S_n^{(i)} \simeq G_{X,k} \hookrightarrow FG_X$$

and hence the group G is embedded into FG_X.
2) We first prove that every finite group G which is embeddable in G_X can be embedded into a subgroup FG_X. Let $G = \{u_1, \ldots, u_m\}$ be a finite subgroup of G_X, where

$$u_k = [u_{1,k}, u_{2,k}(x_1), \ldots, u_{m,k}(x_1, \ldots, x_{m-1}), \ldots], \quad 1 \le k \le m.$$

For every $l \in \mathbb{N}$ we construct the group $G(l) = \left\{u_1^{(l)}, \ldots, u_m^{(l)}\right\}$, where

$$u_k^{(l)} = [u_{1,k}, u_{2,k}(x_1), \ldots, u_{l,k}(x_1, \ldots, x_{l-1})].$$

In this way we obtain a sequence of finite groups $G(1), G(2), \ldots$ such that

$$|G(1)| \leqslant |G(2)| \leqslant \ldots .$$

Since the group G is finite, there exists $k \in \mathbb{N}$ such that for $i > k$ we have $|G(i)| = |G|$. For every l the group $G(l)$ is a homomorphic image of G

(the natural projection of the longer wreath power $\wr_{i=1}^{\infty} S_n^{(i)}$ into shorter ones $\wr_{i=1}^{l} S_n^{(i)}$). Hence, for $i \geq k$ the group $G(i)$ is isomorphic to G. But $G(i)$ is embedded into FG_X in the natural way.
Let G be a finite subgroup of FG_X. There exists $m \in \mathbb{N}$ such that G is a subgroup of $G_{X,m}$, i.e. G is embedded into the wreath power $\wr_{i=1}^{m} S_n^{(i)} = W$. Denote by W_i the i-th base of W. Then $W = W_1 \rhd W_2 \rhd \cdots \rhd W_m \rhd W_{m+1} = \{1\}$ and $U_i = W_i/W_{i+1} \simeq S_n \times \cdots \times S_n$ (n^{i-1} times). For any i ($1 \leq i \leq m$) denote by

$$U_{i,1}, U_{i,2}, \ldots, U_{i,n^{i-1}}, U_{i,n^{i-1}+1}$$

subgroup series of U_i such that

$$U_{i,k} = \{(1, \ldots, 1, \sigma_k, \ldots, \sigma_{n^{i-1}}) | \sigma_k, \ldots, \sigma_{n^{i-1}} \in S_n\}, \quad 1 \leq k \leq n^{i-1} + 1.$$

Then $U_i = U_{i,1}$, $U_{i,k+1} \lhd U_{i,k}$ for $k = 1, \ldots, n^{i-1}$. We construct a subnormal series for G which quotients can be faithful represented by permutations of X in the following way.
Let $H_i = G \cap W_i$. Then

$$G = H_1 \rhd H_2 \rhd \cdots \rhd H_m \rhd H_{m+1} = \{1\}. \tag{1}$$

Without loss of generality, we can suppose that $H_i \neq H_{i+1}$, $i = 1, \ldots, m$. For any i ($1 \leq i \leq m$) we have the natural embedding

$$K_i = H_i/H_{i+1} \hookrightarrow W_i/W_{i+1}.$$

Hence, we can define subgroups $K_{i,l} = K_i \cap U_{i,l}$, ($l \leq i \leq m+1$). Let $\overline{K}_{i,l}$ be a inverse image of $K_{i,l}$ in H_i. Then for all i ($1 \leq i \leq m$) we have subnormal series

$$H_i = \overline{K}_{i,1} \rhd \overline{K}_{i,2} \rhd \cdots \rhd \overline{K}_{i,m} \rhd \overline{K}_{i,m+1} = H_{i+1}. \tag{2}$$

Now we can extend the subnormal series (1) by (2). Which completes the proof. □

4. Proof of Theorem 2

1) If $G < G_X$, $|G| < \infty$, then by theorem 1 the group G is embedded into $\wr_{i=1}^{k} S_n^{(i)}$ for some $k \in \mathbb{N}$. Moreover, $|G| \in E_n$ because $\left| \wr_{i=1}^{k} S_n^{(i)} \right| \in E_n$. The length of an orbit of G on ∂T_X is a divisor of G and consequently is

a $n-$number.
On the other hand, let $m \in E_n$. Then $m = m_1 \cdot m_2 \cdots m_k$, where $m_i|n$ $(1 \leq i \leq k)$. Let $X = \{1, 2, \dots, n\}$, α_i be a cyclic permutation $(1, 2, \dots, m_i) \in S_n$. We construct the automorphism

$$v = [v_1, v_2(x_1), \dots, v_k(x_1, \dots, x_{k-1}), \varepsilon, \varepsilon, \dots] \in G_X$$

as follows: $v_1 = \alpha_1$,

$$v_i(x_1, \dots, x_{i-1}) = \begin{cases} \alpha_i & \text{for} \quad (x_1, \dots, x_{i-1}) = (1, \dots, 1) \\ \varepsilon \end{cases}$$

for $2 \leq i \leq k$. We can directly check that v has the order m and v has a cycle C of the length m on ∂T_X. Let G be the cyclic group generated by v. Then C is an orbit of G, and hence $|C| = m$, $m \in Orb(G, \partial T_X)$ and 1) is proved.

2) Let D be a finite set of $n-$numbers. Then by [1] there exists an automorphism $f \in G_X$ such that the set of the cycle lengths of f is equal to D. Since D is finite, the cyclic group $\langle f \rangle = H$ is finite as well. Every orbit of the group H coincides with the set of elements of some cycle of the automorphism f. Hence, $Orb(H, \partial T_X) = D$ and theorem 2 is proved.

Remark. From the proof follows that for every finite subgroup $G < G_X$ there exists finite cyclic subgroup $H < G_X$ such that $Orb(G, \partial T_x) = Orb(H, \partial T_x)$.

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