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RESEARCH ARTICLE

Structural properties of extremal asymmetric colorings

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Dedicated to R. I. Grigorchuk on the occasion of his 50th birthday

Abstract. Let Ω be a space with probability measure μ for which the notion of symmetry is defined. Given $A \subseteq \Omega$, let $ms(A)$ denote the supremum of $\mu(B)$ over symmetric $B \subseteq A$. An r-coloring of Ω is a measurable map $\chi : \Omega \to \{1, \ldots, r\}$ possibly undefined on a set of measure 0. Given an r-coloring χ, let $ms(\Omega; \chi) = \max_{1 \le i \le r} ms(\chi^{-1}(i))$. With each space Ω we associate a Ramsey type number $ms(\Omega, r) = \inf_\chi ms(\Omega; \chi)$. We call a coloring χ *congruent* if the monochromatic classes $\chi^{-1}(1), \ldots, \chi^{-1}(r)$ are pairwise congruent, i.e., can be mapped onto each other by a symmetry of Ω. We define $ms^\star(\Omega, r)$ to be the infimum of $ms(\Omega; \chi)$ over congruent χ.

We prove that $ms(S^1, r) = ms^\star(S^1, r)$ for the unitary circle S^1 endowed with standard symmetries of a plane, estimate $ms^\star([0, 1), r)$ for the unitary interval of reals considered with central symmetry, and explore some other regularity properties of extremal colorings for various spaces.

1. Introduction

A Ramsey type problem is generated by the following pattern: Given a structure Ω, one has to determine the conditions ensuring that, for any r-coloring $\chi : \Omega \to \{1, \ldots, r\}$, at least one of the monochromatic classes $\chi^{-1}(i)$ contains a regular substructure of a certain prescribed kind. This

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problem is split into two parts. One is to show that, under some conditions, no partition of Ω can completely destroy large regular substructures of Ω, whereas the other part is, under the other conditions, to construct a "bad" partition of Ω destroying all such substructures. We call such "bad" colorings of Ω extremal. Our aim is here to explore properties of extremal colorings for a class of Ramsey problems about symmetric substructures, where the symmetry has the standard geometric meaning.

Banakh and Protasov [2] initiated the study of colorings of the n-dimensional integer grid and, more generally, of an infinite abelian group which destroy all infinite centrally symmetric subsets. Finitary problems of similar flavour were intensively studied in combinatorial number theory (see e.g. [10, 6, 9]). In [4] (see also a survey [5]) we considered the following general problem.

Let Ω be a space with probability measure μ endowed with a family of transformations $\mathcal{S}$ called symmetries. A set $B \subseteq \Omega$ is symmetric if $s(B) = B$ for some $s \in \mathcal{S}$. Given $A \subseteq \Omega$, let $ms(A)$ denote the supremum of $\mu(B)$ over symmetric measurable $B \subseteq A$. Given a measurable coloring $\chi : \Omega \to \{1, \ldots, r\}$, let $ms(\Omega; \chi) = \max_{1 \le i \le r} ms(\chi^{-1}(i))$. With each space Ω we associate a Ramsey type number $ms(\Omega, r) = \inf_\chi ms(\Omega; \chi)$. We call a coloring χ *extremal* if $ms(\Omega, r) = ms(\Omega; \chi)$.

One of the results obtained in [4] is that, if Ω is a circle or an arbitrary figure of revolution in a Euclidean space, then $ms(\Omega, r) = 1/r^2$. Here μ is the normed Lebesgue measure and $\mathcal{S}$ consists of the isometries of the Euclidean space mapping Ω onto itself.

Another result of [4] is that the disc V^2 in a plane has an extremal coloring $\chi : V^2 \to \{1, \ldots, r\}$ with all monochromatic classes $\chi^{-1}(i)$ pairwise congruent, up to a set of measure 0. The latter means that we allow non-empty difference $V^2 \setminus \bigcup_{i=1}^r \chi^{-1}(i)$ provided its measure is 0. Note that this relaxation is necessary because a compact convex set in a Euclidean space has no partition into two congruent parts [11]. It is also observed in [4] that *every* extremal coloring χ of the disc has a weaker regularity property:

$$ms(\chi^{-1}(1)) = \ldots = ms(\chi^{-1}(r)). \tag{1}$$

The present paper is inspired by these results on extremal colorings of the disc. Generally, given a space Ω with a family of symmetries $\mathcal{S}$, we call two sets $A, B \subseteq \Omega$ congruent if $s(A) = B$ for some $s \in \mathcal{S}$. We call an r-coloring χ of Ω *congruent* if the monochromatic classes $\chi^{-1}(1), \ldots, \chi^{-1}(r)$ are pairwise congruent (χ is here allowed to be undefined on a set of measure 0). We consider also weaker regularity properties. We call a coloring χ *balanced* if the equalities (1) hold true and *uniform* if

$$\mu(\chi^{-1}(1)) = \ldots = \mu(\chi^{-1}(r)).$$

Note that an extremal coloring may not exist. As shown in [4], this is the case for the circle S^1. We therefore consider *extremal sequences* of colorings χ_n such that $ms(\Omega, r) = \lim_{n\to\infty} ms(\Omega; \chi_n)$. We treat extremal sequences of uniform, balanced, and congruent colorings in Sections 3, 4, and 5 respectively. Each section contains an expository part surveying known results about the respective properties of extremal (sequences of) colorings. Notice that these results are split into two categories. An *existential* result says that there exists a congruent (uniform or balanced) extremal coloring or a sequence thereof for a space Ω under consideration. A *universal* result says that *all* extremal colorings of Ω have a certain property. In some cases we also show connections of regularity properties of extremal colorings with relations between some Ramsey numbers investigated in the literature independently.

New results proved here concern mainly the circle and the interval $[0, 1)$ of the real line with standard symmetries (i.e. isometries) of, respectively, 2- and 1-dimensional Euclidean space. For the circle we prove the existence of an extremal sequence of congruent colorings. For the interval we prove the existence of an extremal sequence of balanced colorings. The existence of an extremal sequence of congruent colorings for the interval stays open. We therefore define $ms^\star(\Omega, r)$, the infimum of $ms(\Omega; \chi)$ over congruent r-colorings χ, and prove lower and upper bounds for $ms^\star([0, 1), 2)$. Obviously, $ms(\Omega, r) \le ms^\star(\Omega, r)$. In [4] we prove that $ms([0, 1), 2) \ge 1/(4 + \sqrt{6})$. We here find a better bound for $ms^\star([0, 1), 2)$ by showing that $ms^\star([0, 1), 2) \ge M(1/2)$, where $M(x)$ is a continuous version of an Erdős function [12]. An upper bound obtained in [4] is $ms([0, 1), 2) \le 5/24$. We here prove that the same bound holds true for $ms^\star([0, 1), 2)$. This is done with using so-called *blurred colorings* introduced in [4]. While for congruent bicolorings of the interval we have at least as good bounds as those known for any colorings, congruent colorings of the interval in 3 and more colors turn out to be a much more subtle issue. The upper bound for $ms^\star([0, 1), 2)$ immediately implies that $ms^\star([0, 1), 2r) \le 5/(24r)$ and this is all what we know, whereas $ms([0, 1), r) \le 1/r^2$ for all r. In the case of 3 colors we have $ms^\star([0, 1), 3) \le 2/9$ and improvements upon this seem to depend on some unsolved questions about tilings of rectangles by polyominoes (see [7]).

2. Formal framework

Throughout the paper we denote $[n] = \{1, 2, \ldots, n\}$, the set of the first n positive integers, and $I = [0, 1)$, the unitary interval of the real line.

Let $\mathcal{U}$ be a space with measure μ. The space $\mathcal{U}$ is assumed to be endowed with a family $\mathcal{S}$ of one-to-one maps of $\mathcal{U}$ onto itself, that are measurable and preserve the measure. These maps will be called *admissible symmetries.* A set $B \subseteq \mathcal{U}$ is called *symmetric* if $s(B) = B$ for some non-identity symmetry $s \in \mathcal{S}$. Two sets $B, C \subseteq \mathcal{U}$ are called *congruent* if $s(B) = C$ for some symmetry $s \in \mathcal{S}$.

Given $A \subseteq \mathcal{U}$, define

$$ms(A) = \sup \{ \mu(B) \mid B \text{ is a symmetric measurable subset of } A \} .$$

Note that the maximum, with respect to the inclusion, subset of A symmetric with respect to a symmetry s is equal to $\bigcap_{k=-\infty}^{\infty} s^k(A)$. The latter intersection reduces to $A \cap s(A)$ is s is involutive, i.e. $s = s^{-1}$. In the case that all admissible symmetries are involutive, it is easy to derive a relation

$$|ms(A) - ms(A')| \le 2\mu(A \bigtriangleup A') \tag{2}$$

for any sets A and A'.

Clearly, if A is symmetric with respect to s, then it is symmetric with respect to any k-fold composition s^k. In particular, if A in a Euclidean space is symmetric under the rotation by a rational angle $k\pi/l$, where k and l are coprime, then A is symmetric under the rotation by $2\pi/p$ for p any prime divisor of l.

We consider a set $\Omega \subseteq \mathcal{U}$ with $\mu(\Omega) = 1$, i.e. (Ω, μ) is a probability space. Let $r \ge 2$. An r-coloring of Ω is a map $\chi : \Omega \to [r]$ such that each *color class* $\chi^{-1}(i)$ for $i \le r$ is measurable. A subset of Ω is called *monochromatic* if it is included into a color class. Define $ms(\Omega; \chi) = \max_{1 \le i \le r} ms(\chi^{-1}(i))$ and

$$ms(\Omega, r) = \inf_{\chi} ms(\Omega; \chi),$$

where the infimum is taken over all colorings of Ω. To avoid any ambiguity in the presence of several families of admissible symmetries, we will sometimes use more definite notation $ms(\Omega, \mathcal{S}, r)$ and $ms(\Omega, \mathcal{S}; \chi)$.

In the sequel we will deal with the following particular spaces:

- $\Omega = S^{k-1}$, a sphere, and $\Omega = V^k$, a ball, in $\mathcal{U} = \mathbb{R}^k$. μ is the Lebesgue measure on Ω normed so that $\mu(\Omega) = 1$. $\mathcal{S}$ consists of all isometries s of $\mathcal{U}$ such that $s(\Omega) = \Omega$. Note that the circle S^1 as a space with symmetry is equivalent to the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with "axial" symmetries $a(x) = g - x$ and "rotations" $s(x) = g + x$, $g \in \mathbb{T}$. For every space of this family, $ms(\Omega, r) = 1/r^2$ [4].

- $\Omega = I$, the unitary interval $[0, 1)$ in $\mathcal{U} = \mathbb{R}$. μ is the Lebesgue measure. $\mathcal{S}$ consists of all isometries of the real line, i.e., central symmetries and translations. As proved in [4],

$$\frac{1}{4+\sqrt{6}} \leq ms(I, 2) \leq \frac{5}{24} \tag{3}$$

and

$$ms(I, r) = \frac{c}{r^2}(1 + o(1))$$

for a constant c in the range $1/2 \leq c \leq 5/6$. A result of [9] implies for this constant a better lower bound $c > 0.591389$.

- Ω is the vertex set of the regular n-gon. For every $x \in \Omega$, $\mu(x) = 1/n$. $\mathcal{S}$ consists of the symmetries of a plane. This space is equivalent to $\mathbb{Z}_n$, the cyclic group of order n with symmetries $a(x) = g - x$ and $s(x) = g + x$, $g \in \mathbb{Z}_n$. We have $ms(\mathbb{Z}_n, r) = (1 + o(1))/r^2$.

- $\Omega = [n]$ viewed as a set of n points in the real line. This is a discrete analog of the real interval: As shown in [4], $\lim_{n\to\infty} ms([n], r) = ms(I, r)$.

In some cases we need the whole family of symmetries only in order to define the congruence while for the symmetry it is enough to consider a more restricted set of transformations. The first claim below is obvious; The second claim follows from the classic Weyl theorem saying that any rotation by irrational angle is an ergodic transformation, i.e., has invariant sets only of measure 0 or 1.

Proposition 2.1.

1. *A symmetric subset of the interval I is symmetric with respect to a central symmetry.*

2. *A symmetric subset of the circle S^1 whose measure is neither 0 nor 1 is symmetric with respect to an axial symmetry or a rotation by angle $2\pi/p$ for a prime integer p.* □

We call an r-coloring χ of Ω *extremal* if $ms(\Omega, r) = ms(\Omega; \chi)$. Extremal colorings exist for the disc V^2 and do not exist for the circle S^1 and spheres and balls in higher dimensions [4]. The existence of an extremal coloring for the interval I is open. We call a sequence of r-colorings χ_n of Ω *extremal* if $ms(\Omega, r) = \lim_{n\to\infty} ms(\Omega; \chi_n)$.

3. Uniform extremal colorings

We call an r-coloring χ of a space Ω *uniform* if all the monochromatic classes have the same measure: $\mu(\chi^{-1}(1)) = \ldots = \mu(\chi^{-1}(r))$. We call a sequence of r-colorings χ_n *uniform* if $\max_{1\le i\le j\le r} |\mu(\chi_n^{-1}(i)) - \mu(\chi_n^{-1}(j))| = o(1)$ as $n \to \infty$. Notice that a sequence of uniform colorings and a uniform sequence of colorings are two different notions — the former is stronger than the latter. However, using the inequality (2), it is not hard to show that, if (Ω, μ) is a continuous measure space that has a uniform extremal sequence of colorings, then it has an extremal sequence of uniform colorings. Recall that (Ω, μ) is *continuous* (or *atomless*) if there is no atoms, that is, no sets $A \subseteq \Omega$ with $\mu(A) > 0$ such that for every measurable $B \subseteq A$ either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

In [4] we stated the following two properties of a space Ω:

(L) Every measurable set $A \subseteq \Omega$ contains a symmetric subset $B \subseteq A$ of measure $\mu(B) \ge \mu(A)^2$.

(U) $ms(\Omega, r) \le 1/r^2$.

We proved that Property (L) holds for every figure of revolution in a Euclidean space and (U) holds for every compact subset of a connected Riemannian manifold. Due to this, the first claim below applies to the disc V^2 and the second claim applies to all spheres S^{k-1} and balls V^k. The proof of both the claims is straightforward.

Proposition 3.1.

1. *If a space Ω has both Properties (L) and (U), then every extremal coloring of Ω is uniform.*

2. *If a space Ω has both Properties (L) and (U), then every extremal sequence of colorings of Ω is uniform.* □

Question 3.2. Does there exist an extremal sequence of uniform colorings for the interval I?

Define $\Delta(\epsilon) = \inf \{ms(A) \mid A \subset I,\ \mu(A) = \epsilon\}$. In [4, 5] we posed a question if $ms(I, r) = \Delta(1/r)$.

Proposition 3.3. *Assume that $ms(I, r) = \Delta(1/r)$. Then every extremal coloring of I (if such exists) and every extremal sequence of colorings are uniform.*

Proof. We give a proof for extremal colorings. Suppose, to the contrary, that there is a non-uniform extremal coloring χ of I. Then one of the monochromatic classes $\chi^{-1}(i)$ must have measure strictly more than $1/r$ and we obtain a contradiction by

$$ms(I,r) = ms(I;\chi) \geq ms(\chi^{-1}(i)) \geq \Delta(\mu(\chi^{-1}(i))) > \Delta(1/r).$$

The latter inequality follows from a result in [9] that the function $\Delta(\epsilon)$ is strictly monotone. □

4. Balanced extremal colorings

We call an r-coloring *balanced* if the equalities (1) hold true. Using Proposition 3.1, it is easy to prove its analog for balanced colorings.

Proposition 4.1. *If a space Ω has both Properties (L) and (U), then every extremal coloring of Ω is balanced.* □

We do not know if the same conclusion holds true for the interval I. Let us notice a connection of this question with interrelations between some Ramsey type numbers.

Assume that a space Ω is endowed with a family $\mathcal{S}$ of involutive symmetries. Given an r-coloring χ of Ω, let

$$s(\Omega;\chi) = \sup_{s\in\mathcal{S}} \frac{1}{r}\sum_{i=1}^{r} \mu(\chi^{-1}(i) \cap s(\chi^{-1}(i))).$$

Furthermore, $s(\Omega,r) = \inf_\chi s(\Omega;\chi)$. The number $s(\Omega,r)$ is considered in [3] and [6]. We now suggest another variation of the definition. Given an r-coloring χ of Ω, let $\hat{s}(\Omega;\chi) = \frac{1}{r}\sum_{i=1}^{r} ms(\chi^{-1}(i))$. Define $\hat{s}(\Omega,r) = \inf_\chi \hat{s}(\Omega;\chi)$. Obviously,

$$s(\Omega,r) \leq \hat{s}(\Omega,r) \leq ms(\Omega,r).$$

For the sake of completeness, let us show the place in this hierarchy of the function $\Delta_\Omega(\epsilon) = \inf\{ms(A) \,|\, A \subset \Omega,\ \mu(A) = \epsilon\}$ considered in Section 3 for the interval I. Clearly, $\Delta_\Omega(1/r) \leq ms(\Omega,r)$. Moreover, $\Delta_\Omega(1/r) \leq \hat{s}(\Omega,r)$ provided $\Delta_\Omega(\epsilon)$ is a convex function. Indeed, assume the latter condition and, given an arbitrarily small $\delta > 0$, choose χ so that $\hat{s}(\Omega;\chi) \leq \hat{s}(\Omega,r) + \delta$. Then

$$\hat{s}(\Omega,r) \geq \frac{1}{r}\sum_{i=1}^{r} ms(\chi^{-1}(i)) - \delta \geq \frac{1}{r}\sum_{i=1}^{r} \Delta_\Omega(\mu(\chi^{-1}(i))) - \delta$$
$$\geq \Delta_\Omega\left(\frac{1}{r}\sum_{i=1}^{r}\mu(\chi^{-1}(i))\right) - \delta = \Delta_\Omega(\frac{1}{r}) - \delta.$$

Note that the question about the convexity of $\Delta_I(\epsilon) = \Delta(\epsilon)$ is open.

We know neither whether $s(I,r) \neq ms(I,r)$ nor whether $\hat{s}(I,r) = ms(I,r)$.

Proposition 4.2. *Assume that $ms(I,r) = \hat{s}(I,r)$. Then every extremal coloring of I (if such exists) is balanced.*

Proof. Suppose, to the contrary, that there is an imbalanced extremal coloring χ of I. Since not all $ms(\chi^{-1}(i))$ are equal to each other, the largest of them is strictly greater than their average value. Therefore, $\hat{s}(I,r) \leq \hat{s}(I;\chi) < ms(I;\chi) = ms(I,r)$, a contradiction. □

Without any conditions, we are able to prove at least an existential result on balanced extremal colorings of the interval. The folowing theorem can be easily extended over all continuous spaces.

Theorem 4.3. *For the interval I there is an extremal sequence of balanced colorings. Moreover, if I has an extremal coloring, it has an extremal balanced coloring.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_r\}$ and $\mathcal{B} = \{B_1, \ldots, B_r\}$ be two families each consisting of pairwise disjoint measurable sets of reals. We say that $\mathcal{B}$ *balances* $\mathcal{A}$ if

$$\bigcup_{i=1}^{r} A_i = \bigcup_{i=1}^{r} B_i$$

and

$$\min_{i \leq r} ms(A_i) \leq ms(B_1) = \ldots = ms(B_r) \leq \max_{i \leq r} ms(A_i).$$

For a such $\mathcal{B}$ we use notation $ms(\mathcal{B}) = ms(B_1)$. We say that $\mathcal{A}$ can be balanced if there is $\mathcal{B}$ balancing $\mathcal{A}$. The theorem immediately follows from the following fact.

Lemma 4.4. *Every family $A_1, \ldots, A_r$ of pairwise disjoint measurable subsets of I can be balanced.*

To prove the lemma, we will show that I has the following property for each $s \geq 1$.

Property $P(s)$.

Every family $\mathcal{A} = \{A_1, \ldots, A_s, A_{s+1}\}$ of pairwise disjoint subsets of I such that $ms(A_1) = \ldots = ms(A_s) < ms(A_{s+1})$ can be balanced.

Lemma 4.4 easily follows from the properties $P(1), \ldots, P(r-1)$. Indeed, a family $A_1, \ldots, A_r$ such that $ms(A_1) \leq ms(A_2) \leq \ldots \leq ms(A_r)$ can be balanced step by step by balancing first A_1, A_2 on the account of $P(1)$, then A_1, A_2, A_3 (with A_1, A_2 modified in the first step) on the account of $P(2)$ and so on. It remains to prove the property $P(s)$.

Lemma 4.5. *The interval I has property $P(s)$ for every $s \geq 1$.*

Proof. Given a set $X \subseteq I$ and reals $0 \leq u \leq t \leq 1$, we will use notation $X(u) = X \cap [0, u]$ and $X(u, t) = X \cup (u, t]$. Observe that $ms(X(u))$ and $ms(X \setminus [0, u])$ are continuous functions of u. This easily follows from the relation (2). Instead of $P(s)$, we will actually prove a stronger property.

Property $Q(s)$.
Let $A_1, \ldots, A_s, A_{s+1}$, and D be pairwise disjoint subsets of I such that $ms(A_1) = \ldots = ms(A_s) = ms(A_{s+1})$. Given $t \in [0, 1]$, set $A^t_{s+1} = A_{s+1} \cup D(t)$ and $\mathcal{A}_t = \{A_1, \ldots, A_s, A^t_{s+1}\}$. Then each $\mathcal{A}_t$ can be balanced by some $\mathcal{B}_t$ so that $ms(\mathcal{B}_t)$ is a non-decreasing continuous function of t.

Let us see why $Q(s)$ implies $P(s)$. Assume we are given $A_1, \ldots, A_s, A_{s+1}$ with $ms(A_1) = \ldots = ms(A_s) < ms(A_{s+1})$. Note that $ms(A_{s+1}(u))$ is a continuous function on $[0, 1]$ which increases from 0 to $ms(A_{s+1})$. Therefore $ms(A_{s+1}(u^*)) = ms(A_s)$ for some u^*. Then $P(s)$ immediately follows from $Q(s)$ for $A_1, \ldots, A_s, A'_{s+1} = A_{s+1}(u^*)$, and $D = A_{s+1} \setminus A_{s+1}(u^*)$.

We now prove $Q(s)$ using induction on s.

As a base case, we start with $Q(1)$. We are given A_1 and A_2 with $ms(A_1) = ms(A_2)$ and have to balance $\mathcal{A}_t$ consisting of A_1 and $A^t_2 = A_2 \cup D(t)$. Let us show that $\mathcal{A}_t$ is balanced, for some $u \in [0, t]$, by $\mathcal{B}^u$ consisting of $A_1 \cup D(u)$ and $A_2 \cup D(u, t)$. Let $f(u) = ms(A_1 \cup D(u))$ and $g(u) = ms(A_2 \cup D(u, t))$. Both $f(u)$ and $g(u)$ are continuous. It is also clear that $f(u)$ increases starting from $ms(A_1)$ and $g(u)$ decreases up to $ms(A_2) = ms(A_1)$. Therefore $f(u)$ and $g(u)$ meet at some u^* and $\mathcal{A}_t$ is balanced by $\mathcal{B}^{u^*}$.

Since the function $g(u) - f(u)$ is continuous, the set of its zeroes is closed. Therefore the set $\{\, x \mid g(u) = f(u) \,\}$ has the largest element $u^*(t)$ and we set $\mathcal{B}_t = \mathcal{B}^{u^*(t)}$.

Let us show that $ms(\mathcal{B}_t)$ is non-decreasing and continuous. The former follows from the facts that $ms(\mathcal{B}_t) = ms(A_1 \cup D(u^*(t)))$ and $u^*(t)$ is non-decreasing. For the continuity notice that, for $\delta > 0$,

$$
\begin{aligned}
ms(\mathcal{B}_{t+\delta}) = ms(A_2 \cup D(u^*(t+\delta), t+\delta)) \leq \\
ms(A_2 \cup D(u^*(t+\delta), t)) + 2\delta \\
\leq ms(A_2 \cup D(u^*(t), t)) + 2\delta = ms(\mathcal{B}_t) + 2\delta. \quad (4)
\end{aligned}
$$

We here supposed that $u^*(t+\delta) \leq t$; Otherwise we would have $ms(A_2) \leq ms(\mathcal{B}_t) \leq ms(\mathcal{B}_{t+\delta}) < ms(A_2) + 2\delta$.

Let $s \geq 2$. Assume that $Q(s-1)$ is true and prove $Q(s)$. We are given $A_1, \ldots, A_s, A_{s+1}$ with all $ms(A_i)$ equal to each other and we have to balance $\mathcal{A}_t = \{A_1, \dots, A_s, A_{s+1} \cup D(t)\}$. We will show that, for some $u \le t$, $\mathcal{A}_t$ can be balanced as follows:

- Decrease $A_{s+1} \cup D(t)$ to $A_{s+1} \cup D(u,t)$;
- Extend A_s to $A_s \cup D(u)$;
- Balance $\mathcal{A}'_u = \{A_1, \dots, A_s \cup D(u)\}$ according to $Q(s-1)$ (the induction assumption).

Let $g_t(u) = ms(A_{s+1} \cup D(u,t))$ and $f(u) = ms(\mathcal{B}'_u)$, where $\mathcal{B}'_u$ balances $\mathcal{A}'_u$ as guaranteed by $Q(s-1)$. According to $Q(s-1)$, $f(u)$ is continuous and increases starting from $ms(A_s)$. On the other hand, $g_t(u)$ is continuous and decreases from $ms(A_{s+1} \cup D(t))$ to $ms(A_{s+1}) = ms(A_s)$. Let $u^*(t)$ be the maximum u such that $f(u) = g_t(u)$. The above procedure with $u = u^*(t)$ results in a family $\mathcal{B}_t = \{B_1, \dots, B_{s+1}\}$ which balances $\mathcal{A}_t$.

It remains to show that $ms(\mathcal{B}_t)$ is non-decreasing and continuous. Note first that $u^*(t)$ is non-decreasing function of t. This is because, if $t_1 < t_2$, then $g_{t_2}(u) \ge g_{t_1}(u)$ for $u \le t_1$. Since $ms(\mathcal{B}_t) = f(u^*(t))$ and f is non-decreasing, $ms(\mathcal{B}_t)$ is non-decreasing too.

The continuity follows from the analog of (4) with A_{s+1} in place of A_2. □

The proof of the theorem is complete. □

5. Congruent extremal colorings

We call an r-colorings χ of a space Ω *congruent* if all the monochromatic classes $\chi^{-1}(i)$ are pairwise congruent. A reasonable relaxation of this notion is the following. We allow partially defined colorings and call a such coloring χ *congruent up to a set of measure 0* if all $\chi^{-1}(i)$ are pairwise congruent and $\mu\left(\Omega \setminus \bigcup_{i=1}^{r} \chi^{-1}(i)\right) = 0$.

Proposition 5.1. [4] *The disc V^2 has an extremal coloring congruent up to a set of measure 0.*

In this section we obtain a similar result for the circle S^1 and analize congruent colorings of the interval I. To prove our main results about these spaces, we will use the probabilistic method. We will refer to the well-known estimate for the probability of large deviations.

Lemma 5.2. (Chernoff's bound, see e.g. [1, theorem A.16]) *Let $X_1, \dots, X_n$ be mutually independent identically distributed random variables taking on values in $[0,1]$. Let $m = \mathbf{E}[X_i]$ denote the expectation of* *an X_i. Then, for every $\epsilon > 0$, we have*

$$\mathbf{P}\left[\frac{1}{n}\sum_{i=1}^{n} X_i > m + \epsilon\right] < \exp\left(-\frac{\epsilon^2 n}{2}\right).$$

5.1. The circle

Recall that the circle has no extremal coloring.

Theorem 5.3. *For the circle S^1 there is an extremal sequence of congruent r-colorings.*

Proof. With an r-coloring χ of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ we associate an r-coloring $\tilde{\chi}$ of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as follows. We assume that the element sets of the groups $\mathbb{Z}_n$ and $\mathbb{T}$ are, respectively, $\{0, 1, \ldots, n-1\}$ and $[0, 1)$, but for elements $a \in \mathbb{Z}_n$ and $x \in \mathbb{T}$ we will admit also names, respectively, $a + ln$ and $x + l$ for any $l \in \mathbb{Z}$. We set $\tilde{\chi}(x) = \chi(\lfloor nx \rfloor)$.

We will construct a random coloring χ of $\mathbb{Z}_n$ and show that, with high enough probability, $ms(\mathbb{T}; \tilde{\chi})$ is near to $ms(\mathbb{T}, r) = 1/r^2$. We take $n = 6rm$. Define $\chi : \mathbb{Z}_n \to [r]$ as follows. For an x such that $0 \le x < n/r$, $\chi(x)$ takes on each value $i \in [r]$ with probability $1/r$ independently of the other x's. Let $\sigma : [r] \to [r]$ be the cyclic shift $\sigma = (12 \ldots r)$. For $1 \le i < r$ and $0 \le x < n/r$ we deterministically define $\chi(x + in/r) = \sigma^i(\chi(x))$. Thus, $\chi^{-1}(1)$ is mapped onto $\chi^{-1}(i)$ by the rotation over $i2\pi/r$. It is apparent that, as χ is a congruent coloring of the regular n-gon, $\tilde{\chi}$ is a congruent coloring of the circle.

Claim A. *Let $\mathcal{A}$ be the set of the axial symmetries in a plane. Then $ms(\mathbb{T}, \mathcal{A}; \tilde{\chi}) = ms(\mathbb{Z}_n, \mathcal{A}; \chi)$.*

Proof of Claim. For $i \in [r]$, let f_i denote the characteristic function of $\chi^{-1}(i)$ and $\tilde{f}_i$ denote the characteristic function of $\tilde{\chi}^{-1}(i)$. Note that $\tilde{f}_i(x) = f_i(\lfloor nx \rfloor)$. Define $\tilde{f}_i * \tilde{f}_i : \mathbb{T} \to [0, 1]$ by $\tilde{f}_i * \tilde{f}_i(g) = \int_{\mathbb{T}} \tilde{f}_i(x)\tilde{f}_i(g - x)\,dx$ and $f_i * f_i : \mathbb{Z}_n \to [0, 1]$ by $f_i * f_i(g) = \frac{1}{n}\sum_{x=0}^{n-1} f_i(x)f_i(g - x)$, the convolutions over the groups $\mathbb{T}$ and $\mathbb{Z}_n$ respectively. In the respective spaces $\tilde{f}_i * \tilde{f}_i(g)$ and $f_i * f_i(g)$ are equal to the measures of the maximum sets of color i symmetric with respect to $s(x) = g - x$. Similarly to [4, lemma 6.4], we have $\tilde{f}_i*\tilde{f}_i(k/n) = f_i*f_i(k-1)$. Since the function $\tilde{f}_i*\tilde{f}_i(g)$ is continuous and linear on segments $[\frac{k-1}{n}, \frac{k}{n}]$, we have $\sup_{g\in\mathbb{T}} \tilde{f}_i * \tilde{f}_i(g) = \max_{g\in\mathbb{Z}_n} f_i * f_i(g)$, exactly what we need. □

Claim B. *Let $\alpha : \mathbb{N} \to \mathbb{R}$ be an arbitrarily slowly increasing function and $\epsilon(n) = \sqrt{8r \ln n\, \alpha(n)/n}$. Then $ms(\mathbb{T}, \mathcal{A}; \tilde{\chi}) > 1/r^2 + \epsilon(n) + 2/n$ with probability less than $re^{-\alpha(n)}$.*

Proof of Claim. On the account of Claim A, it suffices to estimate the probability of the event that $ms(\mathbb{Z}_n, \mathcal{A}; \chi) > 1/r^2 + \epsilon(n) + 2/n$. Fix a symmetry $a \in \mathcal{A}$ and a color $i \in [r]$. Estimate the probability that

$$\frac{|M_i \cap a(M_i)|}{n} > \frac{1}{r^2} + \epsilon(n) + 2/n, \tag{5}$$

where $M_i = \chi^{-1}(i)$.

Let s_r denote the rotation by angle $2\pi/r$. In the case of $\mathbb{Z}_n$, $s_r(x) = x + n/r$. Consider orbits of $\mathbb{Z}_n$ under the action of the group $\langle s_r, a\rangle$ generated by symmetries s_r and a. Geometrically, the orbit $O(x)$ of an element $x \in \mathbb{Z}_n$ can be obtained as follows. The orbit of x under the action of $\langle s_r\rangle$, $R(x)$, is a regular r-gon. To extend it to $O(x)$, we have to reflect $R(x)$ by a. If a is an axial symmetry of $R(x)$, then $O(x) = R(x)$ consists of r elements. Otherwise $O(x) = R(x) \cup a(R(x))$ consists of $2r$ elements. There are only one or two r-element orbits. Fix a numbering of the $2r$-element orbits $O_1, \ldots, O_t$, where

$$n - 2r \leq 2rt \leq n - r.$$

Denote $M_i^j = M_i \cap O_j$ and define a random variable X_j by $X_j = |M_i^j \cap a(M_i^j)|/|O_j|$. Since every orbit is invariant with respect to a, we have

$$\frac{2r}{n}\sum_{j=1}^{t} X_j \leq \frac{|M_i \cap a(M_i)|}{n} \leq \frac{2r}{n}\sum_{j=1}^{t} X_j + \frac{2}{n}. \tag{6}$$

The term $\frac{2}{n}$ corresponds to a possible contribution of r-element orbits.

A key observation on which our analysis relies is this: Since every orbit is invariant with respect to s_r, the random variables $X_1, \ldots, X_t$ are mutually independent. Let us calculate how each X_j is distributed. Assume that $O_j = O(x)$ and consider the distribution of X_j conditioned on an arbitrarily fixed coloring of $R(x)$. Without loss of generality assume that $\chi(x) = i$. Then for $R(a(x))$ there are r equiprobable colorings, exactly one of which assigns to $a(x)$ the color i. It follows that

$$X_j = \begin{cases} 1/r & \text{with probability 1/r,} \\ 0 & \text{with probability 1-1/r.} \end{cases} \tag{7}$$

The event (5), on the account of the relation (6), implies that

$$\frac{1}{t}\sum_{j=1}^{t} X_j > \frac{n}{2rt}\left(\frac{1}{r^2} + \epsilon\right) \geq \frac{n}{n-r}\left(\frac{1}{r^2} + \epsilon\right) > \frac{1}{r^2} + \epsilon, \tag{8}$$

where $\epsilon = \epsilon(n)$. Since $\mathbf{E}[X_j] = 1/r^2$, by the Chernoff bound we conclude that (8) and hence (5) happens with probability less than

$$\exp\left(-\frac{1}{2}\epsilon^2 t\right) \le \exp\left(-\frac{1}{2}\epsilon^2\left(\frac{n}{2r}-1\right)\right) \le \exp\left(-\ln n\,\alpha(n)\right),$$

the latter if $n \ge 4r$. There are n possible axial symmetries of the regular n-gon and r possible colors. For the probability of the event that (5) happens at least for some a and i we therefore have the upper bound $rn\exp(-\ln n\,\alpha(n)) = r\exp(-\alpha(n))$, exactly what is claimed. □

We now have to treat rotatory symmetries of the circle. On the account of Proposition 2.1, it is enough to consider rotations s_p by angle $2\pi/p$ for p prime. We start with the cases of $p = 2, 3$. Recall that n is chosen to be a multiple of 2 and 3.

Claim C. *If n is divisible by p, then $ms(\mathbb{T}, \langle s_p\rangle; \tilde{\chi}) = ms(\mathbb{Z}_n, \langle s_p\rangle; \chi)$, where in the case of $\mathbb{Z}_n$ we have $s_p(x) = x + n/p$.*

Proof of Claim. The inequality "$\ge$" is evident. The reverse inequality "$\le$" is a consequence of the following observation. Suppose that M is a monochromatic subset of $\mathbb{T}$ that has non-zero measure and is invariant with respect to s_p. If $x \in M \cap (\frac{l-1}{n}, \frac{l}{n})$, then every interval $(\frac{l-1+in/p}{n}, \frac{l+in/p}{n})$ is included into M as this interval contains a point $s_p^i(x)$ of the same color. □

Claim D. *Let p be a prime divisor of n. Let $\alpha : \mathbb{N} \to \mathbb{R}$ be an arbitrarily slowly increasing function and $\epsilon(n) = \sqrt{2rp\alpha(n)/n}$. Then $ms(\mathbb{T}, \langle s_p\rangle; \tilde{\chi}) > 1/r^2 + \epsilon(n)$ with probability less than $re^{-\alpha(n)}$.*

Proof of Claim. On the account of Claim C, it suffices to estimate the probability of the event that $ms(\mathbb{Z}_n, \langle s_p\rangle; \chi) > 1/r^2 + \epsilon$. If p divides r, then $ms(\mathbb{Z}_n, \langle s_p\rangle; \chi) = 0$ by the construction of χ. Assume that p does not divide r. We proceed in the same vein as in the proof of Claim B. Let $O_1, \ldots, O_t$ be the orbits of $\mathbb{Z}_n$ under the action of $\langle s_r, s_p\rangle$. It is not hard to see that each O_j has rp elements and hence $t = n/(rp)$.

Fix a color $i \in [r]$ and denote the respective monochromatic class by M_i. The maximum subset of M_i symmetric with respect to s_p is $\bigcap_{l=0}^{p-1} s_p^l(M_i)$. Denote $M_i^j = M_i \cap O_j$ and define a random variable X_j by $X_j = |\bigcap_{l=0}^{p-1} s_p^l(M_i^j)|/|O_j|$. Since each O_j is invariant with respect to s_p, we have $|\bigcap_{l=0}^{p-1} s_p^l(M_i)|/n = \sum_{j=1}^t \frac{rp}{n} X_j = \frac{1}{t}\sum_{j=1}^t X_j$. Since each O_j is invariant with respect to s_r, the X_j's are mutually independent. Let us calculate the distribution of an X_j.

Note that the orbit $O(x) = O_j$ of an element x consists of a regular r-gon $R(x)$, the orbit of x under the action of $\langle s_r \rangle$, and its rotations by s_p^l. Consider the distribution of X_j conditioned on an arbitrarily fixed coloring of $R(x)$. Without loss of generality assume that $\chi(x) = i$. Then for each $R(s_p^l(x))$, $1 \le l \le p-1$, there are r equiprobable colorings which are chosen independently for each l. Thus, only in one of r^{p-1} cases the color i is assigned to $s_p^l(x)$ for every $l = 1, \ldots, p-1$. It follows that

$$X_j = \begin{cases} \frac{p}{rp} = \frac{1}{r} & \text{with probability } 1/r^{p-1}, \\ 0 & \text{with probability } 1 - 1/r^{p-1}, \end{cases} \tag{9}$$

and hence $\mathbf{E}[X_j] = 1/r^p \le 1/r^2$.

Thus, $|\bigcap_{l=0}^{p-1} s_p^l(M_i)|/n > 1/r^2 + \epsilon$ implies that $\frac{1}{t}\sum_{j=1}^{t} X_j > \mathbf{E}[X_1] + \epsilon$ for a fixed color i and, by the Chernoff bound, this event has probability less than $\exp(-\frac{1}{2}\epsilon^2 t) = \exp(-\alpha(n))$. For at least one color, this event therefore happens with probability less than $r\exp(-\alpha(n))$. $\square$

It remains to consider rotations s_p with prime $p \ge 5$ not dividing r. Given an r-coloring χ of $\mathbb{Z}_n$ and an integer p, we associate with χ an r-coloring $\hat{\chi}$ of $\mathbb{Z}_{pn}$ defined by $\hat{\chi}(x) = \chi(\lfloor x/p \rfloor)$. Similarly to Claim A, we have the following relation.

Claim E. *$ms(\mathbb{T}, \langle s_p \rangle; \tilde{\chi}) = ms(\mathbb{Z}_{pn}, \langle s_p \rangle; \hat{\chi})$, where in the case of $\mathbb{Z}_{pn}$ we have $s_p(x) = x + n$.* $\square$

We split the set of primes p under consideration into 3 classes and treat each of them separately. Let

$$\mathcal{R}(u, v) = \{\, s_p \,|\, u \le p \le v,\ p \text{ is prime},\ p \text{ does not divide } r \,\}.$$

Claim F. *$ms(\mathbb{T}, \mathcal{R}(5, n/r); \chi) > 1/r^2$ with probability at most $1/2$.*

Proof of Claim. On the account of Claim E, it suffices to estimate the probability that $ms(\mathbb{Z}_{pn}, \langle s_p \rangle; \hat{\chi}) > 1/r^2$ at least for some $p \in \mathcal{R}(5, n/r)$. Fix a p in this range. Since p and r are coprime, $\mathbb{Z}_{pn}$ is split into n/r orbits under the action of $\langle s_r, s_p \rangle$, where $s_r(x) = x + 6pm$ in $\mathbb{Z}_{pn}$, each orbit consisting of rp elements. Let $O_1, \ldots, O_{n/r}$ be their numbering. Similarly to the proof of Claim D, fix a color $i \in [r]$ and denote the respective monochromatic class by M_i. Denote $M_i^j = M_i \cap O_j$ and define a random variable X_j by $X_j = |\bigcap_{l=0}^{p-1} s_p^l(M_i^j)|/|O_j|$. Since each O_j is invariant with respect to s_p, we have

$$\frac{|\bigcap_{l=0}^{p-1} s_p^l(M_i)|}{pn} = \frac{r}{n}\sum_{j=1}^{n/r} X_j.$$

Let us calculate the distribution of X_j. Geometrically, $O_j = O(x)$ consists of a regular r-gon $R_j = R(x)$ and its iterated rotations by $2\pi/(rp)$. Consider the distribution of X_j conditioned on an arbitrarily fixed coloring of R_j. Since p is so that angle $2\pi/n$ is no greater than angle $2\pi/(rp)$, $\hat{\chi}$ induces independent colorings of r-gons $s_p^l(R_j)$ for $1 \le l < p$. It follows that

$$X_j = \begin{cases} 1/r & \text{with probability } 1/r^{p-1} \\ 0 & \text{with probability } 1 - 1/r^{p-1} \end{cases}$$

(assuming $\hat{\chi}(x) = i$, $1/r^{p-1}$ is the probability that $\hat{\chi}(s_p^l(x)) = i$ for every $1 \le l < p$).

The random variables X_j's are not independent but we now use the fact that the expectation $\mathbf{E}[X_j] = 1/r^p$ is rather small. By the linearity of the expectation, we have $\mathbf{E}\left[|\bigcap_{l=0}^{p-1} s_p^l(M_i)|/(pn)\right] = 1/r^p$. Using the Markov inequality, we conclude from here that, for a fixed i,

$$\frac{|\bigcap_{l=0}^{p-1} s_p^l(M_i)|}{pn} > \frac{1}{r^2} \tag{10}$$

with probability at most $1/r^{p-2}$. For at least one color i this event therefore happens with probability at most $1/r^{p-3}$. Furthermore, one can find $p \in \mathcal{R}(5, n/r)$ and $i \in [r]$ such that (10) takes place with probability at most

$$\sum_{p=5}^{n/r} \frac{1}{r^{p-3}} < \frac{1}{r(r-1)} \le \frac{1}{2}.$$

The claim follows. □

Claim G. *$ms(\mathbb{T}, \mathcal{R}(\lfloor n/r \rfloor + 1, n-1); \tilde{\chi}) > 0$ with probability at most $n^2 r^{2-n/r^2}$.*

Proof of Claim. We again use Claim E. Let $O_1, \dots, O_n$ be the partition of $\mathbb{Z}_{pn}$ into orbits under the action of $\langle s_p \rangle$, each consisting of p elements. Every set symmetric with respect to s_p is a union of such orbits. An orbit O_j is monochromatic of color i with probability at most $(1/r)^{\lfloor p/r \rfloor}$ because colorings of the elements $x, x+n, \dots, x + \lfloor p/r \rfloor n$ are independent. Consequently, a non-empty monochromatic set of a fixed color i symmetric with respect to s_p exists with probability at most $nr^{1-p/r}$. The same is true at least for some i with probability at most $nr^{2-p/r} \le nr^{2-n/r^2}$. Furthermore, the latter event happens at least for some p in the range $n/r < p < n$ with probability less that $n^2 r^{2-n/r^2}$. □

Claim H. $ms(\mathbb{T}, \mathcal{R}(n,\infty); \tilde{\chi}) > 0$ *with probability at most* $r^{1-n/r}$.

Proof of Claim. If there is a non-empty monochromatic subset of $\mathbb{Z}_{pn}$ symmetric with respect to s_p for some $p \geq n$, then all $\mathbb{Z}_{pn}$ must be monochromatic. The latter is possible with probability at most $r(1/r)^{n/r}$. □

Choose a function $\alpha(n)$ so that $\alpha(n) \to \infty$ as $n \to \infty$ and $\alpha(n) = o(n/\ln n)$. Let $\epsilon(n)$ be as in Claim B. Summing up the bounds of Claims B, D (for $p = 2, 3$), F, G, and H, we conclude that

$$ms(\mathbb{T}, \mathcal{S}; \tilde{\chi}) > \frac{1}{r^2} + \epsilon(n) + \frac{2}{n}$$

with probability at most $1/2 + o(1)$, where $\mathcal{S} = \mathcal{A} \cup \mathcal{R}(2,\infty)$ covers all symmetries in a plane. Thus, if n is large enough, there is an r-coloring χ_n of the circle such that $ms(\mathbb{T}, \mathcal{S}; \chi_n) \leq 1/r^2 + \epsilon(n) + 2/n$ (and, moreover, about a half of all colorings produced by our construction are suitable). □

Remark 5.4. The proof of Theorem 5.3 shows that about a half of congruent colorings of the circle generated in the specified way are suitable for an extremal sequence but gives us no specific extremal sequence. According to communication of Taras Banakh, a construction suggested in [3] can be modified so that it gives an *explicit* extremal sequence of congruent colorings of the circle. However, our approach gains in another respect. Given a coloring χ of the circle, define $F(\chi)$, the *finess* of χ, to be the minimum measure of a connected component of a monochromatic class. It is clear that colorings with lower finess are less preferable. For a function $f : \mathbb{N} \to \mathbb{R}$, we say that an extremal sequence χ_n has finess $f(m)$ if there is an infinite subsequence of indices $n(m)$ such that $ms(S^1; \chi_{n(m)}) = 1/r^2 + O(1/m)$ and $F(\chi_{n(m)}) > c \cdot f(m)$ for a positive constant c. In this setting, the extremal sequence steming from [3] has finess $1/(m2^m)$ while the sequences given by the proof of Theorem 5.3 have finess $1/(m^2 \ln m)$. Note that our approach is quite competetive from algorithmic point of view — it easily translates into a Las-Vegas algorithm finding a congruent coloring χ_n of the circle with $ms(S^1; \chi_n) = 1/r^2 + O(\sqrt{\ln n/n})$ in expected running time $O(n^2)$. It is an interesting open question if a such colorings can be found deterministically in time $n^{O(1)}$, in particular, if our construction can be derandomized.

Question 5.5. Is there an extremal sequence of congruent colorings of the sphere S^2?

5.2. The interval

Let $ms^\star(\Omega, r)$ be the infimum of $ms(\Omega; \chi)$ over congruent (up to a set of measure 0) r-colorings of Ω. Since we do not know if $ms(I, 2) = ms^\star(I, r)$, our task is at least to estimate $ms^\star(I, r)$ from the above and from the below.

Lemma 5.6. *Let $I = A \cup B \cup Z$ be a partition of the interval into three measurable parts so that A and B are congruent and $\mu(Z) = 0$. Assume that $s(A) = B$ for s being an isometry of the real line. Then one of the following two cases must occur:*

1. *$s(x) = 1 - x$ is the central symmetry with center at $1/2$.*
2. *$s(x) = x + 1/(2k)$, with k nonzero integer, is a translation and, moreover, one of the sets $A \cup Z$ and $B \cup Z$ contains the union $\bigcup_{i=0}^{k-1}[2i/(2k), (2i+1)/(2k))$ and the other of them contains the union $\bigcup_{i=1}^{k}[(2i-1)/(2k), 2i/(2k))$.*

Proof. Assume that s is a central symmetry. Since $s(A) = B$ and $s(B) = A$, we have $s(A \cup B) = A \cup B$ and therefore the center of symmetry is at $1/2$.

Assume now that s is a translation over distance t. Without loss of generality suppose that $t > 0$. As esily seen, the relation $B = A + t$ implies that $\inf B = t$. It follows that $[0, t) \subseteq A \cup Z$.

Let l be a positive integer such that $lt < 1$. It is not hard to see that, if $[(l-1)t, lt) \subseteq A \cup Z$, then it must be $[lt, (l+1)t) \subseteq B \cup Z$, and vise versa, if $[(l-1)t, lt) \subseteq B \cup Z$, then $[lt, (l+1)t) \subseteq A \cup Z$. It follows that $t = 1/(2k)$ for some integer k and that A and B, up to a set Z of measure 0, are the unions of intervals $[j/(2k), (j+1)/(2k))$ over even and odd j respectively. □

The Erdős-Świerczkowski function $M(\alpha)$ is defined by

$$M(\alpha) = \inf_A \sup_{g \in \mathbb{R}} \mu(A \cap (\bar{A} + g))$$

where the infimum is taken over subsets A of I with $\mu(A) = \alpha$ and $\bar{A} = I \setminus A$.

Theorem 5.7. *$ms^\star(I, 2) \geq M(1/2)$.*

Proof. Consider a bicoloring of I with congruent monochromatic classes A and B and estimate $ms(A)$ from below. Assume that $B = s(A)$ for an isometry s and use Lemma 5.6. If s is a translation, then $ms(A) = 1/2$.

On the other hand, it is known [8, problem C17] that $M(1/2) < 1/5$ and hence theorem in this case is true.

If s is a central symmetry with center at $1/2$, the maximal subset of A symmetric with respect to a center c can be represented as $A \cap (2c - A) = A \cap (2c - 1 + B)$. It follows that $\mu(A \cap (2c - A)) = \mu(A \cap (2c - 1 + \bar{A}))$. By the definition of the number $M(1/2)$, we can find c with $\mu(A \cap (2c - A))$ arbitrarily close to $M(1/2)$. It follows that $ms(A) \geq M(1/2)$. $\square$

It is known [8, problem C17] that $M(1/2) > 0.178$.

Corollary 5.8. *$ms^\star(I, 2) > 0.178$.* $\square$

To obtain an upper bound for $ms^\star(I, 2)$, we involve the techiques of blurred colorings developed in [4]. Assume that all admissible symmetries are involutive. A *blurred bicoloring* of $\Omega \subseteq \mathcal{U}$ is a pair of measurable functions $\beta_1 : \mathcal{U} \to [0, 1]$ and $\beta_2 : \mathcal{U} \to [0, 1]$ such that $\beta_1 + \beta_2 = \chi_\Omega$, where χ_Ω denotes the characteristic function of Ω. One can think that in a blurred coloring each element x of Ω is colored by a mixture of the two colors at the proportion of $\beta_1(x)$ to $\beta_2(x)$.

Given a measurable function $f : \mathcal{U} \to \mathbb{R}$, we define a map $f \star f : \mathcal{S} \to \mathbb{R}$ by

$$f \star f(s) = \int_{\mathcal{U}} f(x) f(s(x))\, d\mu(x).$$

We use the notation $\| \cdot \|$ for the uniform norm on the set of functions from $\mathcal{S}$ to $\mathbb{R}$, i.e. $\|F\| = \sup_{s \in \mathcal{S}} |F(s)|$ for a function $F : \mathcal{S} \to \mathbb{R}$.

An analog of the maximum measure of a monochromatic symmetric subset under a blurred coloring $\beta = \{\beta_1, \beta_2\}$ is defined by

$$bms(\Omega; \beta) = \max_{i=1,2} \|\beta_i \star \beta_i\|.$$

We set

$$bms(\Omega, 2) = \inf_{\beta} bms(\Omega; \beta),$$

where the infimum is taken over all blurred bicolorings of Ω.

We call a blurred bicoloring $\beta = \{\beta_1, \beta_2\}$ *congruent* if there is a non-identity symmetry s mapping Ω onto itself such that $\beta_1(x) = \beta_2(s(x))$ for all $x \in \Omega$. We define $bms^\star(\Omega, 2)$ to be the infimum of $bms(\Omega; \beta)$ over congruent blurred bicolorings β. As usually, we call a such β extremal if $bms(\Omega; \beta)$ attains $bms^\star(\Omega, 2)$.

In what follows we consider blurred colorings of the discrete segment $\Omega = [k]$. For this space the congruence of a blurred coloring can be ensured by the only symmetry $s_0(x) = k + 1 - x$.

Theorem 5.9. *For every k, $ms^\star(I, 2) \leq bms^\star([k], 2)$.*

Corollary 5.10. $ms^\star(I,2) \le 5/24$.

Proof. As follows from [4, lemma 6.7], $bms^\star([4],2) \le 5/24$. □

Let us call a blurred coloring β of a space Ω *extremal* if $bms(\Omega,2) = bms(\Omega;\beta)$. Note that extremal blurred colorings always exist for finite spaces. In [4, question 7.5] we ask if the spaces $[k]$ have congruent extremal blurred colorings, in other words, if $bms(\Omega,2) = bms^\star(\Omega,2)$.

Corollary 5.11. *If for infinitely many k there are congruent extremal blurred bicolorings of $[k]$, then $ms(I,2) = ms^\star(I,2)$.*

Proof. This follows from the convergence $\lim_{k\to\infty} bms([k],2) = ms(I,2)$ proved in [4, theorem 6.1]. □

Proof of Theorem 5.9. Let β be an extremal congruent blurred coloring of $[k]$. A such coloring exists by the following reason. Any congruent blurred coloring η is determined by $x_1 = \eta_1(1), \ldots, x_l = \eta_1(l)$, where $l = \lceil k/2 \rceil$. Let $f_{b,s}(x_1,\ldots,x_l) = \eta_b \star \eta_b(s)$. Then $bms^\star([k],2) = \min_{x_1,\ldots,x_l} \max_{b,s} f_{b,s}(x_1,\ldots,x_l)$ is attained at some $(x_1,\ldots,x_l)$ because each $f_{b,s}$ is continuous and $(x_1,\ldots,x_l)$ ranges in the compact $[0,1]^l$.

Take $n = km$ with m even and define a blurred coloring β' of $[n]$ by $\beta'_i(x) = \beta_i(\lceil x/m \rceil)$. Furthermore, we define a random bicoloring $\chi : [n] \to \{1,2\}$ by setting

$$\chi(x) = \begin{cases} 1 & \text{with probability } \beta'_1(x), \\ 2 & \text{with probability } \beta'_2(x), \end{cases}$$

for each x independently of the others. We also define a random congruent bicoloring $\chi^\star : [n] \to \{1,2\}$: $\chi^\star(x)$ is defined as $\chi(x)$ for x in the range $1 \le x \le n/2$; If $x > n/2$, $\chi^\star$ is defined deterministically by $\chi^\star(x) = 3-\chi(n+1-x)$. We will consider χ and $\chi^\star$ simultaneously. Any correlation between these colorings is possible and is irrelevant to our argument. To be specific, let χ and $\chi^\star$ be independent (as well we could suppose that $\chi^\star(x) = \chi(x)$ for all $x \le n/2$). It will be essential for our argument that, for each particular $x \in [n]$, $\chi(x)$ and $\chi^*(x)$ are identically distributed (due to the congruence of β).

Claim A. $bms([n];\beta') = bms([k];\beta)$.

Proof of Claim. Though it would be not hard to give a direct proof, we here prefer to rely on work done in [4]. Let $\tilde\beta = (\tilde\beta_1, \tilde\beta_2)$ be a blurred bicoloring of I defined by $\tilde\beta_i(x) = \beta_i(\lceil kx \rceil)$ if $x > 0$ and $\tilde\beta_i(0) = \beta_i(1)$.

If we similarly define another blurred coloring $\tilde{\beta}'$ of I based on β', we obtain

$$\tilde{\beta}'_i(x) = \beta'_i(\lceil nx \rceil) = \beta_i\big(\lceil \lceil nx \rceil / m \rceil\big) = \beta_i(\lceil kx \rceil) = \tilde{\beta}_i(x).$$

Thus, $\tilde{\beta}$ and $\tilde{\beta}'$ coincide. By [4, lemma 6.4], $bms(I; \tilde{\beta}) = bms([k]; \beta)$ and $bms(I; \tilde{\beta}') = bms([n]; \beta')$. The required inequality follows. □

Claim B. *Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrarily slowly increasing function, in particular, $f(n) = o\left(\sqrt{n/\ln n}\right)$. Then*

$$\big|ms([n]; \chi) - bms([n]; \beta')\big| > \sqrt{\frac{\ln n}{n}} f(n)$$

with probability $o(1)$ as $n \to \infty$.

Proof of Claim. This immediately follows from the proof of [4, lemma 5.3]. □

Claim C. *Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrarily slowly increasing function, in particular, $f(n) = o\left(\sqrt[3]{n/\ln n}\right)$. Then*

$$|ms([n]; \chi) - ms([n]; \chi^\star)| > \sqrt[3]{\frac{\ln n}{n}} f(n)$$

with probability $o(1)$ as $n \to \infty$.

As follows from Claims A, B, and C,

$$|ms([n]; \chi^\star) - bms^\star([k], 2)| > 2\sqrt[3]{\frac{\ln n}{n}} f(n)$$

with probability $o(1)$. Choose $f(x) = \frac{1}{2} \ln^{2/3} n$. Then, provided n is large enough, there is a congruent coloring $\chi^\star_n$ such that

$$ms([n]; \chi^\star_n) \le bms^\star([k], 2) + \frac{\ln n}{\sqrt[3]{n}}$$

(actually most colorings $\chi^\star$ generated as described above are such). With $\chi^\star_n$ we associate a bicoloring $\tilde{\chi}^\star_n$ of I by setting $\tilde{\chi}^\star_n(x) = \chi^\star_n(\lceil nx \rceil)$. By [4, lemma 6.4], $ms(I; \tilde{\chi}^\star_n) = ms([n]; \chi^\star_n) \le bms^\star([k], 2) + o(1)$ and therefore $ms^\star(I, 2) \le bms^\star([k], 2)$. The proof of the theorem is complete modulo Claim C.

Proof of Claim C. Fix a symmetry $s(x) = g - x$ and a color $b \in \{1, 2\}$. Denote $M_b = \chi^{-1}(b)$ and $M_b^* = (\chi^*)^{-1}(b)$. The intersections $M_b \cap s(M_b)$ and $M_b^* \cap s(M_b^*)$ are the the maximal monochromatic subsets of $[n]$ that receive color b under χ and χ^* respectively and are symmetric with respect to s. Our task is to show that with high probability the densities of these sets are close to each other. We will do so by finding an expression whose value is close to the average values of both $|M_b \cap s(M_b)|/n$ and $|M_b^* \cap s(M_b^*)|/n$ and estimating the probabilities of deviation of these densities from that expression.

We split $[n]$ into classes $C_{p,q}$, where $0 \le p \le q \le k$, as follows:

$$C_{p,q} = \{ x \in [n] \,|\, \{\lceil x/m \rceil, \lceil s(x)/m \rceil\} = \{p, q\} \} .$$

If g is even, then there is also an especial single-element class $C_{0,0} = \{g/2\}$. Note that $s(C_{p,q}) = C_{p,q}$. It is not hard to see that the number of non-empty classes is at most $2k$. Let

$$t = \frac{f(n)\sqrt[3]{n^2 \ln n}}{28k} \quad \text{and} \quad \epsilon = \frac{f(n)\sqrt[3]{\ln n/n}}{4}.$$

We call a class $C_{p,q}$ *big* if $|C_{p,q}| \ge t$ and *small* otherwise.

We start with $M_b \cap s(M_b)$. If $x \in C_{p,q} \ne C_{0,0}$, then $\chi(x) = \chi(s(x)) = b$ with probability $\beta_b(p)\beta_b(q)$. Denote the set of $x \in C_{p,q}$ for which this event happens by $\hat{C}_{p,q}$. If $C_{p,q}$ is big, then the Chernoff bound implies that

$$\mathbf{P}\left[\left|\frac{|\hat{C}_{p,q}|}{|C_{p,q}|} - \beta_b(p)\beta_b(q)\right| > \epsilon\right] < 2\exp\left(-\frac{\epsilon^2 t}{4}\right). \tag{11}$$

From the expansion

$$\frac{|M_b \cap s(M_b)|}{n} = \frac{1}{n} \sum_{0 \le p \le q \le k} |M_b \cap s(M_b) \cap C_{p,q}|$$

we infer that

$$\sum_{C_{p,q} \text{ big}} \frac{|C_{p,q}|}{n} \frac{|\hat{C}_{p,q}|}{|C_{p,q}|} \le \frac{|M_b \cap s(M_b)|}{n} \le \sum_{C_{p,q} \text{ big}} \frac{|C_{p,q}|}{n} \frac{|\hat{C}_{p,q}|}{|C_{p,q}|} + \frac{2kt}{n}. \tag{12}$$

The term $2kt/n$ here bounds the possible contribution of small classes $C_{p,q}$. Note now that the inequality

$$\left|\frac{|M_b \cap s(M_b)|}{n} - \sum_{C_{p,q} \text{ big}} \frac{|C_{p,q}|}{n} \beta_b(p)\beta_b(q)\right| > \epsilon + \frac{2kt}{n} \tag{13}$$

implies that

$$\frac{|M_b \cap s(M_b)|}{n} < \sum_{C_{p,q} \text{ big}} \frac{|C_{p,q}|}{n} \left(\beta_b(p)\beta_b(q) - \epsilon\right).$$

or

$$\frac{|M_b \cap s(M_b)|}{n} > \sum_{C_{p,q} \text{ big}} \frac{|C_{p,q}|}{n} \left(\beta_b(p)\beta_b(q) + \epsilon\right) + \frac{2kt}{n}.$$

On the account of (12) we see that (13) implies

$$\left| \frac{|\hat{C}_{p,q}|}{|C_{p,q}|} - \beta_b(p)\beta_b(q) \right| > \epsilon$$

for some big class $C_{p,q}$. By (11) we conclude that (13) happens with probability less that

$$4k \exp\left(-\epsilon^2 t/4\right). \tag{14}$$

We next proceed with $M_b^* \cap s(M_b^*)$. We are now in a more difficult situation because $\chi^\star(x)$ are not independent for all $x \in [n]$. We overcome this difficulty by splitting $[n]$ into several (at most 6) parts so that colors of points within each part are independent.

Recall that s_0 is the central symmetry of $[n]$. The case that $s = s_0$ is trivial because then there is no nonempty monochromatic s-symmetric set by the construction of $\chi^\star$. We hence assume that $s \neq s_0$.

Let A be the set of integer points between $(n+1)/2$ and $g/2$, excluding the latter. Consider the orbit of A in $\mathbb{Z}$ under the action of $\langle s_0, s\rangle$. Note that, for distinct u_1 and u_2 in $\langle s_0, s\rangle$, $u_1(A)$ and $u_2(A)$ are disjoint. We classify all elements of $\langle s_0, s\rangle$ as follows:

- Let $l \geq 0$. *The l-fold s_0-composition* is $s_0(ss_0)^{(l-1)/2}$ if l odd or $(s_0 s)^{l/2}$ if l even.
- Let $l \geq 1$. *The l-fold s-composition* is $s(s_0 s)^{(l-1)/2}$ if l odd or $(ss_0)^{l/2}$ if l even.

For $a = 1, 2, 3$, let X_a be the union of all images $u(A)$, where u is a $(3i + a - 1)$-fold s_0-composition, for any $i \geq 0$. Let Y_a be the union of all images $u(A)$, where u is a $(3i + a)$-fold s-composition, for any $i \geq 0$. Notice that

$$s(X_a) = Y_a$$

and that

$$\begin{aligned} s_0(X_a) &\subseteq Y_{(a+1) \bmod 3} \cup X_{(a+1) \bmod 3} \cup X_{(a+2) \bmod 3}, \\ s_0(Y_a) &\subseteq X_{(a+2) \bmod 3} \end{aligned}$$

for each $a = 1, 2, 3$. Regarding a more complicated view of the former inclusion, note that s_0 can take elements in X_a to X_{a+1} or X_{a-1} only in two cases: $s_0(A) \subset X_2$ while $A \subset X_1$ and, vise versa, $s_0(s_0(A)) \subset X_1$ while $s_0(A) \subset X_2$. Define $Z_a = X_a \cup Y_a$. As easily seen,

$$s(Z_a) = Z_a \text{ and } s_0(Z_a) \cap Z_a = \emptyset. \tag{15}$$

The latter property implies that, for each $a = 1, 2, 3$, the colors $\{\chi^\star(x)\}_{x \in Z_a}$ are mutually independent.

If g is odd, Z_1, Z_2, Z_3 is a partition of $\mathbb{Z}$ and, to not abuse the notation, from now on we will use the same characters to denote the induced partition of $[n]$.

If g is even, then $[n] \setminus (Z_1 \cup Z_2 \cup Z_3)$ is the orbit of $g/2$ under the action of $\langle s_0, s\rangle$. Similarly to the above, we split this orbit, excluding $g/2$ itself, into 3 parts Z_4, Z_5, Z_6 so that (15) holds true for $a = 4, 5, 6$. The only difference stems from the fact that $s(g/2) = g/2$. Specifically, we have

$$Z_{3+a} = \left\{ (s_0 s)^{3i+a-1} s_0(g/2) \,|\, i \geq 0 \right\} \cup \left\{ (s s_0)^{3i+a}(g/2) \,|\, i \geq 0 \right\}$$

for $a = 1, 2, 3$.

Thus, we arrive at the partition $[n] = \bigcup_{a=1}^{6} Z_a$, where each Z_a is symmetric with respect to s and colors of elements within each Z_a are independent.

Let $C_{p,q}$ and $\hat{C}_{p,q}$ for $0 \leq p \leq q \leq k$ be as defined above, $\hat{C}_{p,q}$ being defined now with respect to χ^*. Furthermore, we define $C^a_{p,q} = C_{p,q} \cap Z_a$ and $\hat{C}^a_{p,q} = \hat{C}_{p,q} \cap Z_a$ for $1 \leq a \leq 6$. For the exceptional class $C_{0,0}$ we set $C^1_{0,0} = C_{0,0}$. We call a class $C^a_{p,q}$ *big* if it contains at least t elements and *small* otherwise. We use the expansion

$$\frac{|M^\star_b \cap s(M^\star_b)|}{n} = \frac{1}{n} \sum_{\substack{0 \leq p \leq q \leq k \\ 1 \leq a \leq 6}} |M^\star_b \cap s(M^\star_b) \cap C^a_{p,q}|.$$

It follows that

$$\sum_{C^a_{p,q} \text{ big}} \frac{|C^a_{p,q}|}{n} \frac{|\hat{C}^a_{p,q}|}{|C^a_{p,q}|} \leq \frac{|M^\star_b \cap s(M^\star_b)|}{n} \leq \sum_{C^a_{p,q} \text{ big}} \frac{|C^a_{p,q}|}{n} \frac{|\hat{C}^a_{p,q}|}{|C^a_{p,q}|} + \frac{12kt}{n},$$

where the term $12kt/n$ bounds the possible contribution of small classes $C^a_{p,q}$. Assume now that

$$\left| \frac{|M^\star_b \cap s(M^\star_b)|}{n} - \sum_{C_{p,q} \text{ big}} \frac{|C_{p,q}|}{n} \beta_b(p)\beta_b(q) \right| > \epsilon + \frac{12kt}{n}. \tag{16}$$

Then

$$\frac{|M_b^\star \cap s(M_b^\star)|}{n} > \sum_{C_{p,q}\text{ big}} \frac{|C_{p,q}|}{n}(\beta_b(p)\beta_b(q)+\epsilon)+\frac{12kt}{n}$$
$$\geq \sum_{C^a_{p,q}\text{ big}} \frac{|C^a_{p,q}|}{n}(\beta_b(p)\beta_b(q)+\epsilon)+\frac{12kt}{n}$$

or

$$\frac{|M_b^\star \cap s(M_b^\star)|}{n} < \sum_{C_{p,q}\text{ big}} \frac{|C_{p,q}|}{n}(\beta_b(p)\beta_b(q)-\epsilon)-\frac{12kt}{n}$$
$$\leq \sum_{C^a_{p,q}\text{ big}} \frac{|C^a_{p,q}|}{n}(\beta_b(p)\beta_b(q)-\epsilon).$$

It follows that for some big $C^a_{p,q}$ we have

$$\left|\frac{|\hat{C}^a_{p,q}|}{|C^a_{p,q}|}-\beta_b(p)\beta_b(q)\right| > \epsilon.$$

By the Chernoff bound (we here use the congruence of β), the last inequality holds with probability less than $2\exp(-\epsilon^2 t/4)$ for a particular fixed $C^a_{p,q}$ and with probability less than $24k\exp(-\epsilon^2 t/4)$ for at least one $C^a_{p,q}$.

We conclude that (16) happens with probability less than $24k\exp(-\epsilon^2 t/4)$. Combining this with the bound (14) for the probability of (13), we conclude that

$$\left|\frac{|M_b \cap s(M_b)|}{n}-\frac{|M_b^\star \cap s(M_b^\star)|}{n}\right| \geq 2\epsilon+\frac{14kt}{n}$$

with probability at most $28\exp(-\epsilon^2 t/4) = 28n^{-f^3(n)/(1792k)}$. This inequality occurs at least for some of $2n-1$ symmetries and for some of 2 colors with probability less than $112n^{1-f^3(n)/(1792k)} = o(1)$. This readily implies the claim. □

□

Owing to Theorem 5.9, for $ms^\star(I,2)$ we have upper bounds as good as those we know for $ms(I,2)$. However, the next case of three colors seems rather subtle. It is related with some questions on polyomino tilings. A *(one dimensional disconnected) polyomino* is a figure in a plane consisting of several lattice squares in a line. We assume that a single square has size 1 by 1. The smallest number of disjoint copies of a polyomino tiling a rectangle of size n to 1 for some n is called the *order* of the polyomino. Thus, any polyomino of order 3 provides us with a congruent 3-coloring of the interval. The set of known polyominoes of order 3 seems not so rich. This restricts our abilities of estimating $ms^\star(I,3)$.

Proposition 5.12. $ms^\star(I,3) \le 2/9$.

Proof. The bound is given by tiling

$$\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{1}\boxed{2}\boxed{2}\boxed{3} \tag{17}$$

taken from the collection [7]. □

Corollary 5.10 easily implies that $ms^\star(I,2r) \le 5/(24r)$. This bound is fairly weak if compared with the fact that $ms(I,r) \le 1/r^2$. It seems not so easy even to find an infinite sequence of r with $ms^\star(I,r) = O(1/r^2)$. Moreover, till recently we did not know if $ms^\star(I,r) < 1/r$ for all r. This problem is now solved in the affirmative by Alexander Ravsky.

Theorem 5.13. (A. Ravsky) $ms^\star(I,r) \le 2/(3r)$ *for every* $r \ge 2$.

Proof. The key observation is that the polyomino in (17) tiles also a 12×1 rectangle:

$$\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{3}\boxed{1}\boxed{4}\boxed{2}\boxed{3}\boxed{3}\boxed{4}\boxed{4}.$$

Since every $r \ge 6$ is representable as $r = 3a + 4b$ with some non-negative integers a and b, this polyomino tiles every rectangle of size $3r \times 1$ with $r \ge 6$. This implies the theorem for all r but 5. In the case of $r = 5$ there is a suitable polyomino of order 5:

$$\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{4}\boxed{5}\boxed{5}\boxed{1}\boxed{2}\boxed{2}\boxed{3}\boxed{4}\boxed{4}\boxed{5}.$$

The proof is complete. □

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