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STRONGLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS. I

СТРОГО НЕЛІНІЙНІ ВИРОДЖЕНІ ЕЛІПТИЧНІ РІВНЯННЯ З РОЗРИВНИМИ КОЕФІЦІЄНТАМИ. I

This paper is concerned with the existence and uniqueness of variational solutions of the strongly nonlinear equation

$$-\sum_{1}^{m} {}_i \frac{\partial}{\partial x_i}\left(\sum_{1}^{m} {}_j a_{i,j}(x, u(x)) \frac{\partial u(x)}{\partial x_j}\right) + g(x, u(x)) = f(x)$$

when the coefficients $a_{i,j}(x, s)$ satisfy an ellipticity degenerate condition and hypotheses weaker than the continuity with respect to the variable s. Furthermore, we establish under which condition on f the solution is bounded in Ω, a bounded open subset on $\mathbb{R}^m$.

Досліджується існування та єдиність узагальнених розв'язків для строго нелінійного рівняння

$$-\sum_{1}^{m} {}_i \frac{\partial}{\partial x_i}\left(\sum_{1}^{m} {}_j a_{i,j}(x, u(x)) \frac{\partial u(x)}{\partial x_j}\right) + g(x, u(x)) = f(x)$$

з коефіцієнтами $a_{i,j}(x, s)$, які задовольняють умову виродженої еліптичності та умову більш слабку, ніж неперервність відносно змінної s. Більш того, при певній умові відносно f доводиться обмеженість розв'язку на обмеженій множині $\Omega \in \mathbb{R}^m$.

1. Introduction. Let Ω be a bounded open subset of the Euclidean m-space $\mathbb{R}^m$, $m \geq 2$. We shall be concerned with the existence of variational solutions of the equation

$$A u(x) + g(x, u(x)) = f(x), \quad x \in \Omega, \tag{1}$$

with Dirichlet boundary condition. Here A is a quasilinear elliptic partial differential operator in divergence form

$$A u(x) = -\sum_{1}^{m} {}_i \frac{\partial}{\partial x_i}\left(\sum_{1}^{m} {}_j a_{i,j}(x, u(x)) \frac{\partial u(x)}{\partial x_j}\right).$$

The functions $a_{i,j}(x, s)$ satisfy the ellipticity and boundedness condition

$$\begin{cases} \sum_{1}^{m} {}_{i,j} a_{i,j}(x, s)\xi_i\xi_j \geq \nu(x) \sum_{1}^{m} {}_i \xi_i^2, \\ \left|\dfrac{a_{i,j}(x, s)}{\nu(x)}\right| \leq \Lambda_{i,j} \quad (i, j = 1, 2, \ldots, m), \end{cases} \tag{2}$$

for almost all $(x, s) \in \Omega \times \mathbb{R}$ and all $\xi \in \mathbb{R}^m$, with $\nu(x)$, $\nu^{-1}(x)$ satisfying the integrability hypotheses of Murty – Stampaccia's kind (see, e. g., [1]). The term $g(x, s)$ is strongly nonlinear and no such growth restriction is imposed on the size of $g(x, s)$ as a function of s, but we (essentially) impose the weak "sign condition" $g(x, s)\, s \geq 0$.

Existence results for problem (1) are well-known in the literature when the coefficients $a_{i,j}(x, s)$ are functions of Carathéodory type (i.e. measurable in x and continuous in s) and $\nu(x)$ does not depend on x (see for instance, [2–4]). However, equations of the form (1) with discontinuous (with respect to s) coefficients $a_{i,j}(x, s)$

occur in many problems of physics. The purpose of this note is to extend the results of [3] to the degenerate case. By working on the coefficients of principal part, the hypotheses can be made weaker than the continuity with respect to the variable s; in this way we will be able to take, for instance, $a_{i,j}(x,s) = \alpha_{i,j}(x)\beta_{i,j}(s)$, where $\alpha_{i,j}$ and $\beta_{i,j}$ are supposed only to be measurable and satisfying (2). Finally, other interesting results concerning with equation (1), in degenerate case, are established in [5] by assuming the coefficients $a_{i,j}(x,s)$ to be Carathéodory's functions and the functions $g(x,s)$, f, having polynomial growth in s.

2. Function spaces. Let $\mathbb{R}^m$ be the Euclidean m-space with generic point $x = (x_1, x_2, \ldots, x_m)$, Ω a bounded open subset of $\mathbb{R}^m$. We denote by meas_x the m-dimensional Lebesque's measure.

***Hypothesis* 1.** *Let $\nu(x)$ be a positive function defined on Ω; there exist two real numbers $\sigma \in]0,1[$ and $\chi > m/2$ such that:*

$$\nu(x) \in L^{1+\sigma}(\Omega), \quad \frac{1}{\nu(x)} \in L^{\chi}(\Omega).$$

(For instance, if $\Omega = \{x \in \mathbb{R}^m : |x| < 1\}$ we can choose

$$\nu(x) = [d(x, \partial\Omega)]^{\rho}, \quad -\frac{1}{1+\sigma} < \rho < \frac{2}{m}.)$$

The symbol $H^1(\nu, \Omega)$ stands for the space of all $u \in L^2(\Omega)$, whose derivatives (in the distributional sense on Ω) $\partial u/\partial x_i$ are functions such that $\sqrt{\nu(x)}\,\partial u/\partial x_i$ belongs to $L^2(\Omega)$, $i = 1, 2, \ldots, m$. $H^1(\nu, \Omega)$ is a Hilbert space with respect to the norm:

$$\| u \|_1 = \left(\int_\Omega (|u|^2 + \nu(x)|\nabla u|^2)\, dx \right)^{1/2}.$$

$H_0^1(\nu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\nu, \Omega)$; in this space we will take the following equivalent norm:

$$\| u \|_{1,0} = \left(\int_\Omega \nu(x)|\nabla u|^2\, dx \right)^{1/2}.$$

***Remark* 1.** By standard Sobolev's imbedding, there is a constant $C = C(m, \nu(x), \chi)$ such that

$$\left(\int_\Omega |u|^{2^\#}\, dx \right)^{1/2^\#} \leq C \left(\int_\Omega \nu(x)|\nabla u|^2\, dx \right)^{1/2} \quad \text{for} \quad u \in H_0^1(\nu, \Omega);$$

here $2^\# = 2m\chi/(m\chi + m - 2\chi) > 2$.

For more details on these spaces we refer the reader to [6, 7].

Finally, we denote by $H^{-1}(\nu^{-1}, \Omega)$ the dual space of $H_0^1(\nu, \Omega)$.

***Hypotheses* 2.** *The coefficients $a_{i,j}(x,s)$, $i,j = 1, 2, \ldots, m$, are functions defined and measurable in $\Omega \in \mathbb{R}$, fulfilling*

$$\frac{a_{i,j}(x,s)}{\nu(x)} \in L^\infty(\Omega \in \mathbb{R}), \quad i,j = 1, 2, \ldots, m.$$

Hypotheses 3. *For almost every* (x, s) *in* $\Omega \in \mathbb{R}$, *it results*

$$\sum_{1}^{m}{}_{i,j} a_{i,j}(x,s)\xi_i\xi_j \geq \nu(x) \sum_{1}^{m}{}_{i} \xi_i^2 \quad \text{for any} \quad \xi \in \mathbb{R}^m.$$

Let us denote by $a_{i,j,s}(x) = a_{i,j}(x, s)$ for $i, j = 1, 2, \ldots, m$ and $(x, s) \in \Omega \in \mathbb{R}$.

Hypotheses 4. *For every* $\varepsilon > 0$ *there exists a compact subset* $K_\varepsilon \subset \Omega$ *with* $\operatorname{meas}_x(\Omega \setminus K_\varepsilon) < \varepsilon$, *such that for every* $r > 0$ *the functions of the family* $\{a_{i,j,s}(x),\ s \in [-r, r],\ i, j = 1, 2, \ldots, m\}$ *are equicontinuous on* K_ε.

Hypothesis 5. *The function* $g(x, s)$ *is measurable in* x *on* Ω *for fixed* s *in* $\mathbb{R}$, *continuous in* s *for fixed* x. *We, also, suppose*:

(i) *for any* x *in* Ω, $g(x, 0) = 0$, *while for all* s *in* $\mathbb{R}$, x *in* Ω, $g(x, s)$, $s \geq 0$;

(ii) *the function* $g(x, s)$ *is non-decreasing in* s *on* $\mathbb{R}$ *and, for any fixed* s, $g(x, s)$ *belongs to* $L^1(\Omega)$.

Note that hypotheses 4 is fulfilled for instance in the following cases:

(a) $a_{i,j}(x, s)$ is measurable in x and continuous in s, $i, j = 1, 2, \ldots, m$;

(b) $a_{i,j}(x, s) = \alpha_{i,j}(x)\beta_{i,j}(s)$ with $\alpha_{i,j}$, $\beta_{i,j}$ measurable functions. Let $f \in H^{-1}(\nu^{-1}, \Omega)$, hypotheses 1, 2, 5 hold. We will consider the strongly nonlinear elliptic problem with Dirichlet boundary condition:

$$\begin{cases} \int_\Omega \sum_{1}^{m}{}_{i,j} a_{i,j}(x,u) \dfrac{\partial u}{\partial x_j} \dfrac{\partial v}{\partial x_i}\, dx + \int_\Omega g(x,u) v\, dx = \langle f, v \rangle; \\ \qquad \text{for all } v \in H_0^1(\nu, \Omega) \cap L^\infty(\Omega) \text{ and for } v = u, \\ u \in H_0^1(\nu,\Omega),\ g(x,u) \in L^1(\Omega) \text{ and } g(x,u)u \in L^1(\Omega). \end{cases} \tag{3}$$

In Sect. 3 we will show the following statement 1–5.

Theorem 1. *Under hypotheses* 1–5 *there exists a solution of* (3). *Moreover, the solution is unique if* $\mathcal{A}$ *is monotone and* g *is increasing in* s *or if* $\mathcal{A}$ *is strictly monotone (see the next section for the definition of* $\mathcal{A}$).

Next, sect. 4 will be arranged into two parts.

The first will be related to the study of regularity of solutions of problem (3), more precisely we shall give a proof of the following statement.

Theorem 2. *Under the same hypotheses of Theorem* 1, *if* $\chi > m$ *and*

$$f = -\sum_{1}^{m}{}_{i} \frac{\partial f_i}{\partial x_i}$$

with $f_i(x) \in L^t_{1/\nu}(\Omega)$*, $t > m(\chi - 1)/(\chi - m)$, *then we obtain* $u \in L^\infty(\Omega)$ *and*

$$\operatorname*{ess\,sup}_{\Omega} |u| \leq \gamma \|f\|_{H^{-1}(\nu^{-1},\Omega)};$$

(γ *denotes a constant depended on* χ, t, $\nu(x)$, $\operatorname{meas}_x \Omega$).

The second will be devoted to extend the results of the previous sections to variational inequalities.

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* See [1] for the representation of linear continuous functionals on $H_0^1(\nu, \Omega)$. $L^t_{1/\nu}(\Omega)$ denotes the Banach space of all measurable functions, $u(x)$, defined on Ω for which

$$\|u\|_{t,1/\nu} = \left(\int_\Omega \nu(x)^{-1} |u(x)|^t\, dx\right)^{1/t} < +\infty.$$

3. Preliminary Lemmas.

Lemma 1. *Assume that hypotheses 1–4 hold. Then the operator* $\mathcal{A}: H_0^1(\nu, \Omega) \to$ $\to H^{-1}(\nu^{-1}, \Omega)$ such that

$$\langle \mathcal{A}u, v \rangle = \int_\Omega \sum_{i,j}^{m} a_{i,j}(x, u(x)) \frac{\partial u(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_i} dx$$

is bounded, coercive and pseudomonotone.

Proof. We refer the reader to Theorem 2.1 of [8, p. 57].

Lemma 2. *Let* u *be a function belonging to* $H_0^1(\nu, \Omega)$. *Then there exists a sequence* (u_n) *fulfilling the following properties:*

$$u_n \in H_0^1(\nu, \Omega) \cap L^\infty(\Omega) \quad \text{for every} \quad n \in \mathbb{N};$$

$$|u_n(x)| \le |u(x)| \quad \text{and} \quad u_n(x)u(x) \ge 0 \quad \text{a.e. in} \quad \Omega \quad \text{for every} \quad n \in \mathbb{N},$$

$$u_n(x) \to u(x) \quad \text{in} \quad H_0^1(\nu, \Omega) \quad \text{as} \quad n \to +\infty.$$

Proof. For every $n \in \mathbb{N}$, it will be sufficient to define

$$u_n(x) = \operatorname{sgn} u \min(|u|, n) = \begin{cases} n, & \text{if } u \ge n, \\ u, & \text{if } |u| < n, \\ -n, & \text{if } u \le -n \end{cases}$$

(see [1, p. 10] prop. 2.7).

4. Existence and uniqueness Theorem.

***Proof** of Theorem* 1. We observe that the term $g(x, u)$ does not define a map from $H_0^1(\nu, \Omega)$ to $H^{-1}(\nu^{-1}, \Omega)$ because it doesn't satisfy any growth condition. Therefore, for every $n \in \mathbb{N}$, we put

$$g_n(x, s) = \begin{cases} g(x, s), & \text{if } |g(x, s)| < n, \\ n \dfrac{g(x, s)}{|g(x, s)|}, & \text{otherwise.} \end{cases}$$

Then

$$\langle T_n u, v \rangle = \int_\Omega g_n(x, u(x)) v(x) \, dx$$

is defined for all $u, v \in H_0^1(\nu, \Omega)$ and $v \to \langle T_n u, v \rangle$ defines an element $T_n u$ of $H^{-1}(\nu^{-1}, \Omega)$.

We claim that, for every $n \in \mathbb{N}$, T_n is a bounded, pseudomonotone operator. Indeed, by recalling the definition of truncation we obtain

$$|g_n(x, s)| \le n \quad \text{for all} \quad (x, s) \in \Omega \times \mathbb{R}, \quad n \in \mathbb{N}. \tag{4}$$

Also, the imbedding of $H_0^1(\nu, \Omega)$ into $L^2(\Omega)$ is compact (see Lemma 4.3 of [9]). Accordingly to Lemma 1, for every $n \in \mathbb{N}$, the operator $\mathcal{A} + T_n$ is bounded and pseudomonotone.

Next, by hypotheses 2, 3 and inequality (4), it results

$$\langle (\mathcal{A} + T_n) u, u - w \rangle \ge \|u\|_{1,0}^2 - M \|u\|_{1,0} \|w\|_{1,0} -$$

$$- n \left(\operatorname*{meas}_x \Omega \right)^{1/2} \{ \|u\|_{1,0} + \|w\|_{1,0} \},$$

for every $u, w \in H_0^1(\nu, \Omega)$; here

$$M = \max_{i,j=1,\ldots,m} \operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} \frac{|a_{i,j}(x,s)|}{\nu(x)}.$$

Therefore, from Theorem 32C of [10, p. 875], for each integer n and for the given element f of $H^{-1}(\nu^{-1},\Omega)$, there exists an element u_n of $H_0^1(\nu,\Omega)$ such that

$$\langle \mathcal{A}u_n - f, w\rangle + \int_\Omega g_n(x,u_n)w\,dx = 0 \quad \text{for every} \quad w \in H_0^1(\nu,\Omega). \tag{5}$$

Setting $w = u_n$ in (5), for every $n \in \mathbb{N}$, we get

$$\|u_n\|_{1,0}^2 \le \langle \mathcal{A}u_n, u_n\rangle + \int_\Omega g_n(x,u_n)u_n\,dx \le \|f\|_{H^{-1}(\nu^{-1},\Omega)}\|u_n\|_{1,0},$$

according to hypotheses 3 and evident inequality $g_n(x,s)s \ge 0$ in $\Omega\times\mathbb{R}$. Thus, for every $n\in\mathbb{N}$,

$$\|u_n\|_{1,0} \le \|f\|_{H^{-1}(\nu^{-1},\Omega)}. \tag{6}$$

As $\mathcal{A}$ is bounded, by passing to subsequences, we may suppose that $u_n \rightharpoonup u$ in $H_0^1(\nu,\Omega)$ and a.e. in Ω, and $\mathcal{A}u_n \rightharpoonup y$ in $H^{-1}(\nu^{-1},\Omega)$.

Also (6) and $\|\mathcal{A}u_n\|_{H^{-1}(\nu^{-1},\Omega)} \le C_1$ imply that

$$\int_\Omega g_n(x,u_n)u_n\,dx \le \left(\|f\|_{H^{-1}(\nu^{-1},\Omega)} + C_1\right)\|f\|_{H^{-1}(\nu^{-1},\Omega)} = \mathcal{P}$$

for every $n\in\mathbb{N}$. We now proceed to show that the sequence $\{g_n(x,u_n)\}$ in $L^1(\Omega)$ is equi-uniformly integrable.

We get

$$\alpha|g_n(x,u_n)| \le g_n(x,u_n)u_n + \alpha\{g(x,\alpha) + |g(x,-\alpha)|\}$$

foe each positive integer α and all n.

Hence, for any subset E of Ω, we conclude that

$$\int_E |g_n(x,u_n)|\,dx \le \frac{2}{\alpha}\mathcal{P} + \int_E g(x,\alpha)\,dx + \int_E |g(x,-\alpha)|\,dx$$

and finally that for $\operatorname{meas}_x(E)$ sufficiently small, $\int_E |g_n(x,u_n)|\,dx$ may be made small uniformly in n.

In addition, by continuity of $g(x,s)$ in s and definition of truncation, it follows that $g_n(x,u_n(x))$ converges a.e. to $g(x,u(x))$.

Consequently, by Vitali's theorem we have

$$g(x,u)\in L^1(\Omega), \quad g_n(x,u_n(x)) \to g(x,u) \text{ in } L^1(\Omega).$$

Moreover, by Fatou's lemma

$$\int_\Omega g(x,u)u\,dx \le \liminf_{n\to\infty}\int_\Omega g_n(x,u_n)u_n\,dx \le \mathcal{P}.$$

Thus

$$0 \le \int_\Omega g(x,u)u\,dx < +\infty.$$

From (5), for any $w\in H_0^1(\nu,\Omega)\cap L^\infty(\Omega)$, passing to the limit as $n\to+\infty$ we obtain

$$\langle y-f, w\rangle + \int_\Omega g(x,u)w\,dx = 0. \tag{7}$$

We shall show that $y = \mathcal{A}u$. Now, $\langle \mathcal{A}u_n, u_n-u\rangle = \langle \mathcal{A}u_n, u_n\rangle - \langle \mathcal{A}u_n, u\rangle$ so

$$\limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n-u\rangle = \limsup_{n\to\infty}\left\{\langle f, u_n\rangle - \int_\Omega g(x,u_n)u_n\,dx\right\} - \langle y, u\rangle \le$$

$$\le \langle f-y, u\rangle - \liminf_{n\to\infty}\int_\Omega g(x,u_n)u_n dx \le \langle f-y, u\rangle - \int_\Omega g(x,u)u\,dx.$$

Hence, for any $w \in H_0^1(\nu,\Omega)\cap L^\infty(\Omega)$, by virtue of (7),

$$\limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n-u\rangle = \langle f-y, u-w\rangle + \int_\Omega g(x,u)(w-u)\,dx.$$

By Lemma 2, there exists a sequence $w_k \in H_0^1(\nu,\Omega)\cap L^\infty(\Omega)$ such that w_k converges to u in $H_0^1(\nu,\Omega)$ and $|w_k(x)| \le |u(x)|$, a.e. in Ω. Consequently,

$$\langle f-y, u-w_k\rangle \to 0, \quad \int_\Omega g(x,u)w_k dx \to \int_\Omega g(x,u)u\,dx$$

by dominated convergence, since $g(x,u)u \in L^1(\Omega)$.

It follows that

$$\limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n-u\rangle \le 0.$$

By using the pseudomonotone property of $\mathcal{A}$ we get

$$\liminf_{n\to\infty} \langle \mathcal{A}u_n, u_n-w\rangle \ge \langle \mathcal{A}u, u-w\rangle \quad \text{for all} \quad w \in H_0^1(\nu,\Omega).$$

Now, we observe that for all $w \in H_0^1(\nu,\Omega)$ one has

$$\langle \mathcal{A}u, u-w\rangle \le \liminf_{n\to\infty}\langle \mathcal{A}u_n, u_n-w\rangle = \liminf_{n\to\infty}\langle \mathcal{A}u_n, u_n\rangle -$$

$$- \lim_{n\to\infty}\langle \mathcal{A}u_n, w\rangle \le \limsup_{n\to\infty}\langle \mathcal{A}u_n, u_n\rangle - \langle y, w\rangle =$$

$$= \limsup_{n\to\infty}\langle \mathcal{A}u_n, u_n-u\rangle + \lim_{n\to\infty}\langle \mathcal{A}u_n, u\rangle - \langle y, w\rangle \le \langle y, u-w\rangle.$$

Therefore

$$y = \mathcal{A}u, \quad \lim_{n\to\infty}\langle \mathcal{A}u_n, u_n\rangle \le \langle \mathcal{A}u, u\rangle.$$

From (7), in correspondence with $w = w_k$, via another passage to the limit we obtain

$$\langle \mathcal{A}u - f, u\rangle + \int_\Omega g(x,u)u\,dx = 0.$$

Finally, by standard method (see for instance [3]) we get the uniqueness result under strong monotonicity assumptions.

Remarks. **2.** If $a(x) \in L^1(\Omega)$, $a(x) \ge 0$ a.e. in Ω, putting $g(x,s) = a(x)|s|^{p-1}s$, $p > 1$, we obtain a function satisfying hypotheses 5.

3. If we assume that $a_{i,j}(x, s)$ does not depend on s, $i, j = 1, 2, \dots m$, then it is an immediate consequence of hypotheses 3 that the operator $\mathcal{A}$ is strictly monotone. Moreover, the operator $\mathcal{A}$ is monotone if

$$N = \left(\sum_{1}^{m}{}_{i,j} \operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} \left(\frac{|a_{i,j}(x,s)|}{\nu(x)} \right)^2 \right)^{1/2} \le 1,$$

because

$$\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle \ge (\|u\|_{1,0} - \|v\|_{1,0})^2$$

for each u and v belonging to $H_0^1(\nu, \Omega)$.

5. Solution properties and variational inequalities.

Proof of Theorem 2. Let u be a solution of problem (3). For each $k \ge 0$, setting $w_k = \operatorname{sgn} u \min(|u|, k)$, we obtain a sequence of functions $\{w_k\} \in H_0^1(\nu, \Omega) \cap L^\infty(\Omega)$ (see, Lemma 2) such that

$$|w_k(x)| \le |u(x)|, \quad w_k(x)u(x) \ge 0 \quad \text{a.e. in} \quad \Omega \quad \text{for every} \quad k \ge 0.$$

Therefore, denoting by $u_k = u - w_k$ in Ω $(k \ge 0)$, we get $u_k(x)u(x) \ge 0$ in Ω, $k \ge 0$, and so

$$g(x, u(x))u_k(x) \ge 0 \quad \text{in} \quad \Omega, \quad k \ge 0. \tag{8}$$

From (7), choosing $w = w_k$, we have

$$\langle \mathcal{A}u, w_k \rangle + \int_\Omega g(x, u) w_k \, dx = \langle f, w_k \rangle, \quad k \ge 0.$$

By

$$\langle \mathcal{A}u, u \rangle + \int_\Omega g(x, u) u \, dx = \langle f, u \rangle,$$

this implies

$$\langle \mathcal{A}u, u_k \rangle + \int_\Omega g(x, u) u_k \, dx = \langle f, u_k \rangle, \quad k \ge 0,$$

and finally that

$$\langle \mathcal{A}u, u_k \rangle \le \langle f, u_k \rangle, \quad k \ge 0, \quad \text{because of (8)}.$$

Hence, by using the Hölder's inequality, we have

$$\|u_k\|_{1,0} = \sum_{1}^{m}{}_i \left(\int_{\Omega(|u| \ge k)} \nu(x)^{-1} |f_i|^2 \, dx \right)^{1/2} \quad \text{for all} \quad k \ge 0 \tag{9}$$

(we denote by $\Omega(|u| \ge h) = \{x \in \Omega : |u(x)| \ge h\}, h \ge 0$) according to hypotheses 3.

On the other hand, it results

$$\left(\int_{\Omega(|u| \ge k)} \nu(x)^{-1} |f_i|^2 \, dx \right)^{1/2} \le \|f_i\|_{t,1/\nu} \|\nu^{-1}\|_\chi^{(t-2)/2t} \left[\operatorname*{meas}_x \Omega(|u| \ge k) \right]^{(1-1/\chi)(1/2-1/t)},$$

$$\|u_k\|_{2^\#} \le \beta \|u_k\|_{1,0}, \quad k \ge 0$$

(see Remark 1), so, for each $h > k \ge 0$, taking into account that

$$\| u_k \|_{2^\#} \geq (h-k) \left[\operatorname*{meas}_x \Omega(|u| \geq h) \right]^{1/2^\#},$$

from (9) we obtain

$$\left[\operatorname*{meas}_x \Omega(|u| \geq h) \right]^{1/2^\#} \leq$$

$$\leq \frac{\beta}{(h-k)} \sum_1^m{}_i \| f_i \|_{t,1/\nu} \| \nu^{-1} \|_{\chi}^{\frac{t-2}{2t}} \left[\operatorname*{meas}_x \Omega(|u| \geq k) \right]^{\left(1-\frac{1}{\chi}\right)\left(\frac{1}{2}-\frac{1}{t}\right)} =$$

$$= \frac{\beta}{(h-k)} \| f \|_{H^{-1}(\nu^{-1},\Omega)} \| \nu^{-1} \|_{\chi}^{(t-2)/2t} \left[\operatorname*{meas}_x \Omega(|u| \geq k) \right]^{(1-1/\chi)(1/2-1/t)}.$$

Consequently, setting, for all $k \geq 0$, $\varphi(k) = \left[\operatorname*{meas}_x \Omega(|u| \geq k) \right]^{1/2^\#}$, we have

$$\varphi(h) \leq \frac{\gamma}{(h-k)} \|f\|_{H^{-1}(\nu^{-1},\Omega)} \varphi(k)^\theta, \quad h > k \geq 0,$$

where $\theta = 2^\#(1-1/\chi)(1/2-1/t)$ is greater than 1.

The application of Stampacchia's Lemma [11, p. 212] yields to $\varphi(d) = 0$, where

$$d = \beta \| f \|_{H^{-1}(\nu^{-1},\Omega)} \| \nu^{-1} \|_{\chi}^{(t-2)/2t} [\varphi(0)]^{\theta-1} 2^{\theta/(\theta-1)}.$$

Thus, the proof of Theorem 2 is complete.

Now, let V be any closed subspace of $H_0^1(\nu, \Omega)$, K a closed convex subset of V $(0 \in K)$, f a given element of V^*; we can show, using the same method as in Theorem 1, a result of existence of solutions of the following variational inequalities:

$$\begin{cases} \langle \mathcal{A}_V u, v-u \rangle + \int_\Omega g(x,u)(v-u)\,dx \geq \langle f, v-u \rangle \\ \qquad\qquad \text{for every } v \in K \cap L^\infty(\Omega); \\ \int_\Omega G(x,v)\,dx - \int_\Omega G(x,u)\,dx + \langle \mathcal{A}_V u, v-u \rangle \geq \langle f, v-u \rangle \\ \qquad\qquad \text{for every } v \in K \text{ such that } \int_\Omega G(x,v)\,dx < +\infty, \end{cases} \tag{10}$$

where

$$G(x,s) = \int_0^s g(x,\tau)\,d\tau;$$

here $\mathcal{A}_V$ denotes the operator defined on V with value in V^* by the rule $\langle \mathcal{A}_V u, v \rangle = \langle \mathcal{A} u, v \rangle$, $u, v \in V$. (It is important to observe that the operator $\mathcal{A}_V$ is bounded coercive and pseudomonotone.)

The relation between the two classes of problems considered above is clarified by the following result:

In the case $K = V = H_0^1(\nu, \Omega)$, *a solution of problem* (3) *is a solution of* (10).

To this end, we first observe that as

$$0 \leq G(x, u(x)) \leq g(x, u(x))u(x) \quad \text{for every } x \text{ in } \Omega,$$

we have $\int_\Omega G(x,u)\,dx < +\infty$.

Moreover, for all $w \in H_0^1(\nu,\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega G(x,w)\,dx < +\infty$, we get

$$\int_\Omega G(x,w)\,dx - \int_\Omega G(x,u)\,dx \geq \langle f - \mathcal{A}u, v-u \rangle. \tag{11}$$

Suppose that v is an element of $H_0^1(\nu, \Omega)$ with $\int_\Omega G(x, v)\, dx < +\infty$. By Lemma 2 we amy construct a sequence of testing functions $\{w_k\}$ converging to v in $H_0^1(\nu, \Omega)$ and a.e. in Ω such that

$$w_k(x)v(x) \geq 0, \quad |w_k(x)| \leq |v(x)| \quad \text{for every} \quad x \quad \text{in} \quad \Omega,\ k \in \mathbb{N}.$$

It then follows that

$$0 \leq \int_\Omega G(x, w_k)\, dx \leq \int_\Omega G(x, v)\, dx < +\infty.$$

Consequently, we obtain from (11) with $w = w_k$ that

$$\int_\Omega G(x, w_k)\, dx - \int_\Omega G(x, u)\, dx \geq \langle f - \mathcal{A}u, w_k - u \rangle \quad \text{for all} \quad k \in \mathbb{N}. \tag{12}$$

Bearing in mind that

$$\int_\Omega G(x, w_k)\, dx \to \int_\Omega G(x, v)\, dx$$

by dominated convergence, since $0 \leq G(x, w_k(x)) \leq G(x, v(x))$ in Ω, from (12) as $k \to +\infty$

$$\int_\Omega G(x, v)\, dx - \int_\Omega G(x, u)\, dx \geq \langle f - \mathcal{A}u, v - u \rangle$$

so that the last inequality of (10) holds.

Finally, the first inequality of (10) is obvious.

Remark 4. In a forthcoming note we shall extend the existence result of Section 3 to an unbounded open Ω (in this case the imbedding of $H_0^1(\nu, \Omega)$ into $L^2(\Omega)$ is not compact), assuming $g(x, s) = \nu(x)|s|^{p-1}s$, $p > 1$.

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