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On Cohn's embedding of an enveloping algebra into a division ring

Про Конове вкладення обгортуючої алгебри в кільце з діленням

In 1961 P. M. Cohn proved that the universal enveloping algebra of any Lie algebra over a field can be embedded into a division ring. (The Lie algebra is not assumed to be finite dimensional.) Cohn's method is less than direct. We give a more explicit construction. These division rings have recently found uses in the theory of skew linear groups.

В 1961 р. Кон довів, що універсальна обгортуюча алгебра довільної алгебри Лі над полем може бути розширена до кільця з діленням. (Алгебра Лі не обв'язково скінченновимірна.) Метод Кона (побудови кільця) не дуже прямий. Ми наводимо більш явну конструкцію. Такі кільця з діленням останнім часом застосовуються в теорії лінійних груп над тілами.

Let F be a field, L a Lie F-algebra and $U = U(L)$ the universal enveloping algebra of L. In [1] P. M. Cohn constructs an embedding of U into a division ring D. Recently there has been interest in this specific division ring in connection with matrix groups and matrix rings [2—4]. Cohn's construction is less than direct and it seemed useful to have a very explicit description of D, at least for the benefit of group theorists.

My wish therefore was to analyse Cohn's construction, to define the division ring directly and explicitly and then simply to check that everything works. This turns out to be perfectly practicable, although certain checks, for example that for the associative law of addition, become tedious. Fortunately a slightly more devious approach avoids even this discomfort. I should like to emphasize that all the main results below are contained in [1, 2] or [3].

It is a consequence of the Poincaré-Birkhoff-Witt theorem [5, p. 159] that this universal enveloping algebra U has a valuation. Specifically there is a map $|\,| : U \to \mathbf{Z} \cup \{\infty\}$ satisfying

(1) a) $|ab| = |a| + |b|$,

b) $|a+b| \geqslant \min\{|a|, |b|\}$,

c) $|a| = \infty$ if and only if $a = 0$

for all $a, b \in U$. Further we can choose the valuation so that $|\alpha| = 0$ for all $\alpha \in F$ and $|x| = -1$ for all x in $L \setminus \{0\}$. The associated graded algebra of U is a (commutative) polynomial ring over F on a basis of L [5, p. 166]. These are the only facts we need.

More generally let R be any ring with a valuation $|\,|: R \to \mathbf{Z} \cup \{\infty\}$ satisfying (1) for all $a, b \in R$. Set $R_i = \{r \in R : |r| \geqslant i\}$. We embed R into a division ring under the assumption that the associated graded ring $\operatorname{Gr} R = \oplus_i (R_i/R_{i-1})$ satisfies

(2) $\operatorname{Gr} R$ is a right Ore domain.

Now assumption (2) is equivalent to:

(3) for all $a, b \in R \setminus \{0\}$ the map $(x, y) \mapsto |ax - by| - |ax|$ of $(R\setminus\{0\})^{(2)}$ to $\mathbf{Z} \cup \{\infty\}$ is unbounded above.

The equivalence of (2) and (3) is straightforward, and in any case is part of the content of [1] (Theorem 4.2). Thus our basic assumptions on R are (1) and (3). If $|\,|$ only takes the values 0 or ∞ then R is a right Ore domain by (2) and hence R embeds into its division ring of right quotients. From now on assume this is not the case. Then by renumbering we may assume that the image of R under the valuation generates the additive subgroup of $\mathbf{Z}$.

For all $p \geqslant 0$ set $S_{ip} = (R_i \setminus R_{i+1})/R_{i+p} = \{x + R_{i+p} : x \in R_i \setminus R_{i+1}\}$, so S_{ip} is a set of cosets of R_{i+p} in R and $S_{i0} = \{R_i\}$ if $R_i > R_{i+1}$ and $S_{i0} = \varnothing$ otherwise. We regard the S_{ip} as disjoint. Let $S_p = \bigcup_i S_{ip}$. Since the valuation preserves multiplication

$$R_{i+j} \setminus R_{i+j+1} \supseteq (R_i \setminus R_{i+1})(R_j \setminus R_{j+1})$$

as subsets of R. Thus we can make S_p into a multiplicative monoid with identity $1 + R_p$ and $S_{i+j,p} \supseteq S_{ip} \cdot S_{jp}$. ($S_1$ is just the set of homogeneous elements of $\operatorname{Gr} R$). Now S_p satisfies the right Ore condition: given $a, b \in R \setminus \{0\}$ choose $x, y \in R \setminus \{0\}$ by (3) with $|ax - by| > |ax| + p$; using a star to denote cosets in S_p, this says that $a^* x^* = b^* y^*$ in S_p. Then we can form the group $Q_p = S_p S_p^{-1}$ of right quotients. Note that S_0 embeds into $\mathbf{Z}$ via $R_i \mapsto i$, so Q_0 can be formally identified with $\{R_i : i \in \mathbf{Z}\} \cong \mathbf{Z}$, where $R_i . R_j = R_{i+j}$ by definition.

Let $p \geqslant 1$. Each element r of each coset in S_{ip} satisfies $|r^*| = i$. Thus define $|\,|$ on S_p by setting $|r^*| = i$ for all $r \in S_{ip}$. Do this also for $p = 0$. Clearly $|\,|$ is multiplicative. Also for $p \geqslant 1$ there is a natural projection $\pi : S_{ip} \to S_{i,p-1}$ obtained by factorizing by R_{i+p-1}/R_{i+p}. Specifically

$$\pi : r + R_{i+p} \mapsto r + R_{i+p-1} \quad \text{for } i = |r|.$$

Then π gives a monoid homomorphism of S_p onto S_{p-1} satisfying $|x\pi| = |x|$ for any $x \in S_p$. Being homomorphisms $|\,|$ and π extend uniguely to Q_p with

$$|ac^{-1}| = |a| - |c| \text{ and, for } p \geqslant 1,\ (ac^{-1})\pi = (a\pi)(c\pi)^{-1}.$$

These are well-defined and

(4) $$|x\pi| = |x| \text{ for } x \in Q_p, \quad p \geqslant 1.$$

Define π on Q_0 by $\pi : R_i \mapsto R_{i-1}$ for each i.

Set $Q = \bigcup_{p \geqslant 0} Q_p$ and define an addition on Q as follows. Let $x \in Q_p$ and $y \in Q_{p'}$. Assume $p \geqslant p'$. Since common denominators exist in Q_p there exist

$c, r, s \in R \setminus \{0\}$ with valuations h, i, j respectively such that

$$(5)\quad x = (r + R_{i+p})(c + R_{h+p})^{-1} \text{ and } y = (s + R_{j+p'})(c + R_{h+p'})^{-1}.$$

Define $x + y = y + x = (r + s + R_{\min\{i+p, j+p'\}})(c + R_{h+q})^{-1} \in Q_q$

$$\text{where } q = \max\{0, \min\{i + p, j + p'\} - |r + s|\}.$$

Note that $0 \leqslant q \leqslant p$ since by (1) we have

$$|r + s| \geqslant \min\{|r|, |s|\} = \min\{i, j\} \geqslant \min\{i + p, j + p'\} - p.$$

We need to check that addition is well defined.

Clearly $x+y$ depends only the cosets $r + R_{i+p}$, $s + R_{j+p'}$ and $c + R_{h+p}$ and not on the representatives r, s and c. Suppose also that

$$x = (u + R_{|u|+p})(d + R_{|d|+p})^{-1} \text{ and } y = (v + R_{|v|+p'})(d + R_{|d|+p'})^{-1}.$$

By the Ore condition there exist $e, f \in R \setminus \{0\}$ with

$$(c + R_{h+p})(e + R_{|e|+p}) = (d + R_{|d|+p})(f + R_{|f|+p}).$$

Then $ce + R_{|ce|+p} = df + R_{|df|+p}$ and $re + R_{|re|+p} = x(ce + R_{|ce|+p}) = uf + R_{|uf|+p}$. Applying $\pi^{p-p'}$ yields $ce + R_{|ce|+p'} = df + R_{|df|+p'}$ and then $se + R_{|se|+p'} = vf + R_{|vf|+p'}$. Thus we reduce to the case where $u = re$, $v = se$ and $d = ce$. Since $|\ |$ is multiplicative

$$\max\{0, \min\{|re| + p, |se| + p'\} - |re + se|\} = q,$$

where q is as in the definition of $x + y$, and

$$(re + se + R_{\min\{|re|+p, |se|+p'\}})(ce + R_{|ce|+q})^{-1} = (r + s + R_{\min\{|r|+p, |s|+p'\}})(e + R_{|e|+q})(e + R_{|e|+q})^{-1}(c + R_{h+q})^{-1} = x + y.$$

Therefore addition is well defined.

The rules of this addition are now easily checked. Suppose also that $z \in Q_{p''}$. We can choose c, r and s as in (5) and also $t \in R \setminus \{0\}$ such that for $k = |t|$, $z = (t + R_{k+p''})(c + R_{h+p''})^{-1}$. Then $(x + y) + z = (r + s + t + R_{\min\{i+p, j+p', k+p''\}})(c + R_{h+q''})^{-1}$ where $q'' = \max\{0, \min\{i + p, j + p', k + p''\} - |r + s + t|\}$. Therefore addition is associative. It is easily seen that

$$(6)\quad \begin{gathered} x\pi + y\pi = (x + y)\pi, \\ x\pi + y = x + y \text{ or } x\pi + y\pi \end{gathered}$$

and for $p' = 0$,

$$(7)\quad x + y = \begin{cases} x \text{ if } |y| \geqslant |x| + p \text{ and} \\ x\pi^{|x|+p-|y|} \quad \text{otherwise.} \end{cases}$$

Now define a negation on Q by setting $-x = (-r + R_{i+p})(c + R_{h+p})^{-1}$, the notation being as in (5). Then with u, d, e and f as above

$$-x = (-re + R_{|re|+p})(ce + R_{|ce|+p})^{-1}$$

and negation is well defined. Clearly $x + (-x) = R_{|x|+p} \in Q_0$ and if also $x' \in Q_p$ then $(-x)x' = -(xx') = x(-x')$.

The maps π make $\{Q_p : p \geqslant 0\}$ into an inverse system of groups. Let $G = \varprojlim Q_p$; G is the multiplicative group whose elements we can take in the form $g = \{g_p\ g_p \in Q_p,\ g_{p+1}\pi = g_p,\ p \geqslant 0\}$ (since the Q_p are disjoint), the multiplication being componentwise. Note that $|g_p|$ is independent of p by (4). This constant value of $|g_p|$ we denote by $|g|$.

It is easy to see using (6) that for g, h in G the set $g * h = \{x + y : x \in g, y \in h, x + y \notin Q_0\}\pi$ is either empty or a member of G. Set $D = G \cup \{0\}$. Then

D becomes a division ring, where we add and multiply 0 in the obvious way, the multiplication on G is just the group multiplication of G and we add elements g, h of G by the rule

$$g + h = \begin{cases} g*h & \text{if } g*h \neq \varnothing, \\ 0 & \text{if } g*h = \varnothing. \end{cases}$$

This is explicit, but tedious to check that D is in fact a ring. The following alternative construction of D requires less checking.

Define left and right actions of G on Q via the multiplication on each Q_p, viz for $x \in Q_p$ and $g \in G$ set $x.g = xg_p \in Q_p$ and $g.x = g_p x$. Then Q becomes a G—G bisemi-module (satisfies all the axioms of a bimodule except that $(Q, +)$ is only a commutative semigroup, not necessarily a commutative group.) Only the distributative laws need checking. We check first the right distributative law.

Let $x \in Q_p$, $y \in Q_{p'}$ with $p \geq p'$ and let $g = \{g_q : q \geq 0\} \in G$. There exist $r, s, t, c, d \in R \setminus \{0\}$ with valuations i, j, k, h, l respectively such that (5) holds and

$$(c + R_{h+p})^{-1} g_p = (t + R_{k+p})(d + R_{l+p})^{-1},$$

so

$$(c + R_{h+p'})^{-1} g_{p'} = (t + R_{k+p'})(d + R_{l+p'})^{-1}$$

apply $\pi^{p-p'}$). Then

$$x.g + y.g = (rt + R_{i+k+p})(d + R_{l+p})^{-1} + (st + R_{j+k+p'})(d + R_{l+p'})^{-1} =$$
$$= (rt + st + R_{\min\{i+p,\, j+p'\}+k})(d + R_{l+q})^{-1}$$

for $q = \max\{0, \min\{i + p, j + p'\} + k - |rt + st|\} = \max\{0, \min\{i + p, j + p'\} - |r + s|\}$, while

$$(x + y).g = (r + s + R_{\min\{i+p,\, j+p'\}})(c + R_{h+q})^{-1} g_q =$$
$$= (r + s + R_{\min\{i+p,\, j+p'\}})(t + R_{k+q})(d + R_{l+q})^{-1} = x.g + y.g.$$

The left distributative law is proved similarly, the basic step being

$$(f + R_{m+p})^{-1}(r + R_{i+p}) + (f + R_{m+p'})^{-1}(s + R_{j+p'}) = (f + R_{m+q})^{-1} \times$$
$$\times (r + s + R_{\min\{i+p,\, j+p'\}})$$

in the obvious notation, where q is as after (5). To see this note that there exist r', s' and e with

$$(f + R_{m+p})^{-1}(r + R_{i+p}) = (r' + R_{i'+p})(e + R_{|e|+p})^{-1}$$

and

$$(f + R_{m+p'})^{-1}(s + R_{j+p'}) = (s' + R_{j'+p'})(e + R_{|e|+p'})^{-1}.$$

Thus

$$re + R_{i+|e|+p} = fr' + R_{i'+m+p} \text{ and } se + R_{j+|e|+p'} = fs' + R_{j'+m+p'}.$$

Then

$$(r + s)e + R_{\min\{i+p,\, j+p'\}+|e|} = f(r' + s') + R_{\min\{i'+p,\, j'+p'\}+m},$$

so

$$(f + R_{m+q})^{-1}(r + s + R_{\min\{i+p,\, j+p'\}}) = (r' + s' + R_{\min\{i'+p,\, j+p'\}}) \times$$
$$\times (e + R_{|e|+q})^{-1} = (f + R_{m+p})^{-1}(r + R_{i+p}) + (f + R_{m+p'})^{-1}(s + R_{j+p'})$$

as claimed.

Let D_0 denote the set of all subsets of Q. Define addition, negation and two G-actions on D_0 by

$$X + Y = \{x + y : x \in X \text{ and } y \in Y\},$$
$$-X = \{-x : x \in X\},$$
$$X.g = \{x.g : x \in X\}$$

and $g.X = \{g \cdot x : x \in X\}$.

Trivially D_0 is also a G—G bisemi-module. Also G is a subset of D_0 a... the above G—G action on G is just given by the multiplication in the group G.

For each $l \in \mathbf{Z}$ set $H_l = \{R_i : i \geq l\} \subseteq Q_0$, so $H_l \in D_0$. Let D_1 denote the subset of D_0 of all elements of the form $g \cup X$ or $H_l \cup X$ for $g \in G$, X finite and $l \in \mathbf{Z}$. Then D_1 is a G-G sub-bisemi-module of D_0, closed under negation: this follows from the following, whose claims are easy to verify. Set $Q_{ip} = \{x \in Q_p : |x| = i\}$.

(8) Let $g = \{g_p : p \geqslant 0\}$ and h be elements of G and pick $x \in Q_{iq}$ and l, $m \in \mathbf{Z}$. Then

a) $x + H_l = \{x\pi^u : 0 \leqslant u \leqslant \max\{0, i + q - l\}\}$,

b) $g + H_l = g \cup X$ for some finite subset X of Q_0,

c) $x + g = \{x + g_p : p \leqslant i + q - |g|\}$, $x.g = xg_q \in Q$ and $g.x \in Q$,

d) $g + h = k \cup X$ for some k in G and X a finite subset of Q_0 or $= H_l$ for $l = |g| = |h|$,

e) $H_l + H_m = H_{\min\{l,m\}}$, $g.H_l = H_{l+|g|} = H_l.g$ and $-H_l = H_l$ and

f) $-g = \{-g_p\} \in G$, $g + (-g) = H_{|g|}$ and $gh \in G$.

Note that in a) and c) both $x + H_l$ and $x + g$ are finite.

Define an equivalence relation $\approx$ on D_1 by $X \approx Y$ whenever the symmetic difference $(X \setminus Y) \cup (Y \setminus X)$ is finite. There is an obvious bijection between $D_1/\approx$ and $D = G \cup \{0\}$ determined by the map $g \cup X \mapsto g$ and $H_l \cup X \mapsto 0$ of D_1 onto D. Also the equivalence relation respects all four operations by (8) above. Thus $D_1/\approx$ and hence D becomes a G-G bisemi-module with an additional operation negation such that for any $g \in G$

$$g + 0 = g \text{ by 8b)},\ 0 + 0 = 0 \text{ by 8e) and } g + (-g) = 0 \text{ by 8f)}.$$

Therefore D is a division ring.

Define the map $\varphi : R \to D$ by $0\varphi = 0$ and for $r \in R \setminus \{0\}$ by $r\varphi = \{r + R_{|r|+p};\ p \geqslant 0\} \in G$, where G is regarded as a subset of D. Certainly φ is a multiplicative map. If $r, s \in R$ with $r\varphi = s\varphi$ then $r - s \in \bigcap_m R_m = \{0\}$, so φ embeds R into D. Further φ is additive, for $(-r)\varphi = -(r\varphi)$ by definition, so $r\varphi + (-r)\varphi = 0 = (r + (-r))\varphi$, and if $s \neq -r$ then $|r+s| < \infty$ and

$$r\varphi + s\varphi = \{r + s + R_{|r+s|+p} : p \geqslant 0\} = (r + s)\varphi.$$

Therefore φ embeds the ring R into the division ring D and the main theorem is proved.

Suppose F is a central subfield of R, for example suppose R is the universal enveloping algebra $U(L)$ of the Lie F-algebra L. Then φ also embeds F into D and clearly $(F \setminus \{0\})\varphi$ is central in G. Therefore D too becomes an F-algebra. We could have carried the F-structure right through the construction. If $\alpha \in F \setminus \{0\}$ define

$$\alpha.(r + R_{l+p})(c + R_{h+p})^{-1} = (\alpha r + R_{|\alpha r|+p})(c + R_{h+p})^{-1}.$$

This gives an action of $F \setminus \{0\}$ on Q that is well defined. If $|\alpha| = 0$ for all $\alpha \in F \setminus \{0\}$ then this even satisfies $(\alpha + \beta).x = \alpha x + \beta x$ for $\alpha, \beta, \alpha + \beta$ all non-zero. Put $0.x = R_{|x|+p}$ for $x \in Q_p$. Setting $\alpha.X = \{\alpha x : x \in X\}$ defines an action on D_0. This in turn gives an action of F on D_1 and then on $(D_1/\approx) \cong D$. The resulting action of F on D is simply the one given above.

We have already defined the map $|\ | : G \to \mathbf{Z}$ satisfying $|gh| = |g| + |h|$ and $|-g| = |g|$. Set $|0| = \infty$. Then this defines a valuation on D. This will follow from our computation below of the graded ring of D but it is also easily checked directly as follows. We have only to see that $|g + h| \geqslant \min\{|g|, |h|\}$ for all $g, h \in G$.

Assume the notation of (5). Then

$$|x + y| = \min\{i + p, j + p', |r + s|\} - h \text{ and always } |r + s| \geqslant \min\{i, j\}.$$

Thus $|x + y| \geqslant \min\{i, j\} - h = \min\{|x|, |y|\}$. Hence

$$\min\{|g|, |h|\} = \begin{cases} \min\{|g_p|, |h_p|\} \text{ for all } p \geqslant 0, \\ \leqslant |g_p + h_p| \text{ for all } p \geqslant 0, \\ \quad |g + h| \text{ for all large enough } p, \end{cases}$$

unless $g+h=0$ in D, in which case $|g_p+h_p|\leqslant|g+h|$ trivially. Clearly $|r\varphi|=|r|$ for all $r\in R$; that is this valuation of D extends the qiven valuation on R.

We now compute the graded ring $\operatorname{Gr} D=\oplus(D_i/D_{i+1})$ of this valuation on D, where $D_i=\{d\in D:|d|\geqslant i\}$. Denote the ring of right quotients of $\operatorname{Gr} R$, which exists by (2) note, by E. Now $Q_{m1}\cup\{R_{m+1}\}$ is additively a group, for if we assume the notation of (5) with $x, y\in Q_{m1}$, so $i=j=h+m$, then

$$x+y=(r+s+R_{i+1})(c+R_{h+q})^{-1},$$

which either lies in Q_{m1}, or $|r+s|>i$ and it equals R_{m+1}. (In fact this additive group is isomorphic to the additive subgroup $Q_{m1}\cup\{0\}$ of E.)

Suppose $x\in Q_p$ and $y\in Q_{p'}$ with $p\geqslant p'$. Then

(9) $x=y$ if and only if $p=p'$ and $|x|=|y|\leqslant|x-y|-p$,

if and only if $p=p'$, $|x|=|y|$ and $x-y\in Q_0$.

For suppose $p=p'$ and $|x|=|y|\leqslant|x-y|-p$. Then in the notation of (5)

$$x-y=(r-s+R_{i+p})(c+R_{h+q})^{-1}$$

and

$$|r-s|\geqslant|x-y|+h\geqslant|x|+h+p=i+p.$$

Hence $r-s\in R_{i+p}$, $r+R_{i+p}=s+R_{i+p}$ and $x=y$. Now suppose $p=p$ and $|x|=|y|$. Then

$$x-y\in Q_0\Leftrightarrow r-s\in R_{i+p}\Leftrightarrow|r-s|\geqslant i+p\Leftrightarrow|x-y|=i+p-h=|x|+p.$$

If $x=y$ the claims of (9) are trivial and this completes the proof of (9).

Consinder the map $\varphi_p: g\mapsto g_p$ of G onto Q_p. Let $g, h\in G$. Always $|g_p-h_p|\leqslant|g-h|$, with equality unless $g_p-h_p\in Q_0$. Hence by (9) we have $g_p=h_p$ if and only if $|g|=|h|\leqslant|g-h|-p$. Thus φ_p induces bijections of $\cup_i(D_i\setminus D_{i+1})/D_{i+p}$ with Q_p and of $D_i\setminus D_{i+1}$ with $Q_{i1}\cup\{0\}$, and the second bijection respects addition and multiplication. Thus (10)

$$\operatorname{Gr}D=\oplus_i D_i/D_{i+1} \text{ can be identified with the subring } \oplus_i(Q_{i1}\cup\{0\})\supseteq$$

$$\supseteq\oplus_i(S_{i1}\cup\{0\})=\operatorname{Gr}R$$

of E. In particular $\operatorname{Gr}D$ is a right Ore domain with ring of right quotients E.

(11) The following are immediate.

a) $\operatorname{Gr}D$ is a domain, so the map $|\ |$ on D is a valuation, as we have seen.

b) By Theorem 4.2 of [1] for all $a, b\in D\setminus\{0\}$ the map $(x, y)\mapsto|ax-by|-|ax|$ of $(D\setminus\{0\})^{(2)}$ to $\mathbf{Z}\cup\{\infty\}$ is unbounded.

c) If $|xy-yx|>|xy|$ for all $x, y\in R\setminus\{0\}$ then $\operatorname{Gr}R$ is commutative, so $\operatorname{Gr} D$ is commutative and $|xy-yx|>|xy|$ for all $x, y\in D\setminus\{0\}$.

Property (11c) is fundamental to [2]. The filtration $\{D_i : i\in\mathbf{Z}\}$ determines a Hausdorff topology on D so that D becomes a topological ring with each D_i open. If we topologize $\mathbf{Z}\cup\{\infty\}$ by taking all $\{i\}$ and all $\{j: i\leqslant j\leqslant\infty\}$ for $i\in\mathbf{Z}$ as a basis of the open sets, then the valuation is a continuous map. The fibres of the map $\varphi_p: G\to Q_p$ have the form $g+D_{|g|+p}$. Thus the induced topology on G is the topology given by the inverse limit $G=\varprojlim Q_p$ with the discrete topology on the Q_p. By definition G is complete in the latter. Consequently D is complete. Clearly φ_p maps $(R\setminus\{0\})(R\setminus\{0\})^{-1}$ onto Q_p. Hence $(R\setminus\{0\})(R\setminus\{0\})^{-1}$ is dense in G and so $R(R\setminus\{0\})^{-1}$ is dense as a subset of D. This gives an alternative approach to (11b) and, since the topology is given by a $\mathbf{Z}$-valued valuation, (11c).

We now consider the residue class division ring D_0/D_1 of D. We have seen above that D_0/D_1 can be identified with $Q_{01}\cup\{0\}$. Now $S_i=\cup_i S_{i1}$ is the set of homogeneous elements in $\operatorname{Gr}R$ and so $E\supseteq\cup_i S_{1i}S_{i1}^{-1}=Q_{01}$. Suppose for the moment that $R=U(L)$ for L a non-zero Lie F-algebra, F a field and choose any F-basis $\mathbf{B}$ of L. Then $\operatorname{Gr}R$ is isomorphic to the polynomial ring $F[\mathbf{B}]$, where $S_{-i,1}$ is identified with the set of homogeneous polynomials of total degree i. (Here $S_{i1} = \varnothing$ if $i > 0$). This identifies $Q_{01} \cup \{0\}$ and hence D_0/D_1 with the subfield.

$K = \{0, f/g : f, g \in F[\mathbf{B}]$ with f, g homogeneous of the same degree$\}$ of the rational function field $F(\mathbf{B})$. Pick any b_0 in $\mathbf{B}$. If $f, g \in F[\mathbf{B}]$ are homogeneous of degree n then $f/g = b_0^{-n} \cdot f / b_0^{-n} \cdot g$ and $b_0^{-n} \cdot f$ and $b_0^{-n} \cdot g$ are just polynomials in $b_0^{-1}(\mathbf{B} \setminus \{b_0\})$. Thus K, $Q_{01} \cup \{0\}$ and D_0/D_1 are purely transcendental extension fields of F with transcendence bases bijective with $b_0^{-1}(\mathbf{B} \setminus \{b_0\})$ in an obvious way. In particular D_0/D_1 has transcendence degree $(\dim_F L) - 1$ over F.

We now return to our general ring satisfying (1) and (2). Since $|\,|$ is a valuation on D, so D_0 is a valuation ring, whose only ideals, left or right, are the D_i for $i \geqslant 0$, $D_i = u^i D_0 = D_0 u^i$ for any $u \in D_1 \setminus D_2$ and $i \geqslant 1$, D_0 is a principal left and a principal right ideal domain and in particular is Noetherian and (left and right) Ore and D is the division ring of quotients of D_0. Suppose now that R too is right Ore with division ring D_R of right quotients. Then D_R can be identified in a natural way with a subring of D and since the subset $R \cdot (R \setminus \{0\})^{-1}$ of D lies in D_R, the latter is dense in D. The valuation on D restricts to a valuation on D_R, in fact to the unique extension of $|\,|$ on R to D_R, and D is just the completion of D_R with respect to the valuation topology. In particular $D_R = D$ if and only if $(D_R, |\,|)$ is complete.

Finally we consider extensions of the ground field. Let K be an extension field of the field F and suppose L is a Lie F-algebra with universal enveloping algebra $R = U(L)$. Then $L^K = K \otimes_F L$ is a Lie K-algebra and $U(L^K)$ and $R^K = K \otimes_F R$ are naturally isomorphic. The valuation on R^K via $R^K \cong U(L^K)$ is simply the unique extension of the valuation on R subject to $|\alpha| = 0$ for all $\alpha \in K \setminus \{0\}$. Repeat the above construction with R^K, thus obtaining a division over-ring D_K say of R^K. Both K and D can be regarded as subrings of D_K, the latter being identified with the closure of $R \cdot (R \setminus \{0\})^{-1}$ in D_K, and the subring of D_K generated by K and D yields an embedding of $K \otimes_F D$ into D_K. In particular $K \otimes_F D$ is a domain, a fact made use of in [4]. If R is now just an F-algebra satisfying (1) and (2) it is easy to see that $K \otimes_F D$ is a domain whenever $K \otimes_F (D_0/D_1)$ is a domain. Indeed this holds if K is just a division F-algebra and for the same reason. If $R = U(L)$ again then $K \otimes_F (D_0/D_1)$ is just a subring of the rational function field over K in $(\dim_F L) - 1$ variables. This yields a second proof that $K \otimes_F D$ is a domain in this case.

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