Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 2. **(2003).** pp. 1–5

Almost all derivative quivers of artinian biserial rings contain chains

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Communicated by V. V. Kirichenko

Abstract. A lower estimate for the number M_n of all labelled quivers with n–vertex parts of Artinian biserial rings is given and the asymptotic of the relation M_n/B_n, where B_n denotes the number of those quivers all connected components of which are cycles, is studied.

In the beginning of 70-s P.Gabriel [1] introduced a notion of a quiver of a finite dimensional algebra over an algebraically closed field — an directed graph of special type which in concise form preserves some very important information about the algebra. Using these graphs in [1] (see also Krugliak [2]) all finite dimensional algebras of finite type over an algebraically closed field with square zero radical are described. Later V.Kirichenko has expanded the construction of such an directed graph to right Noetherian semiperfect rings [3], and then to several other classes of rings (see, for example, [4], [5] and bibliography there). For some classes of rings it is convenient to consider a so called derivative quiver $RQ(A)$ (see [6]), which for the rings under consideration always turns out to be a simple bipartite graph with equicardinal part, instead of a quiver $Q(A)$ of a ring A.

In this connection there arises a natural problem of investigation of graphs which can be quivers of rings of some class. We will deal with Artinian biserial rings, first introduced by .Fuller [7]. A starting point for this paper is the following statement([4], Corollary 5.15): *An Artinian ring A, with square zero Jacobson radical is biserial if and only if its derivative quiver $RQ(A)$ is a disconnected union of chains and cycles.*

2000 Mathematics Subject Classification: *05C30, 05C38, 16P20.*
Key words and phrases: *artinian rings, quivers, bipartite graphs.*

Therefore, a derivative quiver of an Artinian biserial ring is a simple bipartite graph with parts of the same cardinality, in which the degree of each vertex does not exceed 2. In [8] such graphs have been called *Artinian–biserial*, or just *AB–graphs*. An AB–graph with n–vertex parts is called *labelled*, if the vertices of each part are numbered from 1 to n and it is indicated, which of the parts is *lower*, and which is *upper*. In what follows we consider only labelled AB–graphs.

In [8] the number B_n of those AB–graphs with n–vertex parts, all connected components of which are cycles, is counted:

$$B_n = \sum_{\substack{(l_1,l_2,\dots,l_n)\\ 1l_1+2l_2+\dots+nl_n=n}} \frac{(n!)^2}{(l_1!)^2 \prod\limits_{k=2}^{n} (2^{l_k} \cdot k^{l_k} \cdot l_k!)}, \qquad (1)$$

and an upper bound for the number M_n of all labelled AB–graphs with n–vertex parts is obtained:

$$M_n < \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{n!}{(n-k)!} \left|IS_{n-k}\right|^2,$$

where $\left|IS_n\right|$ — is the order of the inverse symmetric semigroup IS_n of degree n. This estimate, however, is rather rough. Beside this, to give an estimate for the order in the right-hand side of the latter inequality for large values of n is itself a difficult problem.

A more effective lower estimate for M_n is given in the following

Lemma. *The number M_n of all labelled AB–graphs with n–vertex parts satisfies the inequality $M_n > (n!)^2 \cdot \frac{n}{2}$.*

Proof. Since $M_1 = 2$ and $M_2 = 16$ then the statement is obvious for $n = 1$ and for $n = 2$. Let now $n \geq 3$ and suppose that for all $k < n$ the statement of Lemma is true. Consider those AB–graphs, which contain a sufficiently long chain of an odd length. Then exactly one of the endpoints of such a chain will belong to the lower part. To determine a chain of length $2n-2k-1$, one has to choose its endpoint in the lower part, then a vertex in the upper part incident to this endpoint, then the next vertex of a chain in the lower part, and so on, each time switching the part of the next vertex choice, till one reaches the $2n-2k$–s vertex of the chain which is its endpoint from the upper part. Since in this way one will get every chain of length $2n-2k-1$ exactly one time then the number of different chains of length $2n-2k-1$ equals

$$n \cdot n \cdot (n-1) \cdot (n-1) \cdot (n-2) \cdot (n-2) \cdots (k+1) \cdot (k+1) = \frac{(n!)^2}{(k!)^2}.$$

Since for $k \le \dfrac{n-1}{2}$ an AB–graph with n–vertex parts can not contain more than one chain of length $2n-2k-1$ then for such values of k the number of AB–graphs, containing a chain of length $2n-2k-1$, equals $\dfrac{(n!)^2}{(k!)^2}\cdot M_k$. By the inductive assumption $M_k > (k!)^2\cdot\dfrac{k}{2}$. Using the equality $M_1 = 2$, we can assume $M_k > (k!)^2$. It is easily seen, that an AB–graph with n–vertex parts can contain only one chain of length $\ge n$. Therefore,

$$M_n > \sum_{k=0}^{[(n-1)/2]} \left(\frac{n!}{k!}\right)^2 (k!)^2 = (n!)^2\cdot \sum_{k=0}^{[(n-1)/2]} 1 = (n!)^2\cdot\left[\frac{n-1}{2}\right] > (n!)^2\cdot\frac{n}{2}.$$

□

Theorem. *Let M_n be the number of all labelled AB–graphs with n–vertex parts, and B_n — the number of those of such graphs, all connected components of which are cycles. Then* $\lim\limits_{n\to\infty}\dfrac{B_n}{M_n} = 0$.

Proof. Let us calculate an upper bound for B_n. It is known, that the number of permutations of a cycle type $(l_1, l_2, \ldots, l_n)$ equals $n!\cdot\left(\prod_{k=1}^{n}(k^{l_k}\cdot l_k!)\right)^{-1}$. Since the number of all permutations is $n!$, then

$$\sum_{\substack{(l_1,l_2,\ldots,l_n)\\ 1l_1+2l_2+\cdots+nl_n=n}} \frac{n!}{\prod\limits_{k=1}^{n}(k^{l_k}\cdot l_k!)} = n!.$$

After cancellation of both sides by $n!$ we obtain:

$$\sum_{\substack{(l_1,l_2,\ldots,l_n)\\ 1l_1+2l_2+\cdots+nl_n=n}} \frac{1}{\prod\limits_{k=1}^{n}(k^{l_k}\cdot l_k!)} = 1.$$

This equality and an obvious inequality $l_1!\cdot\prod_{k=2}^{n} 2^{l_k} \ge 1$, imply:

$$\sum_{\substack{(l_1,l_2,\ldots,l_n)\\ 1l_1+2l_2+\cdots+nl_n=n}} \frac{1}{(l_1!)^2\prod\limits_{k=2}^{n}(2^{l_k}\cdot k^{l_k}\cdot l_k!)} =$$

$$= \sum_{\substack{(l_1,l_2,\ldots,l_n)\\ 1l_1+2l_2+\cdots+nl_n=n}} \frac{1}{\prod\limits_{k=1}^{n}(k^{l_k}\cdot l_k!)}\cdot\frac{1}{l_1!\prod\limits_{k=2}^{n}2^{l_k}} <$$

$$< \sum_{\substack{(l_1,l_2,\ldots,l_n)\\ 1l_1+2l_2+\cdots+nl_n=n}} \frac{1}{\prod\limits_{k=1}^{n}(k^{l_k}\cdot l_k!)} = 1.$$

This inequality and inequality 1 now imply, that $B_n < (n!)^2$. Therefore, using Lemma, we obtain:

$$0 \leq \frac{B_n}{M_n} \leq \frac{(n!)^2}{(n!)^2 \cdot \frac{n}{2}} = \frac{2}{n}.$$

Thus, $\lim\limits_{n\to\infty} \frac{B_n}{M_n} = 0.$ □

Following the tradition for usage of the expression 'almost all' (see, for example, [9]), we obtain the following

Corollary. *Almost all AB–graphs with n–vertex parts contain chains.*

We conclude by stating the values of B_n and M_n and of the relation B_n/M_n for small values of n:

n	2	3	4	5
B_n	2	16	151	4991
M_n	16	265	7343	304186
B_n/M_n	0.125	0.0603773	0.0205638	0.0164077

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Received by the editors: 06.12.2002.