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RESEARCH ARTICLE

The lower bound for the volume of a three-dimensional convex polytope

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Abstract. In this paper, we provide a lower bound for the volume of a three-dimensional smooth integral convex polytope having interior lattice points. Since our formula has a quite simple form compared with preliminary results, we can easily utilize it for other beneficial purposes. As an immediate consequence of our lower bound, we obtain a characterization of toric Fano threefold. Besides, we compute the sectional genus of a three-dimensional polarized toric variety, and classify toric Castelnuovo varieties.

1. Introduction

Points in $\mathbb{Z}^n$ are called *lattice points* of $\mathbb{R}^n$, and a polytope is said to be *integral* if all its vertices are lattice points. For an integral polytope $\mathcal{P}$, we denote by $\mathrm{vol}(\mathcal{P})$ the volume of $\mathcal{P}$ and by $\partial\mathcal{P}$ the boundary of $\mathcal{P}$, and put $\mathrm{Int}(\mathcal{P}) = \mathcal{P} \setminus \partial\mathcal{P}$. Besides, we define $l(S) = \sharp(S \cap \mathbb{Z}^n)$ for a subset $S \subset \mathbb{R}^n$. In the study of integral polytopes, one of the most significant problem is to compute their volume by using the information of the number of lattice points in them. The following classical theorem gave a clue to the solution of this issue.

Theorem 1.1 (cf. [14]). *Let $\mathcal{P}$ be an integral polygon which is homeomorphic to a closed circle. Then its volume is computed by* $2\mathrm{vol}(\mathcal{P}) = l(\mathcal{P}) + l(\mathrm{Int}(\mathcal{P})) - 2$.

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In the case where an integral polygon is not homeomorphic to a closed circle, Reeve generalized the above Pick's result by employing the Euler characteristics $\chi(\mathcal{P})$ of the polygon and $\chi(\partial\mathcal{P})$ of its boundary.

Theorem 1.2 (cf. [15]). *Let $\mathcal{P}$ be an n-dimensional integral polygon.*

(i) *If $n = 2$, then* $2\mathrm{vol}(\mathcal{P}) = l(\mathcal{P}) + l(\mathrm{Int}(\mathcal{P})) - 2\chi(\mathcal{P}) + \chi(\partial\mathcal{P})$.

(ii) *If $n = 3$, then*

$$\begin{aligned} 2k(k^2-1)\mathrm{vol}(\mathcal{P}) &= l(k\mathcal{P}) + l(\mathrm{Int}(k\mathcal{P})) - k(l(\mathcal{P}) \\ &\quad + l(\mathrm{Int}(\mathcal{P}))) + (k-1)(2\chi(\mathcal{P}) - \chi(\partial\mathcal{P})), \\ l(\partial(k\mathcal{P})) &= k^2 l(\partial\mathcal{P}) - 2(k^2-1) \end{aligned}$$

for any positive integer k, where $k\mathcal{P}$ denotes the dilated polytope $\{k\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{P}\}$.

Moreover, Macdonald established the general formula to compute the volume of an integral polytope of arbitrary dimension in [10]. Concretely, for an n-dimensional integral polytope $\mathcal{P}$,

$$\begin{aligned} (n-1)n!\,\mathrm{vol}(\mathcal{P}) = \sum_{k=1}^{n-1}(-1)^{n-k-1}\binom{n-1}{k}(l(k\mathcal{P}) \\ + l(\mathrm{Int}(k\mathcal{P}))) + (-1)^{n-1}(2\chi(\mathcal{P}) - \chi(\partial\mathcal{P})) \end{aligned} \qquad (1)$$

and

$$n!\,\mathrm{vol}(\mathcal{P}) = \sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k} l(k\mathcal{P}) + (-1)^n\chi(\mathcal{P}). \qquad (2)$$

In addition, some other interesting formulae have been provided by Kołodziejczyk and Reay in [7–9]. One can in principle compute the volume of a polytope by using these results. In fact, however, it is not easy to carry it out. This difficulty comes from the intricate behavior of the number of lattice points in a dilated polytope $k\mathcal{P}$. Therefore, from the application standpoint, it is desirable to find a more simple formula even if not as strong as (1) and (2). Specifically, in this paper, we will give a lower bound for the volume of a three-dimensional integral convex polytope (Theorem 1.4). Although this result gives only an inequality, it is of wide application because of its simplicity. First, Corollary 1.6 provides a characterization of toric Fano threefold. Furthermore, we will apply this corollary to compute the sectional genus of a three-dimensional polarized toric variety and classify so-called Castelnuovo varieties in Section 3 (Theorem 3.8). Before describing our main result, we need to define the *smoothness* of a polytope. A polytope is said to be *convex* if it is a convex hull of a finite number of points in $\mathbb{R}^n$.

Definition 1.3. *Let $\mathcal{P}$ be an n-dimensional integral convex polytope in $\mathbb{R}^n$, and P be a vertex of $\mathcal{P}$. Define $\mathbb{R}_{\geqslant 0}(\mathcal{P}-P)=\{a(Q-P)\in\mathbb{R}^n \mid Q\in\mathcal{P}, a\geqslant 0\}$. If there exists a $\mathbb{Z}$-basis $\{\boldsymbol{m_1},\ldots,\boldsymbol{m_n}\}$ of $\mathbb{Z}^n$ such that*

$$\mathbb{R}_{\geqslant 0}(\mathcal{P}-P)=\mathbb{R}_{\geqslant 0}\boldsymbol{m_1}+\cdots+\mathbb{R}_{\geqslant 0}\boldsymbol{m_n},$$

the vertex P is said to be smooth. We say $\mathcal{P}$ is smooth if all its vertices are smooth. An m-dimensional ($m<n$) integral convex polytope $\mathcal{P}'$ in $\mathbb{R}^n$ is said to be smooth if it is smooth with respect to $\mathbb{R}^m$ which is the smallest affine subspace of $\mathbb{R}^n$ including $\mathcal{P}'$.

Theorem 1.4. *Let $\mathcal{P}$ be a three-dimensional smooth integral convex polytope having at least one interior lattice point. Then $3\mathrm{vol}(\mathcal{P})\geqslant l(\mathcal{P})+l(\mathrm{Int}(\mathcal{P}))-4$, and equality holds if and only if $\mathcal{P}$ is a polytope associated to the anti-canonical bundle on a toric Fano threefold.*

Toric Fano threefolds have been already classified into eighteen types in [2] and [17], independently. Namely, there exist eighteen polytopes (see, e.g., (6) in Section 2) whose volume achieves the lower bound in Theorem 1.4. We remark that the conditions of smoothness and $l(\mathrm{Int}(\mathcal{P}))\geqslant 1$ are essential for the above theorem. Indeed, if we remove these conditions, we can easily find counterexamples as follows.

Example 1.5. *For a subset S of $\mathbb{R}^3$, we denote by $\mathrm{Conv}(S)$ the convex hull of S.*

(i) *For a nonsmooth integral convex polytope*

$$\mathcal{P}_1=\mathrm{Conv}(\{(0,0,\pm 1),(2,1,\pm 1),(1,2,\pm 1),(1,1,2)\}),$$

we have $3\mathrm{vol}(\mathcal{P}_1)=21/2<l(\mathcal{P}_1)+l(\mathrm{Int}(\mathcal{P}_1))-4=11.$

(ii) *For a unit cube $\mathcal{P}_2$, we have* $3\mathrm{vol}(\mathcal{P}_2)=3<l(\mathcal{P}_2)+l(\mathrm{Int}(\mathcal{P}_2))-4=4.$

On the other hand, as is well known, the theory of polytopes is closely related to the toric geometry. For an ample line bundle L on an n-dimensional compact toric variety X, there exists an associated n-dimensional integral convex polytope $\square_L$ from which we can read off many invariants of L. A computation of the dimension of a cohomology group can be reduced to counting lattice points in the polytope. For example, we have $h^0(X, L) = l(\square_L)$ and $h^0(X, L + K_X) = l(\mathrm{Int}(\square_L))$. The degree of L can be computed as $L^n = n!\mathrm{vol}(\square_L)$. These relations, for example, tell us that Pick's formula coincides with the Riemann-Roch theorem on a surface. Indeed, if X is two-dimensional, the equalities $\chi(\mathcal{O}_X) = 1$ and $h^0(X, K_X) = h^1(X, K_X) = 0$ hold. Besides, the general theory of toric varieties gives that $h^1(X, L) = h^2(X, L) = 0$ if $|L|$ has no base points. Therefore, we can deform Theorem 1.1 as

$$h^0(X, L) = L^2 - h^0(X, L + K_X) + 2$$
$$\chi(\mathcal{O}_X(L)) = L^2 - h^0(X, K_L) + 2 = \frac{1}{2}L.(L - K_X) + \chi(\mathcal{O}_X).$$

In this manner, properties of polytopes and that of line bundles on a toric variety can be translated each other.

Using the terminology of the algebraic geometry, we can interpret Theorem 1.4 as a theorem about line bundles.

Corollary 1.6. *Let L be an ample line bundle on a three-dimensional smooth compact toric variety X. If $h^0(X, L + K_X) \geqslant 1$, then $L^3 \geqslant 2(h^0(X, L) + h^0(X, L + K_X) - 4)$ holds, and equality holds if and only if X is a toric Fano threefold and $L = -K_X$.*

2. Proof of the main theorem

First of all, we need to introduce several notations. We denote by $H_{f(x,y,z)}$ the plane in $\mathbb{R}^3$ defined by an equation $f(x, y, z) = 0$. For a lattice point P and a polygon F included in a plane $H_{f(x,y,z)}$, we denote by $h(F, P)$ the *lattice distance*. In concrete terms, we define $h(F, P) = |n|$, where n is an integer such that $H_{f(x,y,z)-n}$ passes through P.

Henceforth, the notation $\mathcal{P}$ always denotes a three-dimensional smooth integral convex polytope having interior lattice points. In addition, we denote by $V(\mathcal{P})$ the set of vertices of $\mathcal{P}$, and $E(\mathcal{P})$ the set of points on edges of $\mathcal{P}$. Note that $\mathrm{vol}(\mathcal{P})$, $l(\mathcal{P})$ and $l(\mathrm{Int}(\mathcal{P}))$ do not change even if we perform an affine linear transformation (i.e., a composition of parallel displacements and linear transformations by unimodular matrices).

Lemma 2.1. *If we place $\mathcal{P}$ in $\mathbb{R}^3_{z\geqslant 0}$ so that $\mathcal{P}$ has a face on H_z, then $\mathcal{P} \cap H_{z-1}$ is an integral convex polygon.*

Proof. We can assume that the origin O is a vertex of $\mathcal{P}$. Then O has three adjacent lattice points $(a_1, b_1, 0), (a_2, b_2, 0), (a_3, b_3, c_3) \in E(\mathcal{P})$. Since the vertex O is smooth, we have $c_3 = 1$ by the equality

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} = (a_1 b_2 - a_2 b_1) c_3 = \pm 1.$$

By similar arguments, we see that any edge of $\mathcal{P}$ which is extending from a vertex on H_z but not lying on H_z has a lattice point on H_{z-1}. The assertion follows from this fact. $\square$

Lemma 2.2. *There exists a plane $H_{f(x,y,z)}$ such that the section $\mathcal{T} = \mathcal{P} \cap H_{f(x,y,z)}$ is an integral convex polygon having a smooth vertex and at least one interior lattice point.*

Proof. Without loss of generality, we can assume that O, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are contained in $E(\mathcal{P})$. Put $P_1 = (0,1,1)$, $P_2 = (1,0,1)$ and $P_3 = (1,1,0)$. In the case where $P_1 \notin \mathcal{P}$, the integral convex polygon $\mathcal{P} \cap H_x$ must be a unit triangle by the smoothness of $\mathcal{P}$. Similarly, if P_2 (resp. P_3) is not contained in $\mathcal{P}$, then $\mathcal{P} \cap H_y$ (resp. $\mathcal{P} \cap H_z$) becomes a unit triangle. Since $l(\mathrm{Int}(\mathcal{P})) \geqslant 1$ and every vertex of $\mathcal{P}$ has just three edges, it is required that at least two points of P_1, P_2 and P_3 are contained in $\mathcal{P}$. Hence we can assume (after permuting the coordinates, if necessary) that $P_1, P_2 \in \mathcal{P}$. It is sufficient to consider the case where $(1,1,1) \notin \mathrm{Int}(\mathcal{P})$, because if $(1,1,1) \in \mathrm{Int}(\mathcal{P})$, we can finish the proof by putting $f(x,y,z) = z - 1$.

Let (x_0, y_0, z_0) be an interior lattice point of $\mathcal{P}$. Suppose that $P_3 \in \mathcal{P}$. Since $(1,1,1)$ is not contained in $\mathrm{Int}(\mathcal{P})$, there exist four integers α, β, γ and δ such that $\mathcal{P} \subset \{(x,y,z) \mid \alpha x + \beta y + \gamma z + \delta \geqslant 0\}$ and $\alpha + \beta + \gamma + \delta \leqslant 0$. By the conditions $P_1, P_2, P_3 \in \mathcal{P}$, we have

$$\beta + \gamma + \delta \geqslant 0, \ \alpha + \gamma + \delta \geqslant 0, \ \alpha + \beta + \delta \geqslant 0,$$

which imply that $\alpha, \beta, \gamma \leqslant 0$. Then we obtain a contradiction

$$\begin{aligned} &\alpha x_0 + \beta y_0 + \gamma z_0 + \delta > 0 \\ &\qquad\qquad \delta > -\alpha x_0 - \beta y_0 - \gamma z_0 \geqslant -\alpha - \beta - \gamma. \end{aligned}$$

Thus we see that P_3 is not contained in $\mathcal{P}$. In this case, the face $\mathcal{P} \cap H_z$ is a unit triangle, and the vertex $(1,0,0)$ has three adjacent lattice points O, $(0,1,0)$ and $(a,0,c)$ in $E(\mathcal{P})$. One can check $c = 1$ by the smoothness of the vertex $(1,0,0)$ in a similar way to that in the proof of Lemma 2.1. Thus $\mathcal{P}$ has a face included in the plane $H_{x+y+(1-a)z-1}$

containing three points $(1,0,0),(0,1,0)$ and $(a,0,1)$. Then we obtain $a \geqslant 2$ by the inequality

$$0 > x_0 + y_0 + (1-a)z_0 - 1 > (1-a)z_0.$$

It follows that $(2,0,1)$ is contained in $\mathcal{P}$. Similarly, one can verify that $(0,2,1)$ is contained in $\mathcal{P}$. By the assumption $(1,1,1) \notin \operatorname{Int}(\mathcal{P})$, the section $\mathcal{P} \cap H_{z-1}$ must be a triangle $\operatorname{Conv}(\{(0,0,1),\ (2,0,1),\ (0,2,1)\})$, which implies $a = 2$. Next we focus on the point $(1,1,2)$. Suppose that $(1,1,2)$ is not contained in $\operatorname{Int}(\mathcal{P})$. Then there exist four integers ε, ζ, η and θ such that $\mathcal{P} \subset \{(x,y,z) \mid \varepsilon x + \zeta y + \eta z + \theta \geqslant 0\}$ and $\varepsilon + \zeta + 2\eta + \theta \leqslant 0$. Since $(0,0,1),(2,0,1),(0,2,1) \in \mathcal{P}$, we have

$$\eta + \theta \geqslant 0,\ 2\varepsilon + \eta + \theta \geqslant 0,\ 2\zeta + \eta + \theta \geqslant 0,$$

which imply that $\varepsilon + \eta + \theta \geqslant 0$, $\zeta + \eta + \theta \geqslant 0$ and $\varepsilon + \zeta + \eta + \theta \geqslant 0$. It follows that $\eta \leqslant -\varepsilon - \zeta - \eta - \theta \leqslant 0$, $\varepsilon + \eta \leqslant -\zeta - \eta - \theta \leqslant 0$ and $\zeta + \eta \leqslant -\varepsilon - \eta - \theta \leqslant 0$. Since $\mathcal{P}$ has a face included in $H_{x+y-z-1}$, the inequality $x_0 + y_0 - z_0 - 1 < 0$ holds, which implies a contradiction

$$\begin{aligned} &\varepsilon x_0 + \zeta y_0 + \eta z_0 + \theta > 0 \\ &\qquad\qquad \theta > -\varepsilon x_0 - \zeta y_0 - \eta z_0 \geqslant -(\varepsilon+\eta)x_0 - (\zeta+\eta)y_0 \\ &\qquad\qquad\quad \geqslant -\varepsilon - \zeta - 2\eta. \end{aligned}$$

Therefore, we can conclude that $(1,1,2)$ is contained in $\operatorname{Int}(\mathcal{P})$, and $\mathcal{P} \cap H_{x-1}$ is the desired section. $\square$

Let $\mathcal{Q}$ be a three-dimensional integral convex polytope, and Q be a vertex of $\mathcal{Q}$. We define

$$\mu(\mathcal{Q}) = \operatorname{vol}(\mathcal{Q}) - \frac{l(\mathcal{Q}) + l(\operatorname{Int}(\mathcal{Q})) - 4}{3}$$

and $\mathcal{Q}^Q = \operatorname{Conv}((\mathcal{Q} \cap \mathbb{Z}^3) \setminus \{Q\})$, and denote by $\overline{\mathcal{Q}}^Q$ the set of points in $\mathcal{Q}^Q$ which are visible from Q, that is,

$$\overline{\mathcal{Q}}^Q = \{P \in \mathcal{Q}^Q \mid (\text{the segment } PQ) \cap \mathcal{Q}^Q = \{P\}\}.$$

Although $\overline{\mathcal{Q}}^Q$ is not a convex polytope but a set consisting of some faces of $\mathcal{Q}^Q$, we formally define $V(\overline{\mathcal{Q}}^Q) = V(\mathcal{Q}^Q) \cap \overline{\mathcal{Q}}^Q$, $\partial\overline{\mathcal{Q}}^Q = \partial\mathcal{Q} \cap \overline{\mathcal{Q}}^Q$ and $\operatorname{Int}(\overline{\mathcal{Q}}^Q) = \overline{\mathcal{Q}}^Q \setminus \partial\overline{\mathcal{Q}}^Q$. The following proposition tells us the precise difference between $\mathcal{Q}$ and $\mathcal{Q}^Q$, which is a central tool in the induction step in the proof of Theorem 1.4.

Proposition 2.3. *Let Q be a vertex of a three-dimensional integral convex polytope $\mathcal{Q}$, and $F_1, \dots, F_k$ be faces of $\mathcal{Q}^Q$ such that $\overline{\mathcal{Q}}^Q = \bigcup_{j=1}^k F_j$. If we put $a_j = l(F_j)$ and $b_j = l(\mathrm{Int}(F_j))$, then*

$$\mu(\mathcal{Q}) = \mu(\mathcal{Q}^Q) + \frac{1}{6} l(\partial \overline{\mathcal{Q}}^Q) - \frac{2}{3} + \frac{1}{6} \sum_{j=1}^k (h(F_j, Q) - 1)(a_j + b_j - 2).$$

Proof. For simplicity, we put $h_j = h(F_j, Q)$. Since $\mathrm{vol}(F_j) = (a_j + b_j - 2)/2$ by Theorem 1.1, we have

$$\begin{aligned} \mu(\mathcal{Q}) - \mu(\mathcal{Q}^Q) &= \frac{1}{6} \sum_{j=1}^k h_j (a_j + b_j - 2) - \frac{1}{3} (l(\mathrm{Int}(\mathcal{Q})) - l(\mathrm{Int}(\mathcal{Q}^Q)) + 1) \\ &= \frac{1}{6} \sum_{j=1}^k (h_j - 1)(a_j + b_j - 2) + \frac{1}{6} \sum_{j=1}^k (a_j + b_j) \\ &\quad - \frac{1}{3} (l(\mathrm{Int}(\overline{\mathcal{Q}}^Q)) + k + 1). \end{aligned}$$

To estimate the right-hand value, let us compute $\sum_{j=1}^k (a_j + b_j)$, which means counting lattice points in $\overline{\mathcal{Q}}^Q$ (with several duplications). Indeed, we can write

$$\sum_{j=1}^k (a_j + b_j) = \sum_{P \in \overline{\mathcal{Q}}^Q \cap \mathbb{Z}^3} c(P), \tag{3}$$

where $c(P)$ denotes the number of times P is counted in the left-hand side of (3). It is clear that $c(P) = 1$ for $P \in \partial \overline{\mathcal{Q}}^Q \setminus V(\overline{\mathcal{Q}}^Q)$ since there exists a unique face containing P in this case. Let us check $c(P) = 2$ for a point P in $\mathrm{Int}(\overline{\mathcal{Q}}^Q) \setminus V(\overline{\mathcal{Q}}^Q)$. This is clear if P is not on any edge of $\mathcal{Q}^Q$. While if P is on some edge of $\mathcal{Q}^Q$, then there exist two faces of $\overline{\mathcal{Q}}^Q$ containing P. Hence, in this case, P is counted two times in the left-hand side of (3). We next consider points in $V(\overline{\mathcal{Q}}^Q)$. Let P be a point in $V(\overline{\mathcal{Q}}^Q) \cap \mathrm{Int}(\overline{\mathcal{Q}}^Q)$ having s edges. Since s faces contain P, P is counted s times in the left-hand side of (3), that is, $c(P) = s$. Meanwhile, if P is contained in $V(\overline{\mathcal{Q}}^Q) \cap \partial \overline{\mathcal{Q}}^Q$ and t edges of $\overline{\mathcal{Q}}^Q$ extend from P, there exist $t - 1$ faces containing P. It follows that $c(P) = t - 1$. Consequently, we obtain

$$\sum_{j=1}^k (a_j + b_j) = 2 l(\mathrm{Int}(\overline{\mathcal{Q}}^Q)) + l(\partial \overline{\mathcal{Q}}^Q \setminus V(\overline{\mathcal{Q}}^Q)) + \sum_{s=3}^{s_0} (s-2) m_s + \sum_{t=2}^{t_0} (t-1) n_t,$$

where we define

$$m_s = \sharp\{P \in V(\overline{\mathcal{Q}}^Q) \cap \mathrm{Int}(\overline{\mathcal{Q}}^Q) \mid \text{there exist } s \text{ edges of } \overline{\mathcal{Q}}^Q \text{ extending from } P\},$$
$$n_t = \sharp\{P \in V(\overline{\mathcal{Q}}^Q) \cap \partial\overline{\mathcal{Q}}^Q \mid \text{there exist } t \text{ edges of } \overline{\mathcal{Q}}^Q \text{ extending from } P\}.$$

Next, to compute the value of k, we take a lattice point $P_0 \notin \mathrm{Conv}(\overline{\mathcal{Q}}^Q)$ such that an integral polytope $\mathcal{Q}_0 = \mathrm{Conv}(\overline{\mathcal{Q}}^Q \cup \{P_0\})$ satisfies $V(\mathcal{Q}_0) = V(\overline{\mathcal{Q}}^Q) \cup \{P_0\}$ (see Fig. 1).

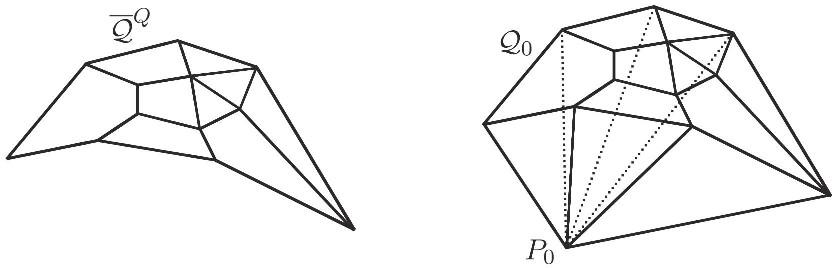

FIGURE 1.

Then the number of vertices, edges and faces of $\mathcal{Q}_0$ are $\sum_{s=3}^{s_0} m_s + \sum_{t=2}^{t_0} n_t + 1$, $\left(\sum_{s=3}^{s_0} sm_s + \sum_{t=2}^{t_0} tn_t\right)/2 + \sum_{t=2}^{t_0} n_t$ and $k + \sum_{t=2}^{t_0} n_t$, respectively. Hence, by Euler's polyhedron formula, we have

$$\sum_{s=3}^{s_0} m_s + \sum_{t=2}^{t_0} n_t + 1 - \frac{\sum_{s=3}^{s_0} sm_s + \sum_{t=2}^{t_0} tn_t}{2} - \sum_{t=2}^{t_0} n_t + k + \sum_{t=2}^{t_0} n_t = 2,$$

which implies that $k = \left(\sum_{s=3}^{s_0}(s-2)m_s + \sum_{t=2}^{t_0}(t-2)n_t\right)/2 + 1$. As a consequence,

$$\begin{aligned}\mu(\mathcal{Q}) - \mu(\mathcal{Q}^Q) &= \frac{1}{6}\left(l(\partial\overline{\mathcal{Q}}^Q \setminus V(\overline{\mathcal{Q}}^Q)) + \sum_{t=2}^{t_0} n_t\right) - \frac{2}{3} \\ &\quad + \frac{1}{6}\sum_{j=1}^{k}(h_j - 1)(a_j + b_j - 2) \\ &= \frac{1}{6} l(\partial\overline{\mathcal{Q}}^Q) - \frac{2}{3} + \frac{1}{6}\sum_{j=1}^{k}(h_j - 1)(a_j + b_j - 2). \qquad \square\end{aligned}$$

By noting that $l(\partial\overline{\mathcal{Q}}^Q) \geqslant 3$, $a_j \geqslant 3$ and $b_j \geqslant 0$, we obtain the following corollary.

Corollary 2.4. *Let $\mathcal{Q}$, Q and F_j be as in Proposition 2.3. If there exists a face F_{j_0} of $\mathcal{Q}^Q$ such that $h(F_{j_0}, Q) \geqslant 2$, then $\mu(\mathcal{Q}) \geqslant \mu(\mathcal{Q}^Q)$.*

Let us show the main result. Since the proof is relatively long, we divide it into two parts.

Proof of the inequality in Theorem 1.4. Let $\mathcal{T}$ be a section of $\mathcal{P}$ as in Lemma 2.2. We take a lattice point $P_0 \in V(\mathcal{P}) \setminus \mathcal{T}$ and put $\mathcal{P}_1 = \mathcal{P}^{P_0}$. By carrying out such operation repeatedly, we construct a sequence of integral convex polytopes

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots, \tag{4}$$

where $P_i \in V(\mathcal{P}_i) \setminus \mathcal{T}$ and $\mathcal{P}_{i+1} = \mathcal{P}_i^{P_i}$. It is sufficient to show that, by going through a suitable process, we can find $\mathcal{P}_n$ such that $\mu(\mathcal{P}_0) \geqslant \mu(\mathcal{P}_n)$, $l(\mathcal{P}_n) = l(\mathcal{T}) + 2$ and $\mathcal{T}$ is not a face of $\mathcal{P}_n$ (see Fig. 2).

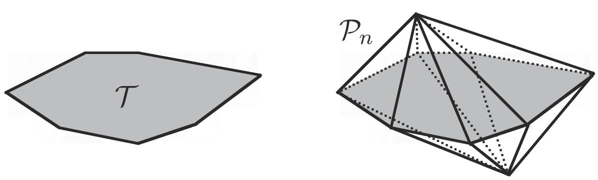

FIGURE 2.

Indeed, for such a $\mathcal{P}_n$, it follows from Theorem 1.1 that

$$\begin{aligned}\mu(\mathcal{P}_n) &= \mathrm{vol}(\mathcal{P}_n) - \frac{l(\mathcal{P}_n) + l(\mathrm{Int}(\mathcal{P}_n)) - 4}{3}\\ &\geqslant \frac{2}{3}\mathrm{vol}(\mathcal{T}) - \frac{l(\mathcal{T}) + l(\mathrm{Int}(\mathcal{T})) - 2}{3}\\ &= \frac{2}{3} \cdot \frac{l(\mathcal{T}) + l(\mathrm{Int}(\mathcal{T})) - 2}{2} - \frac{l(\mathcal{T}) + l(\mathrm{Int}(\mathcal{T})) - 2}{3} = 0.\end{aligned}$$

To verify the existence of $\mathcal{P}_n$, it is sufficient to prove the following claim:

Claim A. *Let i be a nonnegative integer. If $l(\mathcal{P}_i) \geqslant l(\mathcal{T}) + 3$ and $\mathcal{T}$ is not a face of $\mathcal{P}_i$, then we can construct $\mathcal{P}_{i_0}$ $(i_0 > i)$ such that $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i_0})$ and $\mathcal{T}$ is not a face of $\mathcal{P}_{i_0}$.*

We define $A = \{Q \in V(\mathcal{P}_i) \setminus \mathcal{T} \mid \mathcal{T}$ is not a face of $\mathcal{P}_i^Q\}$. If there exists a point $Q \in A$ such that $l(\partial\overline{\mathcal{P}_i}^Q) \geqslant 4$, by putting $P_i = Q$, we obtain the inequality $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1})$ by Proposition 2.3. Hence Claim A is true in this case. We thus assume that

$$l(\partial\overline{\mathcal{P}_i}^Q) = 3 \text{ for any } Q \in A. \tag{5}$$

We take a point $Q_0 \in A$. Note that the inequality $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_i^{Q_0}) - 1/6$ follows from Proposition 2.3. We denote by Q_1, Q_2 and Q_3 the vertices of a triangle $\partial\overline{\mathcal{P}_i}^{Q_0}$, and put $\boldsymbol{v_j} = Q_j - Q_0$ for $j = 1, 2, 3$. We define $\varepsilon_j = \max\{\varepsilon \in \mathbb{N} \mid Q_0 + \varepsilon\boldsymbol{v_j} \in \mathcal{P}_i\}$ and $Q'_j = Q_0 + \varepsilon_j\boldsymbol{v_j}$ for $j = 1, 2, 3$.

(i) We first consider the case where $l(\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\})) \geqslant 5$. We put $t = \sharp(\{Q_1, Q_2, Q_3\} \cap \mathcal{T})$. If $t = 3$, then $\mathcal{T}$ is a triangle $\mathrm{Conv}(\{Q_1, Q_2, Q_3\})$ whose border has no lattice points except for three vertices. This contradicts the property of $\mathcal{T}$ of having a smooth vertex and interior lattice points. Hence we have $t \leqslant 2$.

(i)–(a) If $t \leqslant 1$, we can assume $Q_1, Q_2 \notin \mathcal{T}$ and $\varepsilon_1 \leqslant \varepsilon_2$. In the case where $\varepsilon_1 \geqslant 2$, we put $P_i = Q_0$, $P_{i+1} = Q_1$ and $P_{i+2} = Q_2$. Then $\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}$ contains $Q_1 + \boldsymbol{v_1}$, Q_2, Q_3 and at least one lattice point in $\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\}) \setminus \{Q_0, \ldots, Q_3\}$. It follows from Proposition 2.3 that $\mu(\mathcal{P}_{i+1}) \geqslant \mu(\mathcal{P}_{i+2})$. Similarly, since $\partial\overline{\mathcal{P}_{i+2}}^{P_{i+2}}$ contains $Q_1 + \boldsymbol{v_1}$, $Q_1 + \boldsymbol{v_2}$, $Q_2 + \boldsymbol{v_2}$, Q_3 and at least one lattice point in $\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\}) \setminus \{Q_0, \ldots, Q_3\}$, we have $\mu(\mathcal{P}_{i+2}) \geqslant \mu(\mathcal{P}_{i+3}) + 1/6$. In sum, Claim A is true by

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+2}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+3}).$$

We next consider the case where $\varepsilon_1 = 1$. Since $Q_1 \in A$, we have $l(\partial\overline{\mathcal{P}_i}^{Q_1}) = 3$ by (5), and more concretely, $Q'_1(= Q_1)$ has three adjacent lattice points $Q'_1 - \boldsymbol{v_1}(= Q_0)$, $Q_0 + \alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}$ and $Q_0 + \gamma\boldsymbol{v_1} + \delta\boldsymbol{v_3}$ in $E(\mathcal{P}_i)$, where $\alpha, \gamma \geqslant 0$ and $\beta, \delta \geqslant 1$. We denote by F a face of $\overline{\mathcal{P}_i}^{Q_1}$ containing two points Q_0 and $Q_0 + \alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}$. By an easy computation, we obtain $h(F, Q_1) \geqslant \beta$. If $\beta \geqslant 2$, we can finish the proof by putting $P_i = Q_1$. Indeed, in this case, the inequality $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1})$ holds by Corollary 2.4. Similar arguments can be carried out for the case where $\delta \geqslant 2$. Let us consider the case where $\beta = \delta = 1$. We can assume $\alpha \geqslant \gamma$ without loss of generality, and have $\alpha + \gamma \geqslant 1$ by the existence of $\mathcal{T}$. If $\alpha + \gamma \geqslant 2$, we put $P_i = Q_0$ and $P_{i+1} = Q_1$. Then, since $\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}$

contains $Q_2 + k\boldsymbol{v_1}$ $(k = 0, \ldots, \alpha)$, $Q_3 + k\boldsymbol{v_1}$ $(k = 0, \ldots, \gamma)$ and at least one lattice point in $\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\}) \setminus \{Q_0, \ldots, Q_3\}$, we obtain $\mu(\mathcal{P}_{i+1}) \geqslant \mu(\mathcal{P}_{i+2}) + (\alpha + \gamma - 1)/6$. Hence

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+2}) + \frac{\alpha + \gamma - 2}{6} \geqslant \mu(\mathcal{P}_{i+2}).$$

Let us consider the remaining case where $\alpha = 1$ and $\gamma = 0$. Note that $\mathcal{P}_i$ has a face containing Q_1, Q_3 and $Q_1 + \boldsymbol{v_2}$ (see Fig. 3).

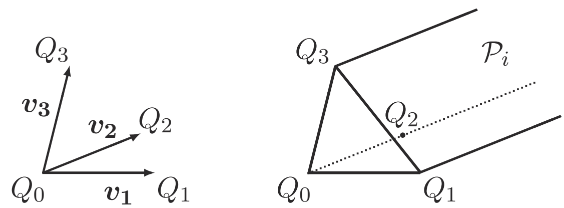

Figure 3.

We define $\zeta_j = \max\{\zeta \in \mathbb{Z}_{\geqslant 0} \mid Q_j + \zeta\boldsymbol{v_2} \in \mathcal{P}_i\}$ for $j = 1, 2, 3$. Considering the properties of $\mathcal{T}$, at least one of $Q_1 + \zeta_1\boldsymbol{v_2}$, $Q_2 + \zeta_2\boldsymbol{v_2}$, and $Q_3 + \zeta_3\boldsymbol{v_2}$ is not on $\mathcal{T}$. Then, by (5), such a point has just three adjacent lattice points in $E(\mathcal{P}_i)$. This implies that $V(\mathcal{P}_i) = \{Q_0, Q_1, Q_3, Q_1 + \zeta_1\boldsymbol{v_2}, Q_2 + \zeta_2\boldsymbol{v_2}, Q_3 + \zeta_3\boldsymbol{v_2}\}$, which contradicts the existence of $\mathcal{T}$.

(i)–(b) If $t = 2$, we can assume that $Q_1 \notin \mathcal{T}$ and $Q_2, Q_3 \in \mathcal{T}$. By the properties of $\mathcal{T}$, we see that $Q_0 + \varepsilon\boldsymbol{v_1}$ is not on $\mathcal{T}$ for $0 \leqslant \varepsilon \leqslant \varepsilon_1$. As we saw in the case (i)-(a), Q_1' has three adjacent lattice points $Q_1' - \boldsymbol{v_1}$, $Q_0 + \alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}$ and $Q_0 + \gamma\boldsymbol{v_1} + \delta\boldsymbol{v_3}$ in $E(\mathcal{P}_i)$, and the proof is finished by putting $P_i = Q_1'$ in the case where $\beta \geqslant 2$ or $\delta \geqslant 2$. We assume $\beta = \delta = 1$. Since $\mathcal{T}$ has interior lattice points, we see that at least one of $Q_0 + \alpha\boldsymbol{v_1} + \boldsymbol{v_2}$ and $Q_0 + \gamma\boldsymbol{v_1} + \boldsymbol{v_3}$ is not on $\mathcal{T}$. Then this case is equivalent to the case (i)-(a) by regarding Q_1' as Q_0.

(ii) We next consider the case where $l(\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\})) = 4$. If $h(\mathrm{Conv}(\{Q_1,\ Q_2, Q_3\}), Q_0) \geqslant 2$, we can finish the proof by putting $P_i = Q_0$. Hence we can assume that Q_0 is a smooth vertex, that is, $Q_0 = O$, $Q_1 = (1, 0, 0)$, $Q_2 = (0, 1, 0)$ and $Q_3 = (0, 0, 1)$. Moreover, we assume that every vertex in A is smooth in order to avoid the duplication with the case (i). We denote by L_j the segment Q_0Q_j' $(j = 1, 2, 3)$, and put $u = \sharp\{L_j \mid L_j \cap \mathcal{T} \neq \varnothing,\ j = 1, 2, 3\}$.

(ii)–(a) In the case where $u \leqslant 1$, we can assume that $L_1 \cap \mathcal{T} = \varnothing$ and $L_2 \cap \mathcal{T} = \varnothing$. Since Q_1' is smooth, it has three adjacent lattice points $(\varepsilon_1 - 1, 0, 0)$, $(\alpha, 1, 0)$ and $(\gamma, 0, 1)$ in $E(\mathcal{P}_i)$. If we put $P_{i+\varepsilon} = (\varepsilon, 0, 0)$

for $\varepsilon = 0, \ldots, \varepsilon_1$, since $\partial\overline{\mathcal{P}_{i+\varepsilon_1}}^{P_{i+\varepsilon_1}}$ contains lattice points $(k, 1, 0)$ with $k = 0, \ldots, \alpha$ and $(l, 0, 1)$ with $l = 0, \ldots, \gamma$, it follows that

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \cdots \geqslant \mu(\mathcal{P}_{i+\varepsilon_1}) - \frac{\varepsilon_1}{6} \geqslant \mu(\mathcal{P}_{i+\varepsilon_1+1}) + \frac{\alpha + \gamma - \varepsilon_1 - 2}{6}.$$

If $\alpha + \gamma \geqslant \varepsilon_1 + 2$, the proof is finished. We next consider the vertex Q_2' and its three adjacent lattice points $(0, \varepsilon_2 - 1, 0)$, $(1, \beta, 0)$ and $(0, \delta, 1)$ in $E(\mathcal{P}_i)$. Similarly to the case of Q_1', we can finish the proof in the case where $\beta + \delta \geqslant \varepsilon_2 + 2$. Hence we assume that $\alpha + \gamma \leqslant \varepsilon_1 + 1$ and $\beta + \delta \leqslant \varepsilon_2 + 1$. Let (x_0, y_0, z_0) be a lattice point in $\mathrm{Int}(\mathcal{P}_i)$. Since $\mathcal{P}_i$ has a face containing three points $(\varepsilon_1, 0, 0)$, $(\alpha, 1, 0)$ and $(\gamma, 0, 1)$ (resp. $(0, \varepsilon_2, 0)$, $(1, \beta, 0)$ and $(0, \delta, 1)$), we have

$$x_0 + (\varepsilon_1 - \alpha)y_0 + (\varepsilon_1 - \gamma)z_0 - \varepsilon_1 < 0$$
$$(\text{resp. } (\varepsilon_2 - \beta)x_0 + y_0 + (\varepsilon_2 - \delta)z_0 - \varepsilon_2 < 0).$$

By noting $x_0, y_0, z_0 \geqslant 1$ and $\alpha, \gamma, \beta, \delta \geqslant 0$, we obtain $(\alpha, \gamma) = (\varepsilon_1 + 1, 0)$ or $(0, \varepsilon_1 + 1)$ and $(\beta, \delta) = (\varepsilon_2 + 1, 0)$ or $(0, \varepsilon_2 + 1)$. Clearly, $\gamma = \delta = 0$ gives a contradiction. Besides, considering the shape of $\mathcal{P}_i$, if either α or β is zero, then the other one also must be zero and $\varepsilon_1 = \varepsilon_2 = 1$. In sum, we have $(\alpha, \gamma) = (\beta, \delta) = (0, 2)$. Then, putting $P_i = Q_0$ and $P_{i+j} = Q_j$ for $j = 1, 2$, we have

$$l(\partial\overline{\mathcal{P}_i}^{P_i}) = l(\mathrm{Conv}(\{(1,0,0), (0,1,0), (0,0,1)\})) = 3,$$
$$l(\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}) = l(\mathrm{Conv}(\{(2,0,1), (0,1,0), (0,0,1)\})) = 4,$$
$$l(\partial\overline{\mathcal{P}_{i+2}}^{P_{i+2}}) = l(\mathrm{Conv}(\{(2,0,1), (0,2,1), (0,0,1)\})) = 6.$$

It follows from Proposition 2.3 that $\mu(\mathcal{P}_i) = \mu(\mathcal{P}_{i+1}) - 1/6 = \mu(\mathcal{P}_{i+2}) - 1/6 = \mu(\mathcal{P}_{i+3}) + 1/6$.

(ii)–(b) If $u = 2$, we can assume that $L_1 \cap \mathcal{T} = \varnothing$, $L_2 \cap \mathcal{T} \neq \varnothing$ and $L_3 \cap \mathcal{T} \neq \varnothing$. As we saw in the case (ii)-(a), it is sufficient to consider the case where Q_1' has three adjacent lattice points $(\varepsilon_1 - 1, 0, 0)$, $(\alpha, 1, 0)$ and $(\gamma, 0, 1)$ with $(\alpha, \gamma) = (\varepsilon_1 + 1, 0)$ or $(0, \varepsilon_1 + 1)$. Here we consider only the former case $(\alpha, \gamma) = (\varepsilon_1 + 1, 0)$. The latter case can be shown in a similar way. First, we remark that $\varepsilon_3 = 1$ holds by the condition $\gamma = 0$. This means that $(0, 0, 1)$ is on $\mathcal{T}$. Denote by L_4 the line passing through Q_1' and $(\alpha, 1, 0)$. Since $\mathcal{T}$ has interior lattice points, L_4 does not contain a lattice point on $\mathcal{T}$. Then, by regarding Q_1' as Q_0, this case can be reduced to the case (ii)-(a).

(ii)–(c) Assume $u = 3$. In this case, we see that $Q_j \notin \mathcal{T}$ for $j = 1, 2, 3$ since $\mathcal{T}$ has interior lattice points. It follows that $(2,0,0), (0,2,0), (0,0,2) \in \mathcal{P}_i$. If we put $P_i = Q_0$ and $P_{i+j} = Q_j$ for $j = 1, 2, 3$, then

$$\begin{aligned} l(\partial\overline{\mathcal{P}_i}^{P_i}) &= l(\mathrm{Conv}(\{(1,0,0),(0,1,0),(0,0,1)\})) = 3, \\ l(\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}) &= l(\mathrm{Conv}(\{(2,0,0),(0,1,0),(0,0,1)\})) = 3, \\ l(\partial\overline{\mathcal{P}_{i+2}}^{P_{i+2}}) &= l(\mathrm{Conv}(\{(2,0,0),(0,2,0),(0,0,1)\})) = 4, \\ l(\partial\overline{\mathcal{P}_{i+3}}^{P_{i+3}}) &= l(\mathrm{Conv}(\{(2,0,0),(0,2,0),(0,0,2)\})) = 6. \end{aligned}$$

Hence we have

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+2}) - \frac{2}{3} \geqslant \mu(\mathcal{P}_{i+3}) - \frac{2}{3} \geqslant \mu(\mathcal{P}_{i+4}).$$

Since $\mathcal{T}$ has interior lattice points, at least one of $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$ is not on $\mathcal{T}$, that is, $\mathcal{T}$ is not a face of $\mathcal{P}_{i+4}$. $\square$

In order to show the latter part of Theorem 1.4, we require results in the theory of toric varieties and the classification theory of polarized varieties. Hence, in the proof below, we take in advance the contents in the next section although not in the proper order. See Section 3 for precise definitions and notations.

Proof of the equivalency in Theorem 1.4. The classification of toric Fano threefolds has been completed, and they are classified into eighteen types (cf. [2, 17]). For each type X of toric Fano threefolds, the polytope $\square_{-K_X}$ associated to the anti-canonical bundle has just one interior lattice point. Moreover, we can obtain

$$\mathrm{vol}(\square_{-K_X}) = \frac{l(\square_{-K_X})}{3} - 1$$

by steady calculations. We list several examples of them for readers' exercise.

$$\begin{array}{rl} X & \square_{-K_X} \\ \mathbb{P}^3 & \text{the fourth dilation of a unit three-simplex} \\ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \text{the twice dilation of a unit cube} \\ \mathbb{P}^2 \times \mathbb{P}^1 & \mathrm{Conv}(\{(0,0,\pm 1),(3,0,\pm 1),(0,3,\pm 1)\}) \\ \Sigma_1 \times \mathbb{P}^1 & \mathrm{Conv}(\{(0,0,\pm 1),(3,0,\pm 1),(1,2,\pm 1),(0,2,\pm 1)\}) \\ \vdots & \quad\vdots \end{array} \tag{6}$$

Let us show the sufficiency. We first consider the case where $l(\mathrm{Int}(\mathcal{P}))$=1 and $\mathrm{vol}(\mathcal{P}) = l(\mathcal{P})/3 - 1$. Let L be an ample line bundle on a three-dimensional toric variety X whose associated polytope $\square_L$ coincides with $\mathcal{P}$. Then our assumptions are equivalent to two equalities $h^0(X, L + K_X) = 1$ and $L^3 = 2h^0(X, L) - 6$. By noting Lemma 3.7, the sectional genus and the Δ-genus of the polarized variety (X, L) is $g(X, L) = h^0(X, L) - 2$ and $\Delta(X, L) = h^0(X, L) - 3$, respectively. On the other hand, since $\lfloor (L^3 - 1)/(L^3 - \Delta(X, L) - 1) \rfloor = 2$, the above sectional genus coincides with the upper bound in Theorem 3.3. Namely, (X, L) is a Castelnuovo variety in this case. Since (X, L) is a Mukai variety by the remark after Theorem 3.3, we can conclude that X is a Fano variety and $L = -K_X$.

In the remaining part, we prove the inequality $\mathrm{vol}(\mathcal{P}) > (l(\mathcal{P}) + l(\mathrm{Int}(\mathcal{P})) - 4)/3$ under the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$. Recall the sequence of integral convex polytopes (4) and Claim A in the proof of Theorem 1.4. Then it is sufficient to show that, by going through a suitable process, we can construct $\mathcal{P}_{i_0}$ such that $\mu(\mathcal{P}) > \mu(\mathcal{P}_{i_0})$ and $\mathcal{T}$ is not a face of $\mathcal{P}_{i_0}$. We place $\mathcal{P}$ in $\mathbb{R}^3$ so that four points O, $Q_1 = (1, 0, 0)$, $Q_2 = (0, 1, 0)$ and $Q_3 = (0, 0, 1)$ are contained in $E(\mathcal{P})$, and define

$$\begin{aligned}
\varepsilon_1 &= \max\{\varepsilon \in \mathbb{N} \mid (\varepsilon, 0, 0) \in \mathcal{P}\},\\
\varepsilon_2 &= \max\{\varepsilon \in \mathbb{N} \mid (0, \varepsilon, 0) \in \mathcal{P}\},\\
\varepsilon_3 &= \max\{\varepsilon \in \mathbb{N} \mid (0, 0, \varepsilon) \in \mathcal{P}\},
\end{aligned}$$

$Q'_1 = (\varepsilon_1, 0, 0)$, $Q'_2 = (0, \varepsilon_2, 0)$ and $Q'_3 = (0, 0, \varepsilon_3)$. Then, by the smoothness of $\mathcal{P}$, we see that Q'_1 has three adjacent lattice points $(\varepsilon_1 - 1, 0, 0)$, $(\alpha_1, 1, 0)$ and $(\alpha_2, 0, 1)$ in $E(\mathcal{P})$. Similarly, Q'_2 (resp. Q'_3) has three adjacent lattice points $(0, \varepsilon_2 - 1, 0)$, $(1, \beta_1, 0)$ and $(0, \beta_2, 1)$ (resp. $(0, 0, \varepsilon_3 - 1)$, $(1, 0, \gamma_1)$ and $(0, 1, \gamma_2)$) in $E(\mathcal{P})$. In the case where $\alpha_1 + \alpha_2 \geqslant \varepsilon_1 + 3$, we can take $\mathcal{T}$ so that it does not contain points on the x-axis by using a similar method to that in the proof of Lemma 2.2. If we put $P_i = (i, 0, 0)$ for $i = 0, \ldots, \varepsilon_1$, then $\partial\overline{\mathcal{P}_i}^{P_i}$ is a triangle with vertices $(i + 1, 0, 0)$, Q_2 and Q_3 for $i = 0, \ldots, \varepsilon_1 - 1$, and $\partial\overline{\mathcal{P}_{\varepsilon_1}}^{P_{\varepsilon_1}}$ is a trapezoid $\mathrm{Conv}(\{Q_2, Q_3, (\alpha_1, 1, 0), (\alpha_2, 0, 1)\})$. We thus obtain

$$\begin{aligned}
\mu(\mathcal{P}_0) &\geqslant \mu(\mathcal{P}_1) - \frac{1}{6} \geqslant \cdots \geqslant \mu(\mathcal{P}_{\varepsilon_1}) - \frac{\varepsilon_1}{6}\\
&\geqslant \mu(\mathcal{P}_{\varepsilon_1+1}) + \frac{\alpha_1 + \alpha_2 - \varepsilon_1 - 2}{6} > \mu(\mathcal{P}_{\varepsilon_1+1})
\end{aligned}$$

by Proposition 2.3. Also in the cases where $\beta_1 + \beta_2 \geqslant \varepsilon_2 + 3$ or $\gamma_1 + \gamma_2 \geqslant \varepsilon_3 + 3$, we can finish the proof in essentially the same way.

We assume henceforth that $\alpha_1 + \alpha_2 \leqslant \varepsilon_1 + 2$, $\beta_1 + \beta_2 \leqslant \varepsilon_2 + 2$ and $\gamma_1 + \gamma_2 \leqslant \varepsilon_3 + 2$. Moreover, without loss of generality, we can assume that OQ'_1 is the shortest edge of $\mathcal{P}$ and $\alpha_1 \geqslant \alpha_2$. Since $\mathcal{P}$ has a face containing three points Q'_1, $(\alpha_1, 1, 0)$ and $(\alpha_2, 0, 1)$, the inclusion

$$\begin{aligned}\mathrm{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, x + (\varepsilon_1 - \alpha_1)(y-1) \\ + (\varepsilon_1 - \alpha_2)(z-1) + \varepsilon_1 - \alpha_1 - \alpha_2 + 1 \leqslant 0\}\end{aligned} \tag{7}$$

is derived from the inequality $x + (\varepsilon_1 - \alpha_1)y + (\varepsilon_1 - \alpha_2)z - \varepsilon_1 < 0$. Similarly, by considering the vertices Q'_2 and Q'_3, we obtain

$$\begin{aligned}\mathrm{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, (\varepsilon_2 - \beta_1)(x-1) + y \\ + (\varepsilon_2 - \beta_2)(z-1) + \varepsilon_2 - \beta_1 - \beta_2 + 1 \leqslant 0\},\end{aligned} \tag{8}$$

$$\begin{aligned}\mathrm{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, (\varepsilon_3 - \gamma_1)(x-1) + (\varepsilon_3 - \gamma_2)(y-1) \\ + z + \varepsilon_3 - \gamma_1 - \gamma_2 + 1 \leqslant 0\}.\end{aligned} \tag{9}$$

(i) Assume that $\alpha_1 \leqslant \varepsilon_1$, and let (x_0, y_0, z_0) be a lattice point in $\mathrm{Int}(\mathcal{P})$. By noting $\alpha_1 \geqslant \alpha_2$ and $\alpha_1 + \alpha_2 \leqslant \varepsilon_1 + 2$, we have $x_0 = 1$ and $\alpha_1 + \alpha_2 = \varepsilon_1 + 2$ by (7). If $\alpha_2 < \varepsilon_1$, we see that $z_0 = 1$ by (7) and $y_0 = 1$ by (8), which contradicts the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$. We thus have $\alpha_2 \geqslant \varepsilon_1$, and similarly $\beta_2 > \varepsilon_2$ and $\gamma_2 > \varepsilon_3$. Since $\varepsilon_1 = \alpha_1 = \alpha_2 = 2$ in this case, $\varepsilon_2, \varepsilon_3 \leqslant 2$ follows from the shortestness of the edge OQ'_1. Moreover, it follows from $(2,1,0), (2,0,1) \in \mathcal{P}$ that $\beta_1 \geqslant \varepsilon_2$ and $\gamma_1 \geqslant \varepsilon_3$. As a consequence, we have $(\varepsilon_2, \beta_1, \beta_2) = (\varepsilon_3, \gamma_1, \gamma_2) = (1,1,2)$. By the shortestness of the edge OQ'_1 again, the faces $\mathcal{P} \cap H_x$ and $\mathcal{P} \cap H_{x-2}$ are one of the four types of polygons as in Fig. 4.

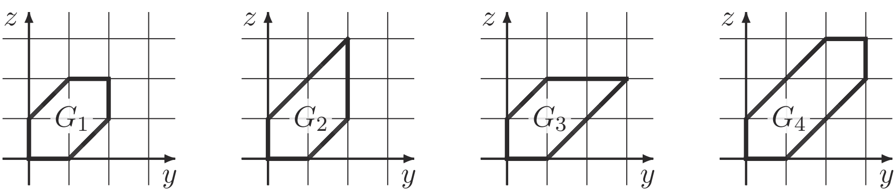

Figure 4.

Then, considering the smoothness of $\mathcal{P}$ and the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$, there exist only three possibilities $(\mathcal{P} \cap H_x, \mathcal{P} \cap H_{x-1}, \mathcal{P} \cap H_{x-2}) = (G_1, G_4, G_4), (G_4, G_4, G_1), (G_4, G_4, G_4)$. In the first two cases, we have $\mathrm{vol}(\mathcal{P}) = 9$, $l(\mathcal{P}) = 27$ and $l(\mathrm{Int}(\mathcal{P})) = 2$. On the other hand, in the last case, $\mathrm{vol}(\mathcal{P}) = 10$, $l(\mathcal{P}) = 30$ and $l(\mathrm{Int}(\mathcal{P})) = 2$. Hence the inequality $\mathrm{vol}(\mathcal{P}) > (l(\mathcal{P}) + l(\mathrm{Int}(\mathcal{P})) - 4)/3$ holds in each case.

(ii) Suppose that $\varepsilon_1 + 1 \leqslant \alpha_1 \leqslant \varepsilon_1 + 2$ and $\beta_1, \beta_2 \leqslant \varepsilon_2$. Let (x_0, y_0, z_0) be a lattice point in $\mathrm{Int}(\mathcal{P})$. We have $y_0 = 1$ and $\beta_1 + \beta_2 = \varepsilon_2 + 2$ by (8). Then, since $\varepsilon_1 - \alpha_2 \geqslant \alpha_1 - 2 \geqslant 0$, we have $x_0 = 1$ and $\alpha_1 + \alpha_2 = \varepsilon_1 + 2$ by (7). By the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$, there must be an interior lattice point such that $z_0 \geqslant 2$. It follows that $\alpha_2 = \varepsilon_1$ and $\beta_2 = \varepsilon_2$. These facts, together with the shortestness of OQ_1', immediately give that $\varepsilon_1 = \varepsilon_2 = \alpha_2 = \beta_2 = 1$ and $\alpha_1 = \beta_1 = 2$. By using the shortestness of OQ_1' again, we see that $\mathcal{P} \cap H_x$ and $\mathcal{P} \cap H_y$ are unit squares, which yields a contradiction $l(\mathrm{Int}(\mathcal{P})) = 0$.

(iii) Suppose that $\varepsilon_1 + 1 \leqslant \alpha_1 \leqslant \varepsilon_1 + 2$ and $\beta_2 \geqslant \varepsilon_2 + 1$. Note that $\alpha_1 \geqslant 2$ and $\beta_1 \leqslant 1$ in this case. Then β_1 must be one since $(2, 1, 0) \in \mathcal{P}$. Hence $\mathcal{P} \cap H_z$ is a trapezoid $\mathrm{Conv}(\{O, Q_1', (\alpha_1, 1, 0), Q_2\})$, which contradicts the shortestness of OQ_1'.

(iv) Suppose that $\varepsilon_1 + 1 \leqslant \alpha_1 \leqslant \varepsilon_1 + 2$ and $\beta_2 = 0$. In this case, ε_2 must be one by the smoothness of the vertex Q_3. We thus have $\varepsilon_3 = 1$ and $\gamma_2 = 0$, which imply $\gamma_1 \geqslant 2$ (namely, $(1, 0, 2) \in \mathcal{P}$) by (9). By noting $(2, 0, 1) \notin \mathcal{P}$, we see that $\mathcal{P} \cap H_y$ is a trapezoid $\mathrm{Conv}(\{O, Q_1, (1, 0, \gamma_1), Q_3\})$, which contradicts the shortestness of OQ_1'.

(v) We finally consider the remaining case where $\varepsilon_1 + 1 \leqslant \alpha_1 \leqslant \varepsilon_1 + 2$, $\beta_1 = \varepsilon_2 + 1$ and $\beta_2 = 1$. If $\varepsilon_1 = 1$, by the shortestness of OQ_1', $\mathcal{P} \cap H_x$ is a unit square, and $\mathcal{P} \cap H_y$ is a unit triangle or a unit square. This contradicts the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$. We thus assume $\varepsilon_1 \geqslant 2$. Note that $\alpha_1 = \varepsilon_1 + 1$ and $\alpha_2 = 1$ in this case. Hence, if (x_0, y_0, z_0) is a lattice point in $\mathrm{Int}(\mathcal{P})$, $x_0 = y_0$ and $z_0 = 1$ follow from (7) and (8). We define $\varepsilon_4 = \max\{\varepsilon \in \mathbb{N} \mid (\varepsilon, \varepsilon_2 + \varepsilon, 0) \in \mathcal{P}\}$, and put $Q_4 = (\varepsilon_4, \varepsilon_2 + \varepsilon_4, 0)$. By the smoothness of $\mathcal{P}$, the vertex Q_4 has three adjacent lattice points $(\varepsilon_4 - 1, \varepsilon_2 + \varepsilon_4 - 1, 0)$, $(\delta_1, \delta_1 + \varepsilon_2 - 1, 0)$ and $(\delta_2, \delta_2 + 1, 1)$ in $E(\mathcal{P})$ (see Fig. 5).

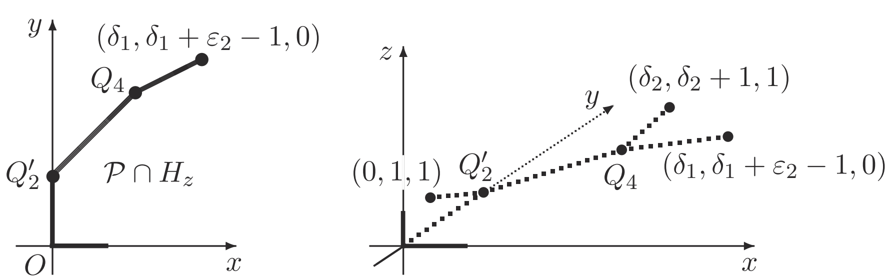

FIGURE 5.

If $\delta_1+\delta_2 \geqslant \varepsilon_4+3$, we put $P_i=(i,\varepsilon_2+i,0)$ for $i=0,\dots,\varepsilon_4$. Then $\partial\overline{\mathcal{P}_i}^{P_i}$ is a triangle with vertices $(i+1,\varepsilon_2+i+1,0)$, $(0,\varepsilon_2-1,0)$ and $(0,1,1)$ for $i=0,\dots,\varepsilon_4-1$, and $\partial\overline{\mathcal{P}_{\varepsilon_4}}^{P_{\varepsilon_4}}$ is a trapezoid

$$\mathrm{Conv}(\{(0,\varepsilon_2-1,0),(\delta_1,\delta_1+\varepsilon_2-1,0),(0,1,1),(\delta_2,\delta_2+1,1)\}).$$

The proof is finished since

$$\begin{aligned}\mu(\mathcal{P}_0) &\geqslant \mu(\mathcal{P}_1)-\frac{1}{6} \geqslant \cdots \geqslant \mu(\mathcal{P}_{\varepsilon_4})-\frac{\varepsilon_4}{6}\\ &\geqslant \mu(\mathcal{P}_{\varepsilon_4+1})+\frac{\delta_1+\delta_2-\varepsilon_4-2}{6} > \mu(\mathcal{P}_{\varepsilon_1+1})\end{aligned}$$

by Proposition 2.3. Finally, let us show that the case where $\delta_1+\delta_2 \leqslant \varepsilon_4+2$ does not occur. Since $\mathcal{P}$ has a face containing three points Q_4, $(\delta_1,\delta_1+\varepsilon_2-1,0)$ and $(\delta_2,\delta_2+1,1)$, the inclusion

$$\begin{aligned}\mathrm{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, &(\varepsilon_4-\delta_1+1)(x-1)-(\varepsilon_4-\delta_1)(y-1)\\ &+(\delta_1\varepsilon_2-\varepsilon_2\varepsilon_4-\delta_1-\delta_2+2\varepsilon_4)(z-1)\\ &-\delta_1-\delta_2+\varepsilon_4+2 \leqslant 0\}\end{aligned}$$

holds. As we have already mentioned, any interior lattice point of $\mathcal{P}$ can be written as $(x_0,x_0,1)$. This contradicts the above inclusion and the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$. □

3. Application

In this section, we apply our result to the computation of the sectional genus of a polarized toric variety. For an n-dimensional (smooth) complex projective variety X and an ample line bundle L on X, the pair (X,L) is called a (*smooth*) *polarized variety*. We remark that, in the case where L is ample, the associated polytope $\square_L$ is smooth if and only if X is smooth. Let us review the classification theory of polarized varieties before getting to the main subject. We first recall the well-known upper bound for the geometric genus of a smooth curve.

Theorem 3.1 (Castelnuovo's bound, [1]). *Let C be a smooth curve of genus g. Assume that C admits a birational map onto a nondegenerate curve of degree d in $\mathbb{P}^r$. Then*

$$g \leqslant \frac{1}{2}a(a-1)(r-1)+a(d-a(r-1)-1),$$

where $a=\lfloor (d-1)/(r-1)\rfloor$.

A smooth curve is said to be *extremal* if its genus is equal to Castelnuovo's bound, which was studied in [4]. As a higher dimensional extension, Fujita established various invariants for polarized varieties and proved a similar inequality (Theorem 3.3).

Definition 3.2. *For an n-dimensional smooth polarized variety (X, L), we define the sectional genus and the Δ-genus by*

$$\begin{aligned} g(X,L) &= \frac{1}{2}L^{n-1}.((n-1)L + K_X) + 1, \\ \Delta(X,L) &= L^n + n - h^0(X,L). \end{aligned}$$

Theorem 3.3 (cf. [5]). *Let L be a line bundle on an n-dimensional smooth projective variety X. If $|L|$ has no base points and the associated morphism $\Phi_{|L|}$ is birational on its image, then*

$$g(X,L) \leqslant a\Delta(X,L) - \frac{1}{2}a(a-1)(L^n - \Delta(X,L) - 1),$$

where $a = \lfloor (L^n - 1)/(L^n - \Delta(X,L) - 1) \rfloor$.

We call (X, L) a *Castelnuovo variety* if its sectional genus achieves the maximum of the above upper bound. Castelnuovo varieties can be roughly classified according to the relation between L^n and $2\Delta(X, L)$. First, the case where $L^n < 2\Delta(X, L)$ has been classified in [5]. If $L^n = 2\Delta(X, L)$, then (X, L) is a Mukai variety (i.e., $K_X \in |(2 - n)L|$), which has been classified in [11]. On the other hand, the case where $L^n > 2\Delta(X, L)$ still has many unknown aspects. Two- or three-dimensional polarized toric varieties, which we consider below, contain examples of this case.

In the two-dimensional case, we do not need to use the results in this paper, but only the Riemann-Roch theorem. To see this, let us introduce the notion of a ladder.

Definition 3.4. *Let (X, L) be an n-dimensional polarized variety, and put $X_0 = X$ and $L_0 = L$. A sequence $X_0 \supset X_1 \supset \cdots \supset X_{n-1}$ of (smooth) subvarieties of X is called a (smooth) ladder of (X, L) if $X_i \in |L_{i-1}|$ for each $i \geqslant 1$, where we put $L_i = L|_{X_i}$.*

Theorem 3.5 (cf. [5]). *Let (X, L) be an n-dimensional polarized variety having a ladder. If $L^n > 2\Delta(X, L)$, then L is very ample and $g(X, L) = \Delta(X, L)$.*

If X is smooth and L is generated by global sections, by virtue of Bertini's theorem, we obtain a smooth ladder of (X, L) by cutting X_i by a general member of $|L_i|$. Since any ample line bundle on a compact toric variety is generated by global sections, a polarized toric variety always has a smooth ladder. Using these results, let us now consider the two-dimensional polarized toric varieties.

Theorem 3.6. *For a smooth compact toric surface X and an ample line bundle L on X, the polarized variety (X, L) is a Castelnuovo variety with $L^2 \geqslant 2\Delta(X, L) + 2$ unless L is a line in $\mathbb{P}^2$.*

Proof. By the general theory of toric varieties, L is very ample, and we have $p_a(X) = 0$, $h^0(X, L) \geqslant 3$ and $h^1(X, L) = h^2(X, L) = 0$. Hence we obtain $L^2 = 2h^0(X, L) + L.K_X - 2$ by the Riemann-Roch theorem. On the other hand, since $-L.K_X$ is equal to $l(\partial\Box_L)$, we have the inequality $-L.K_X \geqslant 3$, where the equality holds if and only if $(X, L) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Consequently, we have $L^2 \leqslant 2h^0(X, L) - 6$, which implies that $L^2 \geqslant 2\Delta(X, L) + 2$. Then, since $g(X, L) = \Delta(X, L)$ by Theorem 3.5, we can conclude that (X, L) is a Castelnuovo variety. $\Box$

Next, in order to investigate the three-dimensional case, we compute the value of $L^2.K_X$. This computation also can be reduced to a matter of the number of lattice points.

Lemma 3.7. *For a three-dimensional smooth polarized toric variety (X, L), the equality $L^2.K_X = 2(h^0(X, L + K_X) - h^0(X, L) + 2)$ holds.*

Proof. We denote by $D_1, \ldots, D_d$ the T_N-invariant divisors of X, and by F_i the face of $\Box_L$ corresponding to D_i. It is known that $K_X \sim -\sum_{i=1}^d D_i$ and $L^2.D_i$ is equal to twice of the area of F_i. Hence the statement of the lemma can be rewritten as

$$\sum_{i=1}^{d} \mathrm{vol}(F_i) = -l(\mathrm{Int}(\Box_L)) + l(\Box_L) - 2 = l(\partial\Box_L) - 2.$$

Let us compute the left-hand side by using Theorem 1.1 and Euler's polyhedron formula. Denote by v and e the number of vertices and edges of $\Box_L$, respectively. It follows from the smoothness of $\Box_L$ that every vertex of $\Box_L$ has three edges. Hence we have $3v = 2e$ and

$$\begin{aligned}\sum_{i=1}^{d} \mathrm{vol}(F_i) &= \frac{1}{2}\sum_{i=1}^{d}(l(F_i) + l(\mathrm{Int}(F_i)) - 2) = \frac{1}{2}(2l(\partial\Box_L) + v) - d \\ &= l(\partial\Box_L) - v + e - d = l(\partial\Box_L) - 2.\end{aligned}$$

$\Box$

Theorem 3.8. *Let X be a three-dimensional smooth compact toric variety, and L be an ample line bundle on X. Assume that $(X, L) \not\simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Then a polarized variety (X, L) is a Castelnuovo variety if and only if*

(i) *X is a Fano variety and $L \sim -K_X$, or*

(ii) *$h^0(X, L+K_X) = 0$ and $h^0(X, 2L+K_X) \leqslant h^0(X, L) - 4$.*

We provide several additional explanations. The former case has been classified into eighteen types in [2] and [17]. Besides, $L^3 = 2\Delta(X, L)$ holds in this case. On the other hand, $L^3 > 2\Delta(X, L)$ holds in the case (ii), and it is known that three-dimensional polarized toric varieties with $h^0(X, L+K_X) = 0$ (not necessarily assume the latter inequality) can be classified into five types (see [13]). We will see further details of this classification after the proof. Incidentally, more generally, Fukuma classified n-dimensional polarized varieties (not necessarily toric) with $h^0(X, (n-2)L + K_X) = 0$ in [6].

Proof of Theorem 3.8. If we rewrite the inequality in Theorem 3.1 by using Lemma 3.7, we see that (X, L) is a Castelnuovo variety if and only if

$$(a^2 + a - 2)h^0(X, L) = 2(2a^2 + a - 3 + (a-1)L^3 - h^0(X, L+K_X)). \tag{10}$$

On the other hand, we have $h^0(X, L) \leqslant (L^3 - 1)/a + 4$ by the definition of a. By combining this inequality with (10), we see that

$$2ah^0(X, L+K_X) \geqslant (a-1)(a-2)(L^3 - 1) \tag{11}$$

holds if (X, L) is a Castelnuovo variety.

(i) We first consider the case where $h^0(X, L+K_X) \geqslant 1$. In order to prove the sufficiency, we assume that (X, L) is a Castelnuovo variety. Note that $a \geqslant 2$ by (10). By Corollary 1.6 and Lemma 3.7, we have

$$\begin{aligned} h^0(X, L+K_X) &\leqslant h^0(X, L+K_X) + \frac{1}{4}(L^3 - 2h^0(X, L) - 2h^0(X, L+K_X) + 8) \\ &= \frac{1}{4}L^3 - \frac{1}{2}h^0(X, L) + \frac{1}{2}\left(\frac{1}{2}L^2.K_X + h^0(X, L) - 2\right) + 2 \\ &= \frac{1}{4}L^3 + \frac{1}{4}L^2.K_X + 1 \leqslant \frac{1}{4}L^3, \end{aligned}$$

where the last inequality follows from the fact that $-L^2.K_X$ is twice the sum of areas of faces of $\Box_L$. Then the inequality (11) induces $(2a^2 - 7a +$

4)$L^3 \leqslant 2(a-1)(a-2)$. If $a \geqslant 3$, we have $\mathrm{vol}(\square_L) = L^3/6 \leqslant 2/3$, which clearly contradicts the assumption $l(\mathrm{Int}(\square_L)) \geqslant 1$. Hence we obtain $a = 2$. Then, since

$$h^0(X, L + K_X) = L^3 - 2h^0(X, L) + 7 \geqslant 2h^0(X, L + K_X) - 1$$

by (10) and Corollary 1.6, we have $h^0(X, L+K_X) = 1$ and $L^3 = 2\Delta(X, L)$. Hence (X, L) is a Mukai variety, which means that $(X, L) \simeq (X, -K_X)$ is a Fano variety. Such varieties, so-called toric Fano three-folds, have been classified into eighteen types in [2] and [17] independently (see (6) in Section 2). Hence the necessity is checked by computing. For all types, in practice, we can confirm that $a = 2$ and the equality (10) holds.

(ii) Assume that $h^0(X, L + K_X) = 0$ (equivalently, $l(\mathrm{Int}(\square_L)) = 0$). In this case, Theorem 1.2 gives

$$\begin{aligned} 12\mathrm{vol}(\square_L) &= l(\square_{2L}) + l(\mathrm{Int}(\square_{2L})) - 2l(\square_L) \\ &= l(\partial\square_{2L}) + 2l(\mathrm{Int}(\square_{2L})) - 2l(\square_L) \\ &= 4l(\partial\square_L) + 2l(\mathrm{Int}(\square_{2L})) - 2l(\square_L) - 6 \\ &= 2l(\square_L) + 2l(\mathrm{Int}(\square_{2L})) - 6, \end{aligned}$$

which implies that $L^3 = h^0(X, L) + h^0(X, 2L + K_X) - 3$. Therefore, the inequality in the statement is equivalent to $L^3 \leqslant 2h^0(X, L) - 7$. If (X, L) is a Castelnuovo variety, we have $a \leqslant 2$ by (11). In the case where $a = 1$, it is clear that $L^3 < 2h^0(X, L) - 7$ by the definition of a. On the other hand, if $a = 2$, the condition (10) is equivalent to the equality $L^3 = 2h^0(X, L) - 7$. Conversely, if $L^3 \leqslant 2h^0(X, L) - 7$, we have

$$a = \begin{cases} 1 & (L^3 < 2h^0(X, L) - 7), \\ 2 & (L^3 = 2h^0(X, L) - 7) \end{cases}$$

by definition. In either case, we can easily check that (X, L) satisfies (10). $\square$

By virtue of [13, Proposition 2.3], we can see the detailed structure of (X, L) in the case (ii) in Theorem 3.8. If $h^0(X, L + K_X) = 0$, then X is one of the following five types.

(a) a toric $\mathbb{P}^1$-bundle over a smooth toric surface.

(b) $(X, L) \simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$ $(k = 1, 2, 3)$.

(c) a toric $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ over $\mathbb{P}^1$.

(d) a blow-up of $\mathbb{P}^3$ at T_N-invariant i points $(i = 1, 2, 3, 4)$.

(e) a blow-up of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ at T_N-invariant i points $(i = 1, 2)$.

In each case, the polytope $\square_L$ is as follows. For simplicity, we put $\boldsymbol{m_1} = (1, 0, 0)$, $\boldsymbol{m_2} = (0, 1, 0)$ and $\boldsymbol{m_3} = (0, 0, 1)$, and assume $a \geqslant b \geqslant c$ in the cases (c) and (e).

(a) $\mathrm{Conv}(F_0 \cup F_1)$, where F_0 and F_1 are parallel smooth faces of distance one such that they define the same two-dimensional smooth fan.

(b) $\mathrm{Conv}(\{O, k\boldsymbol{m_1}, k\boldsymbol{m_2}, k\boldsymbol{m_3}\})$.

(c) $\mathrm{Conv}(\{O, \boldsymbol{m_1}, \boldsymbol{m_2}, (1, 0, a), (0, 1, b), (0, 0, c)\})$ or $\mathrm{Conv}(\{O, 2\boldsymbol{m_1}, 2\boldsymbol{m_2}, (2, 0, 2a - c), (0, 2, 2b - c), (0, 0, c)\})$.

(d) $\mathrm{Conv}(\{k\boldsymbol{m_1}, k\boldsymbol{m_2}, k\boldsymbol{m_3} \mid k = 1, 3\})$, $\mathrm{Conv}(\{\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}, 3\boldsymbol{m_1}, 3\boldsymbol{m_2}, 2\boldsymbol{m_3}, (1, 0, 2), (0, 1, 2)\})$, $\mathrm{Conv}(\{\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}, 3\boldsymbol{m_1}, 2\boldsymbol{m_2}, 2\boldsymbol{m_3}, (1, 2, 0), (0, 2, 1), (1, 0, 2), (0, 1, 2)\})$ or $\mathrm{Conv}(\{k\boldsymbol{m_1}, k\boldsymbol{m_2}, k\boldsymbol{m_3}, (2, 1, 0), (2, 0, 1), (1, 2, 0), (0, 2, 1), (1, 0, 2), (0, 1, 2) \mid k = 1, 2\})$.

(e) a polytope obtained from $\mathcal{Q}$ by cutting of a unit three-simplex at one of O, $2m_1$ and $2m_2$, where $\mathcal{Q}$ denotes the latter polytope in the case (c),
a polytope obtained from the above one by cutting of a unit three-simplex at one of $(2, 0, 2a - c)$, $(0, 2, 2b - c)$ and $(0, 0, c)$.

By computing, we can check $6\mathrm{vol}(\square_L) \leqslant 2l(\square_L) - 8$, that is, (X, L) is a Castelnuovo variety in the latter four cases except for $\mathrm{Conv}(\{O, m_1, m_2, m_3\})$ (in which case L is a hyperplane in $\mathbb{P}^3$). On the other hand, in the case (a), the value of $\mathrm{vol}(\square_L)$ varies greatly depending on the shapes of F_0 and F_1. See the following examples.

Example 3.9. *Let (X, L) be a three-dimensional polarized toric variety of type (a).*

(a_1) *If* $F_i = \mathrm{Conv}(\{(i, 0, 0), (i, 3, 0), (i, 3, 3), (i, 0, 3)\}$, *then* $6\mathrm{vol}(\square_L) = 2l(\square_L) - 10$.

(a_2) *If* $F_i = \mathrm{Conv}(\{(i, 0, 0), (i, 2, 0), (i, 3, 1), (i, 3, 2), (i, 2, 3), (i, 1, 3), (i, 0, 2)\})$, *then* $6\mathrm{vol}(\square_L) = 2l(\square_L) - 7$.

(a_3) *If* $F_i = \mathrm{Conv}(\{(i, 1, 0), (i, 2, 0), (i, 3, 1), (i, 3, 2), (i, 2, 3), (i, 1, 3), (i, 0, 2), (i, 0, 1)\})$, *then* $6\mathrm{vol}(\square_L) = 2l(\square_L) - 6$.

In the first two cases, (X, L) is a Castelnuovo variety, while the last one is not.

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