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HANDLE DECOMPOSITIONS OF SIMPLY-CONNECTED FIVE-MANIFOLDS. I

РОЗКЛАД НА РУЧКИ ОДНОЗВ'ЯЗНИХ П'ЯТИВИМІРНИХ МНОГОВИДІВ. I

The handle decompositions of 1-connected smooth or piecewise linear 5-manifolds are considered. The basic notions and constructions necessary for proving further results are introduced.

Розглядається розклад на ручки однозв'язних гладких або кусково-лінійних п'ятивимірних многовидів. Наведені основні поняття та конструкції, необхідні для одержання подальших результатів.

All manifolds considered below are supposed to be compact and smooth or piecewise linear (*PL*). Any manifold of this type admits a handle decomposition [1, 2]. Each handle decomposition of a manifold can be made proper (see details in [2]). Each proper handle decomposition corresponds to a proper Morse function and satisfies the Morse – Pitcher inequalities [3, 4]. We are interested in the exact handle decompositions which turn the above-mentioned inequalities into equalities. S. Smale proved in [4] the existence of the exact handle decompositions of simply-connected manifolds and cobordisms of dimensionality $n \geq 6$. As was proved by D. Barden in [5], any closed 1-connected 5-manifold admits an exact handle decomposition.

This is the first in a series of three papers containing new results concerning handle decompositions of 1-connected 5-manifolds. Here, we present the basic technique necessary for proving the main result in the next paper:

All incidence indices of 2- *and* 3-*handles of the Barden handle decomposition of a closed* 1-*connected* 5-*manifold are geometric. The Barden handle decomposition is geometrically diagonal.*

1. Connected summing and handle decompositions.

Lemma 1. *Assume that chain complexes* $\{C_\lambda, \partial_\lambda\}$ *and* $\{C'_\lambda, \partial'_\lambda\}$ *are realized by a handle decomposition of n-manifolds* V *and* V', *respectively. Let* $n \geq 3$ *and the manifolds* V, V' *satisfy one of two conditions: i) both* V *and* V' *are closed; ii) both* V *and* V' *are manifolds with a boundary. Then there exists a handle decomposition of* $V \# V'$ *(in case i)) or* $V \natural V'$ *(in case ii)) which realizes the complex* $\{C_\lambda \oplus C'_\lambda, \partial_\lambda \oplus \partial'_\lambda\}$ *with* $0 < \lambda < n$.

Proof. Consider the case *i*). It follows from the notion of a handle decomposition, that there exist a point $x \in \partial h^0(V)$ and its closed neighborhood $D \subset V$ in which the handle $h^n(V)$ is attached directly to $h^0(V)$; D, as a manifold, is diffeomorphic to the standard closed n-disk. The $(n-1)$-disk $d = D \cap \partial h^0(V)$ is a regular neighborhood of x in $\partial h^0(V)$ and divides D into disks $D_1 \subset h^0$, $D_2 \subset h^n$. Similarly, consider the same point x' and disks D', d', D'_1 and D'_2 for V'. The manifolds $\overline{V \setminus D}$ and $\overline{V' \setminus D'}$ have boundaries diffeomorphic to S^{n-1}. We connect them by a summing tube $S^{n-1} \times D^1$ identifying, thus, $S^{n-1} \times 0$ with $\partial(\overline{V \setminus D})$ and $S^{n-1} \times 1$ with $\partial(\overline{V' \setminus D'})$. This tube can be regarded as a double of the ribbon $D^{n-1} \times D^1$ with the first copy of $D^{n-1} \times D^1$ connecting $\overline{h^0(V) \setminus D_1}$ and $\overline{h^0(V') \setminus D'_1}$ along d_1 and d'_1, respectively, and the second copy connecting two n-handles along d_2 and d'_2. Thus, we have constructed $V \# V'$ with the required handle decomposition:

$$h^0(V \# V') = \overline{h^0(V) \setminus D_1} \natural \overline{h^0(V') \setminus D_1'}, \quad h^n(V \# V') = \overline{h^n(V) \setminus D_1'} \natural \overline{h^n(V') \setminus D_1'},$$

all other handles are the handles h^λ of V and V' with $0 < \lambda < n$. The proof in case *ii*) is the same as that in case *i*.

Lemma 2. *Let the n-manifold V' be constructed from V with a boundary M by attaching an m-handle h^m, $0 < m < n$, with the a-sphere lying in some $(n-1)$-disk in M. Then V' is diffeomorphic to $V \natural W$ for some W with a handle decomposition $W = h^0 \cup h^m$.*

Proof. Obviously, $V \simeq V \natural D$ and $M \simeq M \# S$, where D is the standard closed n-disk and $S = \partial D \simeq S^{n-1}$. The a-sphere (and hence, the a-tube) of h^m is contained in some $(n-1)$-disk, $d \subset M$. By the Disk Lemma, d can be isotopped to the $(n-1)$-disk, $d' \subset S$; therefore,

$$V' = V \cup h^m \simeq (V \natural D) \cup h^m \simeq V \natural (D \cup h^m) \simeq V \natural W,$$

where $W = h^0 \cup h^m$ with $h^0 = D$.

***Corollary* 1.** *$\partial V' \simeq M \# \partial W$, where V', M and V are the same as in Lemma 2.*

Lemma 3 [6]. *Let a closed m-manifold M' be constructed by surgery of a closed m-manifold M on an imbedding $f: S^k \times D^{m-k} \to M$ such that $f(S^k \times D^{m-k})$ is contained in some m-disk in M. Then $M' \simeq M \# Q$, where Q is formed by surgery on some imbedding f': $S^k \times D^{m-k} \to S^m$ and can be considered as a boundary of some m-manifold $W = h^0 \cup h^m$ with h^m attached along $f'(S^k \times D^{m-k})$.*

The proof is similar to that of Lemma 2.

Consider an n-manifold $W = h^0 \cup_f h^m$ such that the a-sphere $S = f(S^{m-1} \times 0)$ of h^m is unknotted in $S^{n-1} = \partial h^0$. For the manifold $q = \partial W$, we have

$$Q = \chi(\partial h^0, f) \simeq cl(\partial h^0 \setminus f(S^{m-1} \times D^{n-m})) \cup_{f'} R,$$

where $R = D^m \times S^{n-m-1}$ is the b-tube of h^m and f': $S^{m-1} \times S^{n-m-1} \to \partial h^0$ is obtained from the embedding f of the a-tube of h^m by restriction to its boundary which is also a boundary of the b-tube of h^m. Since the attaching sphere $S = f(S^{m-1} \times 0)$ of h^m is unknotted in ∂h^0, the pair $(\partial h^0, S)$ is standard. It follows from this that S bounds an m-disk d in ∂h^0 and since $f(S^{m-1} \times D^{n-m}) = S \times D^{n-m}$, we have

$$P = cl\,(\partial h^0 \setminus f(S^{m-1} \times D^{n-m})) \simeq d \times S^{n-m-1}.$$

For some $x \in S^{n-m-1}$, consider the sphere $S_1 = S \times x$. It bounds the m-disk $D_1 = d \times x$ in P. Consider a tubular neighborhood $P_1 = D_1 \times D^{n-m-1}$ of D_1 in P and $P_2 = \overline{P \setminus P_1} \simeq D^m \times D^{n-m-1}$. Thus, we have $P = P_1 \cup_T P_2$, where $T = \partial P_1 \cap \partial P_2 = D_1 \times S^{n-m-2}$. For ∂P, we have

$$\partial P = S_1 \times D^{n-m-2} \cup S_1 \times D^{n-m-2},$$

where the first term is $\overline{\partial P_1 \setminus T}$, and the second is $\overline{\partial P_2 \setminus T}$. By analogy with P, consider $S_2 = f'^{-1}(S_1) \subset \partial R$. Clearly, $S_2 = S^{m-1} \times y$ bounds some disk $D_2 = D^m \times y$ for some $y \in S^{n-m-1}$. Thus, we have $R = R_1 \cup R_2$, where R_1 is a tubular neighborhood of D_2 in R, $R_2 = \overline{R \setminus R_1}$, and

$$\partial R_1 = S_2 \times D^{n-m-1} \cup D_2 \times D^{n-m-2},$$

where the second term is $\partial R_1 \cap \partial R_2$. Since $f'(S_2) = S_1$, we can set $f'(S_2 \times D^{n-m-1}) = S_1 \times D^{n-m-1}$ by changing the embedding f for the ambiently isotopic embedding in ∂h_0. This enables us to consider the union Q_1 of P_1 and R_1 along $S^{m-1} \times D^{n-m-1}$ and the similar union Q_2 of P_2 and R_2. Clearly, $Q_1 \simeq Q_2$ and Q can be obtained as a double of $P_1 \cup_{f'} R_1$. Thus, we have

$$Q = h^0 \cup h^m \cup h^{n-m-1} \cup h^{n-1},$$

where the dimensionality of each handle is $n-1$, $h^0 = P_1$, $h^{n-1} = P_2$, $h^m = R_1$, $h^{n-m-1} = R_2$ and $R_1 \cup R_2$ is the b-tube of the n-dimensional handle h^m of W. We have thus proved the following lemma.

Lemma 4. *Let $W = h^0 \cup_f h^m$ be an n-manifold with the embedding f of the a-tube of h^m such that the a-sphere of h^m is unknotted in ∂h^0. Then the $(n-1)$-manifold $Q = \partial W$ can be decomposed into handles in such a way:*

$$Q = h^0 \cup h^m \cup h^{n-m-1} \cup h^{n-1},$$

where $h^m \cup h^{n-m-1}$ is the b-tube of the n-dimensional handle h^m of W.

2. Standard 5- and 4-manifolds. Consider a 5-manifold $W = h^0 \cup_f h^2$; W is diffeomorphic to a smooth bundle on S^2 with the fiber D^3 and the group SO_3. Since $\pi_1(SO_3) \simeq \mathbb{Z}_2$, W is diffeomorphic to $A = S^2 \times D^3$ or $B = S^2 \underset{\sim}{\times} D^3$. The corresponding embedding $f: S^1 \times D^3 \to D^4$ is called the embedding of type A or type B, respectively. Clearly, $H_0(W) \simeq H_2(W) \simeq \mathbb{Z}$, where W is A or B and $w^2(A) = 0$, $w^2(B) \neq 0$, where $w^2 \in H^2(W, \mathbb{Z}_2)$ is the second Shtiefel–Whitney class. There exists a canonical generator a of $H_2(W)$, which can be realized by a 2-sphere $\tilde{a}$ embedded into ∂W. This sphere $\tilde{a}$ can be obtained as the union of a copy $D = D^2 \times x$ of the core of h^2 taken on its b-tube and of a disk in ∂h^0 with the boundary $S = f(\partial D)$, which always exists because $S = f(\partial D) = f(S^1 \times x)$ is unknotted in $\partial h^0 \simeq S^4$ [8]. By the definition of a, we have $w^2(a) = 0$ for $W = A$ and $w^2(a) \neq 0$ for $W = B$; A and B will be called *elementary* 5-manifolds.

Consider also *standard* 5-manifolds, i.e., manifolds with the handle decomposition $V = h^0 \cup h_1^2 \cup \ldots \cup h_r^2$. Clearly, $V \simeq kB \,\natural\, lA$, where $k + l = r$. Conversely, by Lemma 1, every 5-manifold $V = kB \,\natural\, lA$ admits a handle decomposition $V = h^0 \cup h_1^2 \cup \ldots \cup h_r^2$, with $r = k + l$. The core of each of the 2-handles $h_1^2, \ldots, h_r^2$ determines each of r canonical generators $a_1, \ldots, a_r$ of $H_2(V) \simeq r\mathbb{Z}$. By using the handle subtraction in the handle decomposition of $B \,\natural\, B$, one can easily show that $B \,\natural\, B \simeq B \,\natural\, A$. Thus, we see that for $V = kB \,\natural\, lA$, we can choose $k = 0$ if $w^2(V) = 0$, or $k = 1$ if $w^2(V) \neq 0$; this means that w^2 and $r = k + l$ determine the standard 5-manifold V up to a diffeomorphism.

Define an *elementary* 4-manifold as the 4-manifold $M = \partial V$, where V is elementary 5-manifold. Then M is a bundle on S^2 with the fiber S^2, and hence,

$$M = \partial A \simeq S^2 \times S^2 \text{ or } M = \partial B \simeq S^2 \underset{\sim}{\times} S^2 \simeq \mathbb{CP}^2 \# (-\mathbb{CP}^2).$$

It readily follows from the definition of $M = \partial V$ that there exists a basis $\{a, b\}$ of

$H_2(M)$ realized by 2-spheres $\{\tilde{a}, \tilde{b}\}$ embedded into M, in which the intersection form $Q(M)$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $M = A$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for $M = B$. The sphere $\tilde{a}$ can be taken to be the above-mentioned sphere $\tilde{a}$, determined by the core of 5-dimensional 2-handle h^2 of V, and the sphere $\tilde{b}$ can be taken to be the b-sphere of this 5-dimensional 2-handle. By Lemma 4, the basis $\{a, b\}$ can be also determined by the handle decomposition $M = h^0 \cup h_1^2 \cup h_2^2 \cup h^4$, where $h_1^2 \cup h_2^2$ is the b-tube of the 5-dimensional 2-handle $h^2(V)$. This definition of $\tilde{a}$ and $\tilde{b}$ implies at once that all coefficients in the intersection form $Q(M)$ are geometric.

The basis $\{a, b\}$ of $H_2(M)$ considered above is called the *canonical basis*. We also say that a handle decomposition of an elementary 5-manifold $V = h^0 \cup h^2$ induces a *canonical handle decomposition* of $m = \partial V$.

Since the normal bundle over $\tilde{b}$ in M is trivial, we can consider the manifold $\chi(M, f)$, where $f\colon S^2 \times D^2 \to M$ is an embedding with $f(S^2 \times 0) = \tilde{b}$. It is easily seen that choosing f to be the embedding of b-tube of $h^2(V)$, we obtain $\chi(M, f) \simeq S^4$.

The properties of elementary 4-manifolds given above are combined in the following proposition.

Proposition 1. *Let M be an elementary 4-manifold. Then there exists a canonical basis $\{a, b\}$ of $H_2(M)$, realized by 2-spheres $\{\tilde{a}, \tilde{b}\}$ embedded into M, in which the intersection form $Q(M)$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $M = S^2 \times S^2$ or $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for $M = S^2 \tilde{\times} S^2$ and all coefficients of $Q(M)$ are geometric. This basis can be determined by the induced canonical handle decomposition of $M = \partial V$, and $\chi(M, f) \simeq S^4$, where $f\colon S^2 \times D^2 \to M$ is an embedding of the b-tube of $h^2(V)$ into M with $f(S^2 \times 0) = \tilde{b}$.*

Now let V be a standard 5-manifold. Then

$$M = \partial V \simeq k\partial V \# l\partial A \simeq kS^2 \tilde{\times} S^2 \# lS^2 \times S^2,$$

where $k + l = r = \operatorname{rk} H_2(V)$ and k can be chosen equal to 0 or 1 depending on whether $w^2(V) = 0$ or $w^2(V) \neq 0$. Since $w^2(M)$ is zero or nonzero together with $w^2(V)$, $w^2(M)$ also determines whether k is equal to 0 or 1.

M is said to be a *standard 4-manifold*. By using Lemma 1 and 4, M, similarly to an elementary 4-manifold, can be supplied with a canonical basis of $H_2(M)$ and the induced handle decomposition; therefore, Proposition 1 can be reformulated for the standard 4-manifold.

Since $M = \partial V$, where V is a standard 5-manifold, the natural embedding $i\colon M \to V$ induces a homomorphism $i_*\colon H_2(M) \to H_2(V)$. Then the definition of the canonical basis $\{a_j, b_j, j = 1, \ldots, r\}$ of $H_2(M)$ implies that $i_*(b_j) = 0$ for $j = 1, \ldots, r$ and $\{i_*(a_j); j = 1, \ldots, r\}$ is a basis of $H_2(V)$.

Let (W, V_0, V_1) be a 5-dimensional cobordism with 1-connected V_1. Thinking in the same way as for standard 5-manifold V, we can prove the following proposition.

Proposition 2. *Assume that a 5-dimensional cobordism*

$$W' = W \cup h_1^2 \ldots \cup h_m^2$$

is obtained from a cobordism (W, V_0, V_1) with 1-connected V_1 by means of *gluing handles* $h_1^2, \ldots, h_m^2$ *along* 1-*connected* V_1. *Then* $W' \simeq W \natural_{V_1} V'$, *where*

$$V' = h^0 \cup h_1^2 \cup \ldots \cup h_m^2$$

is a standard 5-*manifold*, $\partial_+ W' \simeq V_1 \# M'$, *where* $M' = \partial V'$ *is a standard* 4-*manifold and*

$$H_2(\partial_+ W') \simeq H_2(V_1) \oplus \langle a_1', b_1', \ldots, a_m', b_m' \rangle,$$

where $\{a_j', b_j', j = 1, \ldots, m\}$ *is a canonical basis of* $H_2(M')$, *induced by the handle decomposition of* V'.

3. Special handle decompositions of 1-connected 5-manifolds. It is known from [8] that a 5-dimensional cobordism (W, V_0, V_1) with 1-connected W, V_0, and V_1 admits a proper handle decomposition which does not contain handles with indices 1 and 4. Such decompositions will be called *special*.

Given a special handle decomposition of the cobordism W, we define m_2 as a number of 2-handles and m_3 as a number of 3-handles. Since V_i, $i = 0, 1$, are 1-connected, the groups $H_2(V_i)$ are free and we can define $n_i = \operatorname{rk} H_2(V_i)$. If W is a 5-manifold with a boundary, we set $V_0 = \varnothing$, and if W is a closed 5-manifold, we set $V_1 = V_0 = \varnothing$.

Define a 2-*skeleton* of a given special handle decomposition

$$W = V_0 \times D^1 \cup h_1^2 \cup \ldots \cup_{m_2}^2 \cup h_1^3 \cup \ldots \cup h_{m_3}^3 \cup V_1 \times D_1$$

as the cobordism

$$W_2 = V_0 \times D^1 \cup h_1^2 \cup \ldots \cup h_{m_2}^2 .$$

A *dual* 2-*skeleton* is defined as

$$W_2' = V_1 \times D^1 \cup \overline{h_1^2} \cup \ldots \cup \overline{h_{m_3}^2},$$

where $\overline{h_i^2}(W_2') = h_i^3(W)$. It follows from the definition that $\partial_+ W_2' = \partial_- W_2$ and $W \simeq W_2 \cup_\varphi (-W_2')$, where φ is a gluing diffeomorphism of $M_2 = \partial_+ W_2$. By Proposition 2,

$$W_2 \simeq V_0 \times D^1 \natural_{V_0 \times 1} X, \quad W_2' \simeq V_1 \times D^1 \natural_{V_1 \times 1} X',$$

where

$$X = h^0 \cup h_1^2 \cup \ldots \cup h_{m_2}^2, \; X' = h^0 \cup \overline{h_1^2} \cup \ldots \cup \overline{h_{m_3}^2}$$

are standard 5-manifolds. Since $M_2 \simeq V_0 \# \partial X \simeq V_1 \# \partial X'$, we easily obtain $n_0 + m_2 = n_1 + m_3$.

Consider also a homomorphism $\partial_3 \colon C_3(W) \to C_2(W)$ of the chain complex $(C_*(W), \partial_*)$ associated with a given special handle decomposition of W. The homomorphism ∂_3 can be represented by a matrix with coefficients $\partial_3^{ij} = \varepsilon(h_i^3, h_j^2)$, $j = 1, \ldots, m_2$, $i = 1, \ldots, m_3$, where $\varepsilon(h^3, h^2)$ is the incidence index of the handles h^3 and h^2. But the b-sphere of each h_j^2 is the sphere $\tilde{b}_j$ which determines the cycle b_j in the canonical basis $\{a_j, b_j;\ j = 1, \ldots, m_2\}$ for the induced canonical handle decomposition of ∂X. The a-sphere of each $h_i^3(W)$ is the b-sphere of $\overline{h_i^2}(W_2')$ and coincides with $\varphi(\tilde{b}_i')$, where $\{a_i', b_i', i = 1, \dots, m_3\}$ is the canonical basis for the induced handle decomposition of $\partial X'$. Thus, we have $\partial_3^{ij} = \varphi(\tilde{b}_i') \cdot \tilde{b}_j$.

The properties of a special handle decomposition of a 5-manifold given above are summarized in the following proposition.

Proposition 3. *Assume that a special handle decomposition of a 5-dimensional cobordism (W, V_0, V_1) with 1-connected W, V_0, and V_1 is given. Then $m_2 + n_0 = m_3 + n_1$, where $n_j = \mathrm{rk} H_2(V_j)$, $j = 1, 2$, m_i is the number of i-handles for $i = 2, 3$. The homomorphism $\partial_3 \colon C_3(W) \to C_2(W)$ can be represented by the matrix with coefficients $\partial_3^{ij} = \varphi(\tilde{b}_i') \cdot \tilde{b}_i$, where $\{a_j, b_j, j = 1, \dots, m_2\}$ is the canonical basis of $H_2(M_2)$ induced by the handle decomposition of W_2, $\{a_i', b_i', i = 1, \dots, m_3\}$ is the canonical basis of $H_2(M_2)$ induced by the handle decomposition of W_2', and φ is a gluing diffeomorphism of M_2.*

It follows from Proposition 3 that for a special handle decomposition of a closed 1-connected 5-manifold, $m_2 = m_3$. For a 1-connected 5-manifold W with a 1-connected boundary, we obtain $\rho(H_2(W)) - \rho(H_3(W)) = m_2 - m_3 = m$, where $m = \mathrm{rk}\, H_2(M)/2$. The number $\rho(G)$, where G is a finitely generated Abelian group, signifies $\mathrm{rk}\, G + \mu(\mathrm{fors}\,(G))$.

We can also define a minimal special handle decomposition of a 5-manifold, i.e., a special handle decomposition which contains the 2-handles and 3-handles in minimal numbers.

Corollary 2. *A 1-connected 5-dimensional cobordism with 1-connected ends admits a minimal special handle decomposition.*

By Proposition 3, each special handle decomposition with minimal m_2 is also a minimal special handle decomposition.

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