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Generalized $\oplus$-supplemented modules

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Abstract. Let R be a ring and M be a left R-module. M is called *generalized* $\oplus$*- supplemented* if every submodule of M has a generalized supplement that is a direct summand of M. In this paper we give various properties of such modules. We show that any finite direct sum of generalized $\oplus$-supplemented modules is generalized $\oplus$-supplemented. If M is a generalized $\oplus$-supplemented module with $(D3)$, then every direct summand of M is generalized $\oplus$-supplemented. We also give some properties of generalized cover.

1. Introduction

In this note R will be an associative ring with identity and all modules unital left R-modules. Let M be an R-module. The notation $N \leq M$ means that N is a submodule of M. Rad (M) will indicate Jacobson radical of M. A submodule N of an R-module M is called *small* in M (notation $N << M$), if $N + L \neq M$ for every proper submodule L of M. An epimorphism $f : K \to M$ is called a *small cover* (cover in [9]) if $Ker\, f << K$. Let M be an R-module and let N and K be any submodules of M. K is called a *supplement* of N in M if K is minimal with respect to $M = N + K$. K is a supplement of N in M iff $M = N + K$ and $N \bigcap K << K$(see [8]). Following [8], M is called *supplemented* if every submodule of M has a supplement in M, and is called *amply supplemented* (supplemented in [6]) if for any two submodules U and V of M with $M = U + V$, V contains a supplement of U in M. Cleary amply supplemented modules are supplemented. If $M = N + K$

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and $N \bigcap K << M$, then K is called a *weak supplement* of N in M(see [5]). Then clearly N is a weak supplement of K, too. A module M is called *weakly supplemented* if every submodule of M has a weak supplement in M.

Let M be an R-module and let N and K be any submodules of M with $M = N + K$. If $(N \bigcap K \subseteq \text{Rad}\,(M))N \bigcap K \subseteq \text{Rad}\,(K)$ then K is called a *(weak) generalized supplement* of N in M. Since Rad (K) is the sum of all small submodules of K, every supplement submodule is a generalized supplement in M. Following [9], M is called *generalized supplemented* or briefly GS- module if every submodule N of M has a generalized supplement K in M, and it is called *generalized amply supplemented* or briefly GAS-module in case $M = K + L$ implies that K has a generalized supplement $L' \leq L$. Clearly every (amply) supplemented module is generalized (amply) supplemented. In [7], a module M is called *weakly generalized supplemented* or briefly WGS-module if every submodule K of M has a weak generalized supplement N in M. For characterizations of generalized (amply) supplemented and weakly generalized supplemented modules we refer to [7] and [9].

Recall from [1] that an epimorphism $f : P \to M$ is called a *generalized cover* if $Ker\, f \subseteq \text{Rad}\,(P)$, and a generalized cover $f : P \to M$ is called *generalized projective cover* in case P is a projective module. Clearly every small cover is a generalized cover. In [1], M is called *(generalized) semiperfect* if every factor module of M has a (generalized) projective cover. The concepts of (generalized) semiperfect modules were introduced in [1] and [9].

This note consists of two sections. We obtain some properties of generalized cover in section 2. In section 3 we introduce generalized $\oplus$-supplemented modules. We show that every finite direct sum of generalized $\oplus$-supplemented modules is generalized $\oplus$-supplemented.

2. Generalized cover

It was shown in [9, Lemma 1.1] that if $f : M \to N$ and $g : N \to K$ are generalized covers, then $gf : M \to K$ is a generalized cover, too. We prove that the converse of this fact is also true.

Proposition 2.1. *If $f : M \to N$ and $g : N \to K$ are two epimorphisms, then f and g are generalized covers if and only if $gf : M \to K$ is a generalized cover.*

Proof. $(\Rightarrow)$ Let $m \in Ker\, gf$. Then $(gf)\,(m) = 0$ and $f\,(m) \in Ker\, g \subseteq \text{Rad}\,(N)$. Note that $R\,f\,(m) << N$. Suppose that $m \notin \text{Rad}\,(M)$. Then there exists a maximal submodule P of M such that $P + Rm = M$. Then

$f(P) + Rf(m) = N$, and since $Rf(m) << N$ it follows that $f(P) = N$. Hence $P = f^{-1}(f(P)) = P + Ker\, f = M$. This is a contraction.

($\Leftarrow$) Let $m \in Ker\, f$. Then $g(f(m)) = 0$ and by assumption, $m \in Ker\, gf \subseteq \mathrm{Rad}(M)$, i.e. $Ker\, f \subseteq \mathrm{Rad}(M)$.

Let $n \in Ker\, g$. Since f is an epimorphism there exists an element m of M such that $f(m) = n$. Then $(gf)(m) = g(n) = 0$ and hence $m \in Ker\, gf \subseteq \mathrm{Rad}(M)$, which implies $n = f(m) \in f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(N)$ by [8, 21.6]. Hence $Ker\, g \subseteq \mathrm{Rad}(N)$. $\square$

Theorem 2.2. *An epimorphism $f : M \to N$ is a generalized cover if and only if for every homomorphism $h : L \to M$ such that $fh : L \to N$ is epic, $h(L)$ is a weak generalized supplement of $Ker\, f$.*

Proof. ($\Rightarrow$) Let $f : M \to N$ be a generalized cover and let $m \in M$. Since fh is epic there exists $l \in L$ such that $f(m) = (fh)(l)$. Then $m - h(l) \in Ker\, f$ and hence $m \in h(L) + Ker\, f$, which means that $M = Ker\, f + h(L)$. By assumption, $Ker\, f \bigcap h(L) \subseteq \mathrm{Rad}(M)$ and so $h(L)$ is a weak generalized supplement of $Ker\, f$.

($\Leftarrow$) It is clear that $1_M f = f$ is epic, for the identity homomorphism $1_M : M \to M$. By the hypothesis, $1_M(M) = M$ is a weak generalized supplement of $Ker\, f$, that is, $Ker\, f \subseteq \mathrm{Rad}(M)$. Hence $f : M \to N$ is a generalized cover. $\square$

Proposition 2.3. *Any homomorphic image of a WGS-module is a WGS-module.*

Proof. Let $f : M \to N$ be a homomorphism and M be a $WGS-$module. Suppose that U is a submodule of $f(M)$. Then $f^{-1}(U)$ is a submodule of M. Since M is a WGS-module, $f^{-1}(U)$ has a weak generalized supplement V in M, i.e. $f^{-1}(U) + V = M$ and $f^{-1}(U) \bigcap V \subseteq \mathrm{Rad}(M)$. Then $f\left(f^{-1}(U)\right) + f(V) = f(M)$. It follows that $U + f(V) = f(M)$. Note that $U \bigcap f(V) = f\left(f^{-1}(U) \bigcap V\right) \subseteq f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(f(M))$ by [8, 23.2]. Hence $f(M)$ is a WGS-module. $\square$

3. Generalized $\oplus$-supplemented modules

Recall from [6] that a module M is called $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M. Clearly $\oplus$-supplemented modules are supplemented.

In this section, we define the concept of generalized $\oplus$-supplemented modules, which is adapted from Xue's generalized supplemented modules, and give the properties of these modules.

Definition 3.1. *A module M is called* generalized $\oplus$-supplemented *if every submodule of M has a generalized supplement that is a direct summand of M.*

Clearly $\oplus$-supplemented modules are generalized $\oplus$-supplemented. Also, finitely generated generalized $\oplus$-supplemented modules are $\oplus$-supplemented by [8, 19.3], but it is not generally true that every generalized $\oplus$-supplemented module is $\oplus$- supplemented. Let R be a non-local dedekind domain with quotient field K. Then the module K is generalized $\oplus$-supplemented, but it is not $\oplus$-supplemented. If K is $\oplus$-supplemented, R is a local ring by [10]. This is a contradiction by assumption. Later we shall give other examples of such modules (see Example 3.11).

To prove that a finite direct sum of generalized $\oplus$-supplemented modules is generalized $\oplus$-supplemented, we use the following standard lemma (see [8, 41.2]).

Lemma 3.2. *Let N and K be submodules of a module M such that $N+K$ has a generalized supplement X in M and $N \bigcap (K+X)$ has a generalized supplement Y in N. Then $X+Y$ is a generalized supplement of K in M.*

Proof. Let X be a generalized supplement of $N+K$ in M. Then $M = (N+K)+X$ and $(N+K) \bigcap X \subseteq \mathrm{Rad}\,(X)$. Since $N \bigcap (K+X)$ has a generalized supplement Y in N, we have $N = N \bigcap (K+X) + Y$ and $(K+X) \bigcap Y \subseteq \mathrm{Rad}\,(Y)$. Then

$$M = N+K+X = \Big[N \bigcap (K+X)+Y\Big] + K + X = K + (X+Y)$$

and

$$\begin{aligned} K \bigcap (X+Y) &\leq X \bigcap (K+Y) + Y \bigcap (K+X) \\ &\leq X \bigcap (K+N) + Y \bigcap (K+X) \\ &\leq \mathrm{Rad}\,(X) + \mathrm{Rad}\,(Y) \\ &\leq \mathrm{Rad}\,(X+Y)\,. \end{aligned}$$

Hence $X+Y$ is a generalized supplement of K in M. □

Theorem 3.3. *For any ring R, any finite direct sum of generalized $\oplus$-supplemented R-modules is generalized $\oplus$-supplemented.*

Proof. Let n be any positive integer and M_i $(1 \leq i \leq n)$ be any finite collection of generalized $\oplus$-supplemented R-modules. Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$.

Suppose that $n = 2$, that is, $M = M_1 \oplus M_2$. Let K be any submodule of M. Then $M = M_1 + M_2 + K$ and so $M_1 + M_2 + K$ has a generalized supplement 0 in M. Since M_1 is generalized $\oplus$- supplemented,

$M_1 \bigcap (M_2+K)$ has a generalized supplement X in M_1 such that X is a direct summand of M_1. By Lemma 3.2, X is a generalized supplement of M_2+K in M. Since M_2 is generalized $\oplus$-supplemented, $M_2 \bigcap (K+X)$ has a generalized supplement Y in M_2 such that Y is a direct summand of M_2. Again applying Lemma 3.2, we have that $X+Y$ is a generalized supplement of K in M. Since X is a direct summand of M_1 and Y is a direct summand of M_2, it follows that $X \oplus Y$ is a direct summand of M. The proof is completed by induction on n. □

We prove the following proposition, which is a modified form of Proposition 2.5 in [3]. We need the following lemma.

Lemma 3.4. *Let M be a module and N be a submodule of M. If U is a generalized supplement of N in M, then $\frac{U+L}{L}$ is a generalized supplement of $\frac{N}{L}$ in $\frac{M}{L}$ for every submodule L of N.*

Proof. By the hypothesis, $M = N+U$ and $U \bigcap N \subseteq \mathrm{Rad}\,(U)$. Hence $\frac{M}{L} = \frac{N}{L} + \frac{U+L}{L}$ for every submodule L of N. Consider that the natural epimorphism $\phi : N \to \frac{N}{L}$. Then by [8, p. 191], $\phi\,(\mathrm{Rad}\,(U)) \subseteq \mathrm{Rad}\left(\frac{U+L}{L}\right)$. Since $U \bigcap N \subseteq \mathrm{Rad}\,(U)$ it follows that

$$\frac{N}{L} \bigcap \frac{U+L}{L} = \frac{L+(N \bigcap U)}{L} =$$
$$= \phi\left(N \bigcap U\right) \subseteq \phi\,(\mathrm{Rad}\,(U)) \subseteq \mathrm{Rad}\left(\frac{U+L}{L}\right).$$

Hence $\frac{U+L}{L}$ is a generalized supplement of $\frac{N}{L}$ in $\frac{M}{L}$. □

Proposition 3.5. *Let M be a nonzero generalized $\oplus$-supplemented R-module and let U be a submodule of M such that $f\,(U) \leq U$ for each $f \in End_R\,(M)$. Then*

(1) *The factor module $\frac{M}{U}$ is generalized $\oplus$-supplemented.*

(2) *If, moreover, U is a direct summand of M, then U is also generalized $\oplus$-supplemented.*

Proof. (1) Let $\frac{L}{U}$ be any submodule of $\frac{M}{U}$. Since M is generalized $\oplus$-supplemented, there exist submodules N and N' of M such that $M = L+N$, $L \bigcap N \subseteq \mathrm{Rad}\,(N)$ and $M = N \oplus N'$. By Lemma 3.4, $\frac{N+U}{U}$ is a generalized supplement of $\frac{L}{U}$ in $\frac{M}{U}$. Since $f\,(U) \leq U$ for each $f \in End_R\,(M)$, it follows from [3, Lemma 2.4] that $U = (U \bigcap N) \oplus (U \bigcap N')$. Hence $(N+U) \bigcap (N'+U) \leq U$ and so $\frac{N+U}{U} \bigcap \frac{N'+U}{U} = 0$, i.e. $\frac{N+U}{U}$ is a direct summand of $\frac{M}{U}$. Thus $\frac{M}{U}$ is generalized $\oplus$-supplemented.

(2) Let U be a direct summand of M and let X be a submodule of U. Since M is generalized $\oplus$-supplemented, there exist submodules Y and Y' of M such that $M = X + Y$, $X \bigcap Y \subseteq \mathrm{Rad}\,(Y)$ and $M = Y \oplus Y'$. Hence $U = X + (U \bigcap Y)$. Again applying [3, Lemma 2.4], we have that $U = (U \bigcap Y) \oplus (U \bigcap Y')$. Now we show that $X \bigcap (U \bigcap Y) = X \bigcap Y \subseteq \mathrm{Rad}\,(U \bigcap Y)$. Let m be any element of $X \bigcap Y$. Then $m \in \mathrm{Rad}\,(Y)$ and so Rm is small in Y. Since U is a direct summand of M, by [8, 19.3], Rm is small in U. Again by [8, 19.3], Rm is also small in $U \bigcap Y$ because $U \bigcap Y$ is direct summand of U. Hence $m \in \mathrm{Rad}\,(U \bigcap Y)$. Consequently, U is generalized $\oplus$-supplemented. □

Corollary 3.6. *Let M be a nonzero generalized $\oplus$-supplemented module. If* $\mathrm{Rad}\,(M)$ *is a direct summand of M, then* $\mathrm{Rad}\,(M)$ *is also generalized $\oplus$-supplemented.*

For a positive integer n, the modules M_i $(1 \leq i \leq n)$ are called *relatively projective* if M_i is M_j-projective for all $1 \leq i \neq j \leq n$.

Theorem 3.7. *Let M_i $(1 \leq i \leq n)$ be any finite collection of relatively projective modules and let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. Then M is generalized $\oplus$-supplemented module if and only if M_i is generalized $\oplus$-supplemented for each $1 \leq i \leq n$.*

Proof. $(\Leftarrow)$ It follows from Theorem 3.3.

$(\Rightarrow)$ Clearly, it suffices to prove that M_1 is generalized $\oplus$-supplemented. Let U be any submodule of M_1. Since M is generalized $\oplus$-supplemented, there exist submodules V and V' of M such that $M = U + V$, $U \bigcap V \subseteq \mathrm{Rad}\,(V)$ and $M = V \oplus V'$. By [6, Lemma 4.47], there exists a submodule V_1 of V such that $M = M_1 \oplus V_1$. Then $V = (M_1 \bigcap V) \oplus V_1$ and so $M_1 \bigcap V$ is a direct summand of M_1. Now $U \bigcap (M_1 \bigcap V) = U \bigcap V \subseteq \mathrm{Rad}\,(V)$ and thus $U \bigcap V \subseteq \mathrm{Rad}\,(M_1 \bigcap V)$ because $M_1 \bigcap V$ is a direct summand of V. Hence M_1 is generalized $\oplus$-supplemented. □

Let R be a ring and M be an R-module. We consider the following condition.

$(D3)$ If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \bigcap M_2$ is also a direct summand of M (see [6, p. 57]).

Proposition 3.8. *Let M be a generalized $\oplus$-supplemented module with $(D3)$. Then every direct summand of M is generalized $\oplus$-supplemented.*

Proof. Let N be a direct summand of M and U be a submodule of N. Then there exists a direct summand V of M such that $M = U + V$ and

$U \bigcap V \subseteq \operatorname{Rad}(V)$. It follows that $N = U + (N \bigcap V)$. Since M has $(D3)$ $N \bigcap V$ is a direct summand of M and so it is also a direct summand of N. Note that $U \bigcap (N \bigcap V) = U \bigcap V \subseteq \operatorname{Rad}(V)$. Since $N \bigcap V$ is a direct summand of M, it follows that $U \bigcap V \subseteq \operatorname{Rad}(N \bigcap V)$. Hence N is generalized $\oplus$-supplemented. □

Proposition 3.9 (see [2, Proposition 2.10]). *Let M be a $\oplus$-supplemented module. Then $M = M_1 \oplus M_2$, where M_1 is a module with $\operatorname{Rad}(M_1)$ small in M_1 and M_2 is a module with $\operatorname{Rad}(M_2) = M_2$.*

We give an analogous characterization of this fact for generalized $\oplus$-supplemented modules.

Proposition 3.10. *Let M be a generalized $\oplus$-supplemented module. Then $M = M_1 \oplus M_2$, where M_1 is a module with $\operatorname{Rad}(M_1) = M_1 \bigcap \operatorname{Rad}(M)$ and M_2 is a module with $\operatorname{Rad}(M_2) = M_2$.*

Proof. Since M is generalized $\oplus$-supplemented, there exist submodules M_1 and M_2 of M such that $M = \operatorname{Rad}(M) + M_1$, $\operatorname{Rad}(M) \bigcap M_1 \subseteq \operatorname{Rad}(M_1)$ and $M = M_1 \oplus M_2$. Then $\operatorname{Rad}(M_1) = M_1 \bigcap \operatorname{Rad}(M)$ and $M = M_1 \oplus \operatorname{Rad}(M_2)$. It follows that $M_2 = \operatorname{Rad}(M_2)$. □

Now we give some examples of module, which is generalized $\oplus$-supplemented, but not $\oplus$-supplemented.

Example 3.11. Let M be a non-torsion $\mathbb{Z}$-module with $\operatorname{Rad}(M) = M$. It is clear that $M = \operatorname{Rad}(M)$ is a generalized supplement of every submodule of M. Hence M is generalized $\oplus$-supplemented, but M is not supplemented by [10].

Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}}$, for any prime p. Note that M has a unique maximal submodule, i.e. $\operatorname{Rad}(M) \neq M$. By Theorem 3.3, M is generalized $\oplus$-supplemented. If M is $\oplus$-supplemented, then $\mathbb{Q}$ is supplemented. It is a contradiction by [10].

Theorem 3.12. *Let M be a module with $(D3)$. Then the following statements are equivalent.*

(1) *M is generalized $\oplus$-supplemented.*

(2) *Every direct summand of M is generalized $\oplus$-supplemented.*

(3) *There exists decomposition $M = M_1 \oplus M_2$ such that M_1 is semisimple and M_2 is a generalized $\oplus$-supplemented module with $\operatorname{Rad}(M_2)$ essential in M_2.*

(4) *There exists a decomposition* $M = M_1 \oplus M_2$ *of* M *such that* M_1 *is a generalized* $\oplus$*-supplemented module and* M_2 *is a module with* $\mathrm{Rad}\,(M_2) = M_2$.

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 3.8.

(2) $\Rightarrow$ (3) By [7, Proposition 2.3], $M = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is a module with $\mathrm{Rad}\,(M_2)$ essential in M_2. By (2), M_2 is a generalized $\oplus$-supplemented.

(3) $\Rightarrow$ (1) By Theorem 3.3, M is generalized $\oplus$-supplemented.

(1) $\Rightarrow$ (4) By Proposition 3.10, there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$ and $\mathrm{Rad}\,(M_2) = M_2$. Since M has $(D3)$, by Proposition 3.8, M_1 is generalized $\oplus$-supplemented.

(4) $\Rightarrow$ (1) Since $\mathrm{Rad}\,(M_2) = M_2$, M_2 is generalized $\oplus$-supplemented. By (4) and Theorem 3.3, M is generalized $\oplus$-supplemented. □

A ring R is *semiperfect* if $\frac{R}{\mathrm{Rad}(R)}$ is semisimple and idempotents can be lifted modulo $\mathrm{Rad}(R)$. It is known that a ring R is semiperfect if and only if every simple left R-module has a projective cover (see [8, 42.6]). Therefore it is shown in [4, Theorem 2.1] that R is semiperfect if and only if every finitely generated free R-module is $\oplus$-supplemented.

Remark 3.13. For a ring R if every finitely generated free R-module is generalized $\oplus$-supplemented, then R is semiperfect. If ${}_RR$ is generalized $\oplus$-supplemented, ${}_RR$ is $\oplus$-supplemented because ${}_RR$ is a finitely generated R-module. It follows from [4, Theorem 2.1] that R is semiperfect.

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