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RESEARCH ARTICLE

On the generators of the kernels of hyperbolic group presentations

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Abstract. In this paper we prove that if $\mathcal{R}$ is a (not necessarily finite) set of words satisfying certain small cancellation condition in a hyperbolic group G then the normal closure of $\mathcal{R}$ is free. This result was first presented (for finite set $\mathcal{R}$) by T. Delzant [Delz] but the proof seems to require some additional argument. New applications of this theorem are provided.

1. Introduction

In the founding paper [Gro] M. Gromov defined the notion of hyperbolic groups and outlined a number of research directions in this (now well established) area. In particular, one finds the following Statement 5.3E in [Gro]:

There exists a constant $m = m(k, \delta)$ such that for every k hyperbolic elements $x_1, \ldots, x_k$ in a word δ–hyperbolic group G the normal subgroup generated by $x_1^{m_1}, \ldots, x_k^{m_k}$ is free for all $m_i \geq m$.

Although not correct in full generality (as a counter-example in the appendix to [Delz] shows) the following theorems are true:

Theorem 1.1 (Delzant [Delz], Theoreme I)**.** *Let G be a non-elementary hyperbolic group. There exists an integer N such that for any elements $f_1, \ldots, f_n$ such that $[[f_i]] = [[f_j]] \geq 1000\delta$ (where $[[f]] = lim_{n\to\infty} \frac{|f^n|}{n}$), the normal subgroup $\mathcal{N}(f_1^{kN}, \ldots, f_n^{kN})$ is free for every k. Moreover, (for every k) the group $G/\mathcal{N}(f_1^{kN}, \ldots, f_n^{kN})$ is hyperbolic.*

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The Theorem 1.1 is obtained in [Delz] from Theorem 1.2 by arguing that for sufficiently large N (independent of choice f_i) the system of relations $f_1^N, \ldots, f_n^N$ can be completed to that satisfying small cancellation $C'(\mu)$ (see definition 2.9).

Theorem 1.2 (Delzant [Delz], Theoreme II)**.** *Let $\mathcal{R}$ be a finite set of elements satisfying the the small cancellation condition $C'(\mu)$. A normal subgroup $\mathcal{N}(\mathcal{R})$ generated by $\mathcal{R}$ is free. The quotient $G/\mathcal{N}(\mathcal{R})$ is hyperbolic.*

However we think that the proof of Theorem 1.2 requires some additional arguments. To be more precise, the proof of the Theorem 2.1 (iii) [Delz] pp 677-678 (stating that if a (finite) system $\mathcal{R}$ satisfies condition $C'(\mu),\ \mu < 1/8$ then the normal subgroup $\mathcal{N}(\mathcal{R})$ generated by $\mathcal{R}$ is free) is incomplete. We provide a proof of essentially the same fact in somewhat different setting (in particular, the set $\mathcal{R}$ can be infinite) using both techniques of Delzant (such as Lemmas 2.11, 2.15) and diagram techniques of A. Olshanskii from [Olsh], [Olsh93]. We would like to note that the Lemma 5.10 of this paper provides justification for the formula on top of page 678 of [Delz]. One may replace Theorem 1.2 with the following statement:

Theorem 1.3. *There exists $\mu_0 > 0$ such that for any $\mu < \mu_0$ there are ϵ and ρ such that if $\mathcal{R}$ is a set of geodesic words satisfying $\tilde{C}(\epsilon, \mu, \rho$–condition (see Definition 3.9) in the hyperbolic group G then:*

(i) the normal subgroup $\mathcal{N} = \mathcal{N}(\mathcal{R})$ is free;

(ii) if G is non-elementary and $\mathcal{R}$ is finite then $G/\mathcal{N}(\mathcal{R})$ is non-elementary hyperbolic.

As a corollary we get:

Theorem 1.4. *Let G be a non-elementary hyperbolic group. For any finite set of elements $x_1, \ldots, x_m$ there exists an integer N such that the normal closure $\mathcal{N} = \mathcal{N}(x_1^{s_1 N}, \ldots, x_m^{s_m N})$ in G of elements $x_1^{s_1 N}, \ldots, x_m^{s_m N}$ is free for any integer $s_i > 0$ and the quotient $G/\mathcal{N}$ is non-elementary hyperbolic.*

Let us note that in our result 1.4, the choice of constant N depends on the elements $x_1, \ldots, x_m$ rather then being an absolute constant as in Theorem 1.1. On the other hand we do not assume any significant restrictions on the set of elements $x_1, \ldots, x_m$.

The following corollary somewhat strengthens the theorem proved by T. Delzant and A. Olshanskii independently (see [Delz], [Olsh95]) stating that every non-elementary hyperbolic group is SQ–universal.

Corollary 1.5. *Let G be a hyperbolic group. Then:*

(i) there exists a free normal subgroup $\mathcal{N}$ of G of rank greater then 1;

(ii) for any free normal subgroup $\mathcal{N}$ of rank greater then 1 and any countable group H there exists a free subgroup $M < \mathcal{N}$, $M \triangleleft G$ such that H embeds in quotient G/M.

In conclusion we would like to mention the following

Open problem ([Kour], 15.69). *Does every hyperbolic group G have a free normal subgroup N such that the quotient G/N is a torsion group of bounded exponent?*

The above problem is motivated by the result of Ivanov and Olshanskii [IvOl] stating that for every non-elementary hyperbolic group G there is a number $n = n(G)$ such that the quotient group G/G^n is infinite.

2. Hyperbolic spaces and hyperbolic groups

Hyperbolic Spaces. We recall some definitions and properties from the founding article of Gromov [Gro]. Let $(X, |\ |)$ be a metric space. We sometimes denote the distance $|x-y|$ between $x, y \in X$ by $d(x, y)$. We assume that X is geodesic, i.e. every two points can be connected by a geodesic line. We refer to a geodesic between some point x, y of X as $[x, y]$. For convenience we denote $|x|$ distance $|x - y_0|$ to some fixed point y_0 (usually the identity element of the group).

For a path γ in X we denote the initial (terminal) vertex of γ by γ_- (γ_+), denote by $\|\gamma\|$ the length of path γ and by $|\gamma|$ the distance $|\gamma_+ - \gamma_-|$. Recall that if $0 < \lambda \leq 1$ and $c \geq 0$ then a path γ in X is called (λ, c)*–quasigeodesic* if for every subpath γ_1 of γ the following inequality is satisfied:

$\|\gamma_1\| \leq \frac{1}{\lambda}|\gamma_1| + c.$

We call the path γ *geodesic up to* c, if it is $(1, c)$-quasigeodesic.

Define a scalar (Gromov) product of x, y with respect to z by formula

$$\langle x, y\rangle_z = \frac{1}{2}(|x-z| + |y-z| - |x-y|).$$

We call the space X δ–hyperbolic if there exists a non-negative integer δ such that the following inequality holds:

$$\forall x, y, z, t \in X, \quad \langle x, y\rangle_z \geq min(\langle x, t\rangle_z, \langle y, t\rangle_z) - \delta.$$

We will need a few properties of hyperbolic groups and Gromov products:

Lemma 2.1 ([Delz], Lemma 1.3.3)**.** *Let K be a nonnegative real number, $[x, y]$ and $[x', y']$ – two segments in a δ–hyperbolic space of length at least $2K + 20\delta$ and suppose that $|x - x'| \leq K$, $|y - y'| \leq K$. Choose points u and v on $[x, y]$ at distance $K + 2\delta$ from x and y respectively. Then every point P on $[u, v]$ is in the 6δ-neighborhood of the segment $[x', y']$.*

Lemma 2.2 ([Ghys], Chapter 3, §17)**.** *For any three points x, y, z in a δ–hyperbolic space X, we have $d(x, [y, z]) - \delta \leq \langle y, z\rangle_x \leq d(x, [y, z])$.*

We will use the following easy remark.

Remark 2.3. Let X be a hyperbolic space. Then:

(i) In the notations of Lemma 2.1 it is immediate that the segment $[x, y]$ is within $K + 2\delta + 6\delta$–neighborhood of $[x', y']$.

(ii) Suppose γ is a path, geodesic up to some $c \geq 0$ in X, and o is an arbitrary point on γ. Then

$$\langle \gamma_-, \gamma_+ \rangle_o \leq c/2. \tag{1}$$

Combining the previous inequality with Lemma 2.2 we get that:

$$d(o, [\gamma_-, \gamma_+]) - \delta \leq c/2. \tag{2}$$

We recall the notion of the *metric tree* T ([Ghys], Chapter 2, §1). Let T' be a tree (i.e. graph without cycles), we construct the geometric realization T in the following way. For every edge a of T' we choose a real positive number $l(a)$. Then there exists a unique (up to isometry) metric d on T maximal with respect to the following condition: edge a is isometric to interval $[0, l(a)]$ on the real line. Then T with the metric d is a metric tree.

Various versions of the following Gromov's theorem provide an approximation of a finite set of geodesics in hyperbolic space by metric trees:

Theorem 2.4 ([Ghys], Chapter 2, Theorem 12)**.** *Let F be a δ-hyperbolic metric space. Suppose that $F = \cup_{i=1}^{n} F_i$, where each $F_i = [w, w_i]$ is a geodesic and $n \leq 2^k$.*

Then there exists a metric tree T and function $\Phi : F \to T$ such that
(i)$|[\Phi(x), \Phi(w)]| = |[x, w]|, \ \forall x \in F$;
(ii)$|x - y| - 2(k+1)\delta \leq |\Phi(x) - \Phi(y)| \leq |x - y|$ for all $x, y \in F$.

It is clear that if x is some vertex in a metric graph T in the theorem above then either

(i) there exist some indexes i, j such that the images of F_i and F_j under Φ depart at x: $\Phi([w, w_i]) \cap \Phi([w, w_j]) = [\Phi(w), x]$ (in this case we call vertex x a branching point), or

(ii) there exists some index i such that $\Phi(w_i) = x$ or $\Phi(w) = x$. In this case we call x a leaf (because it is adjacent to a single vertex).

When we talk about an approximation tree for a set of vertices $w, w_1, \ldots, w_n$ in the hyperbolic space X, we mean an approximation of the set $F = \cup_{i=1}^{n} F_i$ in the sense of the previous theorem.

By a tripod we mean a metric tree with one branching point (center o) and three edges (pods).

Remark 2.5 ([Ghys] Chapter 2, §1)**.** Let x, y, z be some points in a δ–hyperbolic space X, and o_1 be a point on $[x, y]$ at distance $s \leq \langle y, z\rangle_x$ from x, o_2 be on $[x, z]$ at distance s from x. Then there exists a tripod T and a map $\Phi : [x, y] \cup [x, z] \longrightarrow T$ such that:

(i) a restriction of the map Φ on each segment $[x, y], [x, z]$ is an isometry which sends x, y, z to different ends of pods of T and $\Phi(o_1) = \Phi(o_2)$;

(ii) Φ, T satisfies the previous theorem.

Hyperbolic Groups. Let G be a finitely presented group with presentation $gp(S|\mathcal{D})$. We consider G as a metric space with respect to the distance function $|g - h| = \left|gh^{-1}\right|$ for every g and h. We denote by $|g|$ the length of a minimal (geodesic) word with respect to the generators S equal to g. The notation $\langle g, h\rangle$ is the Gromov product $\langle g, h\rangle_e$ with respect to the identity vertex 1.

We denote the (right) Cayley graph of the group by $Cay(G)$. Graph $Cay(G)$ has a set of vertices G, and a pair of vertices g_1, g_2 is connected by an edge of length 1 labeled by s if and only if $g_1^{-1}g_2 = s$ in G for some $s \in S^{\pm1}$. We define a label function on paths in $Cay(G)$. By a path in $Cay(G)$ we mean a path $p = p_1 \ldots p_n$, where p_i is an edge between some g_i, g_{i+1} for every $1 \leq i \leq n$. We can define a label $lab(p)$ (a word in alphabet $S^{\pm1}$) by:

$lab(p) = lab(p_1) \ldots lab(p_n)$.

It is clear that $Cay(G)$ may be considered as a geodesic space: we may identify every edge of $Cay(G)$ with interval $[0, 1]$ and choose a maximal metric d which agrees with metric on every edge.

We have assigned a unique word $lab(p)$ to the path p in $Cay(G)$. On the other hand for every word w in alphabet $S^{\pm1}$ there exists a unique path p in $Cay(G)$ starting from the identity vertex with label w. Hence there is a one-to-one correspondence between paths with initial vertex 1 (the identity vertex in G) and words in alphabet $S^{\pm1}$, so we will not distinguish between a word in the alphabet $S^{\pm1}$ and it's image in $Cay(G)$ – a path starting from the identity vertex. Thus, when considering some words X, Y, Z in the alphabet $S^{\pm1}$, we can talk about the path $\gamma = XYZ$ in the Cayley graph of G originating in the identity vertex 1. To distinguish a path Y with initial vertex 1 from the subpath of γ with label Y we denote the latter as $_\gamma Y$ (similar notations will be used for paths in van Kampen diagrams, see Section 3). We will talk about values $|X|, \|X\|$ for a word X in alphabet over $S^{\pm1}$ meaning these values on corresponding paths in $Cay(G)$.

A group G is called *δ-hyperbolic* for some $\delta \geq 0$, if it's Cayley graph is δ-hyperbolic. It is well known that hyperbolicity of the group does not depend on choice of a finite presentation of the group G (while δ does depend on presentation).

In this section we recall some definitions and lemmas from [Delz], but with certain modifications. We would like to formulate all the statements in the language of (geodesic) and cyclically reduced words rather then group elements and cyclically reduced group elements (element g of the group G is called *a cyclically reduced element* if g has a minimal length in it's conjugacy class in G). The proofs of these lemmas can be repeated while changing the terminology.

We first recall the following lemmas:

Lemma 2.6 ([Delz], Lemma 1.2.1)**.** *Let V, W be geodesic words in G; their scalar product is an integer or $\frac{1}{2}$ times integer. If $V \equiv AB$ such that $|A| = [\langle V, W\rangle_1]$ and C is defined by equality $AC = W$ in G then the path AC is geodesic up to constant 2δ (we denote by $[x]$ a maximal integer smaller or equal to x).*

Lemma 2.7 ([Delz] Lemma 1.5.1)**.** *Let V be a geodesic word in G which is shortest in it's conjugacy class and of length no less then 20δ. Assume that W is conjugate to V. Then there exists a geodesic word U and a cyclic conjugate V' of V such that $W = UV'U^{-1}$ and the path $UV'U^{-1}$ is geodesic up to 10δ.*

Let us mention the following property of metric trees with finite number of vertices. If a metric tree T is a union of n segments $\cup_{i=1}^{n}[l_0, l_i]$ originating from a fixed vertex w_0, it is easy to see that an addition of a new segment $[l_0, l_{n+1}]$ to T can increase the number of edges by at most 2. To be more precise we can prove by induction on n that $|E(T)| \leq 2n - 1$, where $E(T)$ is a set of edges in T.

The proposition below provides a "pull-back" of the tree approximation T for the set F in the situation of Theorem 2.4 in the original hyperbolic space X. It will be formulated for hyperbolic groups. In order to formulate this proposition we need to add some edges of zero length to $E(T)$. The reason for this adjustment is that a trivial edge in T may correspond to a nontrivial group word ("edge in the pullback tree") in the Proposition 2.8. For every $k \leq n$ we consider a subtree $T_k = \Phi(\cup_{s=1}^{k}[w_0, w_s])$. For every $i \leq n$, if $\Phi(w_i) \in T_{i-1}$, then we add to the set of edges $E(T)$ a new edge of zero length $[\Phi(w_i), \Phi(w_i)]$. The inequality $|E(T)| \leq 2n - 1$ still holds if we take into account edges of zero length. We choose an (arbitrary) orientation on every edge $\alpha \in E(T)$. When we consider a segment $[\Phi(w_i), \Phi(w_j)] = \alpha_{s_1}^{\epsilon_1} \dots \alpha_{s_m}^{\epsilon_m}$ ($\alpha_{s_i} \in E(T)$) in Proposition 2.8 such that a zero length edge was defined for i (for j), we assume that α_{s_1} is the edge $[\Phi(w_i), \Phi(w_i)]$ (respectively, α_{s_m} is the edge $[\Phi(w_j), \Phi(w_j)]$). After described conventions, we may formulate the following:

Proposition 2.8 ([Delz] Lemma 1.3.2)**.** *Let $g_0, g_1, \dots, g_n$ be elements in G, $n \leq 2^k$ and let Φ, T be the corresponding approximation tree and* *function provided by Theorem 2.4. Denote by* $E(T) = \{\alpha_1, \dots, \alpha_{2n-1}\}$ *the set of edges of* T*. Let* W *be a geodesic word such that* $W = g_0^{-1}g_1$ *in* G*. Then there exist geodesic words* $A_1, \dots, A_{2n-1}$ *in* G *satisfying the following properties:*

(i)$||\alpha_i| - |A_i|| \leq 2\delta(k+1)+2$.

(ii) If the geodesic $[\Phi(g_i), \Phi(g_j)]$ *is a path* $\alpha_{s_1}^{\epsilon_1} \dots \alpha_{s_m}^{\epsilon_m}$ *in the tree* T*, then* $g_i^{-1}g_j = A_{s_1}^{\epsilon_1} \dots A_{s_m}^{\epsilon_m}$ *in* G *,* $\epsilon_i = \pm 1$ *and* $A_{s_1}^{\epsilon_1} \dots A_{s_m}^{\epsilon_m}$ *is geodesic up to* $n(2\delta(k+1)+2)$*.*

(iii) The word $A_{s_1}^{\epsilon_1} \dots A_{s_m}^{\epsilon_m}$ *defined in (ii) for* $g_0^{-1}g_1$ *is geodesic and* $W \equiv A_{s_1}^{\epsilon_1} \dots A_{s_m}^{\epsilon_m}$*.*

Small cancellation properties on the Cayley graph of hyperbolic groups. The following definitions can be found in [LSch]. We call the set of words $\mathcal{R}$ *symmetrized* if it is a set of freely cyclically reduced words in alphabet $S^{\pm 1}$, i.e.

(i) $R \in \mathcal{R} \implies R^{-1} \in \mathcal{R}$,

(ii)$R \in \mathcal{R}, R \equiv R_1R_2 \implies R_2R_1 \in \mathcal{R}$.

We will sometimes talk about cyclic word R meaning R or one of it's cyclic conjugates. Denote by G_1 the factor group $G/\mathcal{N}(\mathcal{R})$ of G by the normal closure (in G) of the set $\mathcal{R}$. For a pair of words X, Y in the alphabet $S^{\pm 1}$ let us denote by $X \equiv Y$ a letter-by-letter equality of X and Y.

Definition 2.9. *Let* $\mathcal{R}$ *be a symmetrized set of geodesic words in the* δ*–hyperbolic group* G *and* $\mu < 1/8$*. Assume furthermore that every* $R \in \mathcal{R}$ *is a cyclically reduced element of* G*. The family* $\mathcal{R}$ *satisfies a small cancellation condition* $C'(\mu)$ *if:*

(i) For every words A, B *in* G *,* $|A|, |B| \leq 100\delta$*,* $\forall R_1, R_2 \in \mathcal{R}$*, if* $\langle AR_1B, R_2\rangle > \mu min(|R_1|, |R_2|)$*, then* $R_2 = AR_1A^{-1}$ *in* G*;*

(ii) $min_{R \in \mathcal{R}}(|R|) \geq 5000\delta/(1-8\mu)$*.*

The previous definition is essentially the same as that in [Delz], 2.1 up to some adjustment of constants (the difference between them is that $b = 1$ in [Delz]).

Definition 2.10 ([Delz]). *We say that a geodesic word* U *of* G *contains more then half of a relation if there exists* $R \equiv r_1r_2$ *from* $\mathcal{R}$ *such that*

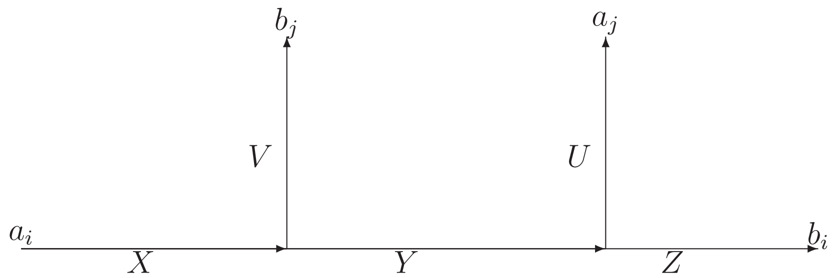
(i) $R \equiv r_1r_2$ *is geodesic,* $|r_1| \geq |r_2| + 60\delta$ *and*

(ii) U *equals to the word* $U_1r_1U_2$ *in* G*, which is geodesic up to* 50δ*.*

We denote the set of all geodesic words U *which do not contain more then half of a relation by* $\mathcal{U}$*.*

Lemma 2.11 ([Delz], Lemma 2.2). *Consider the set* $\mathcal{X}$ *of words* URU^{-1} *geodesic up to* 10δ *in* G *such that* U *does not contain more then half of a relation from* $\mathcal{R}$*. Then every element* g *in the normal closure* $\mathcal{N}(\mathcal{R})$ *is a product of words from* $\mathcal{X}$*.*

Figure 1:



The proof of the Lemma 2.11 follows immediately from the remark below.

Remark 2.12. (i) Suppose that a geodesic word U contains more then half of a relation (i.e. $U = U_1 r_1 U_2$ for some geodesic words U_1, U_2, r_1 satisfying Definition 2.10). Then

$URU^{-1} = (U_1 r_1 r_2 U_1^{-1})[(U_1 r_2^{-1} U_2) R (U_1 r_2^{-1} U_2)^{-1}](U_1 r_1 r_2 U_1^{-1})^{-1}$ in G and, evidently,

$\left|U_1 r_2^{-1} U_2^{-1}\right|, |U_1| < |U|$.

(ii) Suppose that $R \in \mathcal{R}$, and URU^{-1} is not geodesic up to 10δ. Then by Lemma 2.7 there exists $R' \in \mathcal{R}$ (so $|R| = |R'|$) and a geodesic word V such that $URU^{-1} = VR'V^{-1}$ in G and $VR'V^{-1}$ is geodesic up to 10δ.

We introduce some notation and conventions. Let g be an element in the normal closure of $\mathcal{R}$, choose n minimal such that

$$g = U_1 R_1 U_1^{-1} \dots U_n R_n U_n^{-1} \text{ with } U_i R_i U_i^{-1} \in \mathcal{X}.$$

Then we denote: $g_0 = 1,\ g_1 = U_1 R_1 U_1^{-1}, \dots, g_n = g$. Also we set $a_i = g_{i-1} U_i$ and $b_i = a_i R_i = g_i U_i$.

Assume that for some indices $i < j$ the approximation tree T for for vertices a_i, b_i, a_j, b_j is of shape on the Figure 1 (T is provided by Gromov's theorem 2.4 where $w = a_i, k = 2,\ n = 3$). For convenience we label vertices of the tree on Figure 1 by corresponding group elements. Proposition 2.8 provides us with with five geodesic words X, Y, Z, U, V such that $R_i = XYZ$, where XYZ is geodesic and $R_j = U^{-1} Y^{-1} V$, where $U^{-1} Y^{-1} V$ is geodesic up to $3(2 \cdot 3\delta + 2) = 18\delta + 6$. We label edges of the tree T with X, Y, Z, U, V for convenience of the reader. Note that Φ and T determine the exponents of X, Y, Z, U, V in equalities for R_i, R_j uniquely.

The following lemma is an application of the small cancellation, we provide a proof of it (following [Delz]) for future references.

Lemma 2.13 ([Delz], Lemma 2.3)**.** *Suppose that a fixed element g is equal to a word $W = U_1R_1U_1^{-1}\dots U_nR_nU_n^{-1}$ in G and that for some indices $i < j$ the tree approximation of vertices a_i, b_i, a_j, b_j in $Cay(G)$ (with geodesic words X, Y, Z, U, V provided by Proposition 2.8) has the shape on Figure 1.*

(i) Assume that n is a minimal possible number among all words W equal to g. Then the following inequality holds:

$$|Y| \leq \mu min(|R_i|\,;|R_j|) + 10\delta + 3. \tag{3}$$

(ii) If the equality (3) is violated then n is not minimal and the following equality holds in G:

$$U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_{j-1}R_{j-1}U_{j-1}^{-1} = U_iR_iU_i^{-1}\dots U_jR_jU_j^{-1}. \tag{4}$$

Proof. Assume that the inequality (3) does not hold. In notations used in Figure 1 we have $R_i = XYZ$ and XYZ is geodesic, $R_j = U^{-1}Y^{-1}V$, where the right-hand side is geodesic up to $3(2\cdot 3\delta + 2) = 18\delta + 6$. We consider the conjugate $R_i' = YZX$ of R_i, which is also geodesic: $|R_i'| \geq |R_i|$ (since R_i is a cyclically reduced geodesic word), but on the other hand $|R_i'| \leq |Y| + |Z| + |X| = |R_i|$. Consider also the conjugate $R_j' = YUV^{-1}$ of R_j^{-1} which is geodesic up to $3(2\cdot 3\delta+2) = 18\delta+6$ (we have $|R_j| \leq \left|R_j'\right| \leq |Y| + |U| + |V| \leq |R_j| + 18\delta + 6$).

By Lemma 2.7, there exists a geodesic word $R'' = AR_j'A^{-1}$ cyclically conjugate to R_j such that $2\,|A| + |R''| \leq \left|R_j'\right| + 10\delta$ and $|R''| = |R_j|$. Now the computation

$$2\,|A| + \left|R''\right| \leq \left|R_j'\right| + 10\delta \leq |R_j| + 28\delta + 6$$

implies that $|A| \leq 14\delta + 3$. We also have that $R'' \in \mathcal{R}$: it is a cyclic conjugate of R_j.

By definition of hyperbolicity, we have that

$$\langle R_i', R_j'\rangle \geq min(\langle Y, R_j'\rangle, \langle R_i', Y\rangle) - \delta.$$

Both Gromov products on the right side of the last equation are not greater then $|Y|$ and the second is actually equal to $|Y|$ because $R_i' = YZX$ is geodesic. So $\langle R_i', R_j'\rangle \geq \langle Y, R_j'\rangle - \delta = |Y| - \delta - \langle 1, R_j'\rangle_Y$, where the last equality follows from $\langle Y, R_j'\rangle_1 + \langle 1, R_j'\rangle_Y = |Y|$. Since $R_j' = YUV^{-1}$ is geodesic up to $18\delta + 6$ we have by inequality (1) that $\langle 1, R_j'\rangle_Y \leq 9\delta + 3$ and finally

$$\langle R_i', R_j'\rangle \geq |Y| - 10\delta - 3.$$

We hence obtained that $\langle AR''A^{-1}, R_i'\rangle \geq \mu min(|R_i|\,;|R_j|)$ and by the condition $C'(\mu)$ we get that $A^{-1}R''A = YUV^{-1} = R_i' = YZX$. Thus $UV^{-1} = ZX$, hence $Z^{-1}U = XV$ and so ${b_i}^{-1}a_j = a_i^{-1}b_j$, which in turn is equivalent to $U_i^{-1}g_i^{-1}g_{j-1}U_j = U_i^{-1}g_{j-1}^{-1}g_jU_j$ and hence $g_i^{-1}g_{j-1} = g_{i-1}^{-1}g_j$. Rewriting the last equality in the explicit form, we get precisely equation (4).

The left-hand side of the last equality contains fewer elements of $\mathcal{X}$ contrary to the minimality of number n for g. Contradiction. □

The following definition utilizes the lemma

Definition 2.14 ([Delz]). *A word (or, equivalently, a path in $Cay(G)$) $U_1R_1U_1^{-1}\dots U_nR_nU_n^{-1}$ is called* reduced *if for every pair of indices $i<j$ such that the approximating tree for a_i, b_i, a_j, b_j is of shape on Figure 1, the inequality (3) holds. If for a pair of indexes $i<j$ the tree approximation is of shape on Figure 1, the inequality (3) is violated, then we call $i<j$ a* reducible pair of indexes.

Note that if we switch the labels a_j and b_j on Figure 1, the pair $i<j$ will no longer be a reducible pair. The following corollary summarizes [Delz] Lemma 2.4.

Lemma 2.15. *Suppose G is hyperbolic and $\mathcal{R}$ satisfies $C'(\mu)$, $\mu \le 1/8$. Let $\gamma = \prod_{i=1}^{n} U_iR_iU_i^{-1}$ be a reduced path in $Cay(G)$, $U_iR_iU_i^{-1} \in \mathcal{X}$ and denote by $\bar{\gamma}$ some geodesic between γ_-, γ_+. Then there exist an index $1 \le i_0 \le n$, a subsegment x of geodesic segment ${}_\gamma R_{i_0}$ such that x is in 30δ-neighborhood of $\bar{\gamma}$ and $|x| \ge (1-3\mu)\,|R_{i_0}| - 1500\delta$.*

3. Diagrams and small cancellation

Suppose we are given a hyperbolic group G with a combinatorial presentation $G = gp(S|\mathcal{D})$. For technical purposes we assume that $\mathcal{D}$ contains all relations of the group G.

For $\epsilon \ge 0$ a subword U is called an *ϵ-piece* of a word R in a symmetrized set $\mathcal{R}$ with respect to G if there exists a word $R' \in \mathcal{R}$ such that

(i) $R \equiv UV,\ R' \equiv U'V'$ for some U', V', V;

(ii) $U' = YUZ$ in G for some words Y, Z where $\|Y\|, \|Z\| \le \epsilon$;

(iii) $YRY^{-1} \ne R'$ in the group G.

We say that the system $\mathcal{R}$ satisfies the $C(\epsilon, \mu, \rho)$–condition (with respect to G) for some $\epsilon \ge 0,\ \mu \ge 0,\ \rho \ge 0$ if

(i)$\|R\| \ge \rho$ for any $R \in \mathcal{R}$;

(ii) any word $R \in \mathcal{R}$ is geodesic;

(iii) for any ϵ-piece of any word $R \in \mathcal{R}$ the inequalities $\|U\|, \|U'\| < \mu\,\|R\|$ hold (using notations of the definition of the ϵ-piece).

Definition 3.1. *Consider a finite, two dimensional complex Δ with directed edges such that:*

(i) The underlying topological space of complex M is a disc with a boundary P.

(ii) For any path in Δ there defined a label function $\phi(*)$. If x is an edge in Δ, $\phi(x) \in S \cup S^{-1} \cup 1$ and $\phi(x^{-1}) = \phi(x)^{-1}$. For a path q in Δ, $q = q_1 ... q_n$, where q_i is an edge for every i, we define $\phi(q) = \phi(q_1)...\phi(q_n)$. If q is a simple closed path we choose a base vertex o and read off the labels of edges in the clockwise direction.

*(iii) A boundary label of any 2-cell of M is either an element of $\mathcal{R}$ (then we call it an $\mathcal{R}$-*face*) or has a label D where $D = 1$ in the hyperbolic group G ($\mathcal{D}$-*face*).*

We call the triple $(M, \phi(*), P)$ a *(disc) diagram Δ with respect to $gp(S|\mathcal{D} \cup \mathcal{R})$ with a boundary path P.*

Similarly we may define notions of *annular or spherical diagrams.*

For convenience we often fix a base point o of the diagram Δ – a vertex on one of the boundary components of Δ. We may also choose a base point o_1 on the boundary of a face Π and write $\partial_{o_1}\Pi = r$ where r is a simple closed boundary path of Π with a initial (terminal) vertex o_1.

Consider a path γ in Δ as a path in the underlying topological space M. We say that γ is a *simple path* in Δ if for every open set U in M containing γ there exists a homotopy (in U) from γ to a simple curve $\gamma' = \gamma'(U)$. A simple closed path γ in Δ bounds a subdiagram Δ_1 with boundary $\partial\Delta_1 = \gamma$ consisting of all edges, vertices and faces which are inside the simple closed curve $\gamma' = \gamma'(U)$ for every open set U containing γ. Subdiagrams Δ_1, Δ_2 are called disjoint if for every neighborhood of $\partial\Delta_1 \cup \partial\Delta_2$ (in the underlying space for Δ) there exists a homotopy inside U of $\partial\Delta_1$ to a simple γ_1 such that $\Delta_2 \cap \gamma_1 = \emptyset$.

The following operations (and their inverses) are referred to as *elementary transformations* of diagram Δ over G_1:

1. Let Π_1, Π_2 be $\mathcal{D}$-faces in Δ with a common boundary subpath p. Then we can erase p making Π_1, Π_2 into a single $\mathcal{D}$-face.

2. Let p be a simple path in Δ. Then we cut the diagram Δ along p (i.e. consider the path pp^{-1} as a new boundary component) and glue in a $\mathcal{D}$-face labeled by $\phi(p)\phi(p)^{-1}$.

It is clear that elementary transformations define an equivalence relation on the set of all reduced diagrams over G_1. We say that Δ is equivalent to Δ' if there exists a finite sequence of elementary transformations starting from Δ and ending with Δ'.

Definition 3.2 ([Olsh93]). *Let Π_1, Π_2 be different $\mathcal{R}$–faces of a diagram Δ having boundary labels R_1, R_2 reading in a clockwise direction, starting from vertices o_1, o_2 respectively. Suppose also that there exists a simple* *path t in Δ such that $t_- = o_1, t_+ = o_2$. Call* Π_1, Π_2 opposite *(with respect to the path t) if the following equality holds:*

$$\phi(t)^{-1} R_1 \phi(t) R_2 = 1 \text{ in } G. \tag{5}$$

If a diagram Δ contains no opposite faces then we call it reduced.

Lemma 3.3 (van Kampen, see [Olsh93])**.** *Let w_0 be an nonempty word in the alphabet S. Then $w_0 = 1$ in G_1 if and only if there exists a reduced disc diagram over $gp(S|\mathcal{D} \cup \mathcal{R})$ with boundary label equal to w_0.*

Let p be a path in Δ over G, define $\|p\| = \|\phi(p)\|$ and $|p| = |\phi(p)|$. We call a path p geodesic if $\|p\| = |p|$ (recall that $|p|$ equals the distance $|p_+ - p_-|$ in G).

One can define a map ϕ' (see [Olsh93], §5) from a disc diagram Δ over G with the base point o to Caley graph $Cay(G)$. Set $\phi'(o) = 1$, where 1 is the identity vertex of $Cay(G)$. For an arbitrary vertex a in Δ we define $\phi'(a)$ to be the vertex of $Cay(G)$ labeled by the geodesic word $\phi(p)$ where p is a path in Δ connecting o and a (it follows from the van Kampen Lemma that $\phi'(a)$ does not depend on the choice of p). If p is an edge in Δ labeled by $s \in S^{\pm 1}$, then define $\phi'(p)$ to be the edge labeled by s in Cayley graph $Cay(G)$ with vertices $\phi'(p_-), \phi'(p_+)$. If $\phi(p) \equiv 1$ for an edge p of Δ then $\phi'(p) = \phi'(p_-) = \phi'(p_+)$. One can verify that $|p| = |\phi'(p)|$, $\|p\| = \|\phi'(p)\|$ for any path p in diagram Δ over G ([Olsh93], Lemma 5.1).

When Δ is a diagram over G_1 we still use functions $\|p\|, |p|$, where p is a path in Δ.

In the following remark we translate some hyperbolic properties of $Cay(G)$ into the context of diagrams over G.

Remark 3.4. (i) Suppose Δ is a reduced diagram over G, p_1 and p_2 are disjoint paths in Δ, vertices $(p_i)_\pm$ are on the boundary $\partial\Delta$. Then there exists a diagram Δ' equivalent to Δ, such that $\partial\Delta' = \partial\Delta$, vertices $(p_i)_\pm$ are connected by a *geodesic* path p_i' for $i = 1, 2$, and paths p_1', p_2' are disjoint. Furthermore, a point x of the path p_i' is on $\partial\Delta'$ if and only if it is an initial or terminal vertex of p_i'.

(ii) Suppose Γ is a diagram over G, $\partial\Gamma = p_1 q_1 p_2 q_2$, where q_i are geodesic in G and $\|p_i\| \le K$, $|q_i| \ge 2K + 20\delta$ for $i = 1, 2$ and some $K \ge 0$. Then (after elementary transformations) there exists a subdiagram Γ' in Γ with boundary $\partial\Gamma' = p_1' q_1' p_2' q_2'$ such that $\|p_i'\| \le 6\delta$, q_i' are geodesic subpaths of q_i and $|(q_1)_+ - (q_1')_+| = |(q_1)_- - (q_1')_-| = K + 2\delta$. In particular,

$$|q_1'| = |q_1| - 2K - 4\delta.$$

(iii) If a subdiagram Γ satisfies the conditions of part (ii), then every vertex x of q_1 is at distance not greater then $K + 8\delta$ from q_2 (i.e. there exist a vertex y on q_2 such that $|x - y| \le K + 8\delta$).

Proof. (i) Consider the map ϕ' from diagram Δ to $Cay(G)$. For $i = 1, 2$ we pick a geodesic in $Cay(G)$ with label P'_i between vertices $\phi'(p_{i\pm})$ in $Cay(G)$. We apply an elementary transformation of type (ii) to p_i: cut Δ along p_i to get a new boundary component $p_i\tilde{p}_i$, $\phi(\tilde{p}_i) = \phi(p_i)^{-1}$ in G and glue inside a $\mathcal{D}$-face Π_i with boundary $p_i\tilde{p}_i$. Then apply the inverse type (ii) to Π_i: replace it with a pair of faces Π_{i1}, Π_{i2} with common subpath p'_i labeled by P'_i ($\partial\Pi_{i1} = p_i {p'}_i^{-1}$, $\partial\Pi_{i2} = p'_i\tilde{p}_i$). We have constructed the desired diagram Δ'. It remains to notice that no vertex belongs to both closed paths $p_1\tilde{p}_1$ and $p_2\tilde{p}_2$ since $p_i, \tilde{p}_i$ are copies of disjoint paths p_i in Δ. Also, all vertices of p'_i except for $p'_{i\pm}$ are interior in a subdiagram bounded by $p_i\tilde{p}_i$, and the remark is proved completely.

(ii) We consider $\phi'(\Gamma)$, and apply Lemma 2.1 to the pair of geodesic paths $\phi'(q_1)$, $\phi'(q_2)$ in $Cay(G)$ to find the subpath q''_1 of $\phi'(q_1)$ such that $|(q''_1)_\pm - \phi'((q'_1)_\pm)| = K + 2\delta$ and vertices $(q'_1)_\pm$ are in 6δ–neighborhood of geodesic $\phi'(q_2)$. Define a subpath q''_2 of $\phi'(q'_1)$ so that the inequality $|(q''_1)_\pm - (q''_2)_\pm| \leq 6\delta$ holds. It remains to choose a subpath q'_i on q_i satisfying equality $\phi'(q'_i) = q''_i$. Now apply part (i) to two pairs of points $(q'_2)_+, (q'_1)_-$ and $(q'_1)_+, (q'_2)_-$ in Γ which provides paths p'_i and observe that the path $p'_1q'_1p'_2q'_2$ bounds the desired diagram Γ'.

(iii) Follows from remark 2.3 and properties of the mapping ϕ'. □

We will need the following:

Lemma 3.5. *Suppose we have a diagram Δ consisting of cells Π_1, Π_2, a simple path t between them such that Π_1, Π_2 is pair of opposite cells with respect to a path t. Then, for any vertices o_1, o_2 on $\partial\Pi_1, \partial\Pi_2$ respectively, there exists a path s_1ts_2 such that $\phi(s_1ts_2) = P\phi(a)$ in G, where $|a| \leq \frac{1}{2}|\partial\Pi_2|$, P is a geodesic word and $|P| \leq |t| + 8\delta$, s_i is a subpath of $\partial\Pi_i$ ($i = 1, 2$), a is a subpath of $\partial\Pi_2$ and $s_{1-} = o_1, s_{2+} = o_2$. Moreover, the following equality holds in G:*

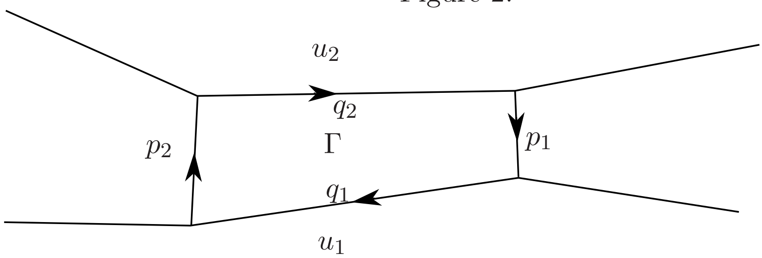
$$(P\phi(a))^{-1}\phi(\partial_{o_1}\Pi_1)(P\phi(a))\phi(\partial_{o_2}\Pi_2) = 1 \text{ in } G. \tag{6}$$

Proof. We denote r_1 to be the boundary path $\partial_{t-}\Pi_1$, r_2 to be the boundary path $\partial_{t+}\Pi_2$. By definition of an opposite pair (bounded by $r_1tr_2t^{-1}$) and the van-Kampen Lemma, there exists a diagram Γ over G with boundary $r_1tr_2t_1^{-1}$, where $\phi(t_1) = \phi(t)$. Since each path r_i is geodesic, by Remark 3.4 (iii) the distance between a vertex on r_1 and r_2 is not greater then $|t| + 8\delta$, hence there exists a vertex o'_1 on r_2 such that $|o_1 - o'_1| \leq |t| + 8\delta$.

Consider a subpath of the form $s_1t's'_2$ on $\partial\Gamma$, where s_1 is a subpath of $r_1^{\pm 1}$, s'_2 is a subpath of $r_2^{\pm 1}$, $(s_1)_- = o_1$, $(s'_2)_+ = o'_1$, t' is either t or t_1.

Let P be a geodesic word equal in G to the label of the path $s_1t's'_2$, so $|P| \leq |t| + 8\delta$. Now we consider $s_1t's'_2$ as a subpath of boundary $\partial\Delta$, so t' is t. We choose a path a on $\partial\Pi_2$ between o'_1 and o_2 satisfying inequality

Figure 2:



$|a| \leq \frac{1}{2}|\partial\Pi_2|$. Define the path s_2 to be $s_2'a$ after elimination of returns, hence $\phi(s_2'a) = \phi(s_2)$ in a free group generated by S. Since the boundary labels of Δ and Γ are the same, we may consider the path $s_1t's_2'$ as a path s_1ts_2' in Δ. We have that $\phi(s_1ts_2') = P$ in G, and so the following first two equalities hold in the free group generated by S while the last one holds in G:

$$\phi(s_1ts_2) = \phi(s_1t's_2's) = \phi(s_1ts_2')\phi(a) = P\phi(a).$$

To establish (6), we observe that the path $(s_1^{-1}\partial_{o_1}\Pi_1 s_1)t(s_2\partial_{o_2}\Pi_2 s_2^{-1})t^{-1}$ coincide with $(\partial_{t_-}\Pi_1)t(\partial_{t_+}\Pi_2)t^{-1}$ after the elimination of returns. Thus

$$\phi((s_1^{-1}\partial_{o_1}\Pi_1 s_1)t(s_2\partial_{o_2}\Pi_2 s_2^{-1})t^{-1}) = (\partial_{t_-}\Pi_1)t(\partial_{t_+}\Pi_2)t^{-1} = 1 \text{ in } G,$$

which after conjugation provides $\phi^{-1}(s_1ts_2)\phi(\partial_{o_1}\Pi_1)\phi(s_1ts_2)\phi(\partial_{o_2}\Pi_2) = 1$ in G providing (6).□

The following notion of ϵ-contiguity subdiagram will be used extensively. Let Δ be a diagram over G_1. Let u_1 and u_2 be a pair of paths in Δ with subpaths q_1 and q_2 respectively, such that there exists a pair of simple paths p_1, p_2, $|p_1|, |p_2| \leq \epsilon$ and suppose that a path $p_1q_1p_2q_2$ bounds a disc diagram Γ which does not contain any $\mathcal{R}$–faces (see Figure 2). Then we call Γ an *ϵ-contiguity subdiagram* between paths u_1 and u_2. When we talk about the contiguity subdiagram Γ between u_1 and u_2 we use the formula $\partial(u_1, \Gamma, u_2) = p_1q_1p_2q_2$ to define notation for arcs of Γ. In this case q_1, q_2 are referred to as *contiguity arcs* and p_1, p_2 as *side arcs* of the ϵ-contiguity subdiagram Γ. We usually consider contiguity subdiagrams between a pair of $\mathcal{R}$–faces or between an $\mathcal{R}$–face and a boundary path (i.e. u_1 is the boundary path of $\mathcal{R}$–face Π_1 and u_2 is the boundary path of $\mathcal{R}$–face Π_2 or is a subpath of the boundary of Δ). If u_1 is the boundary of an $\mathcal{R}$–face Π_1, u_2 is a path of a boundary of an $\mathcal{R}$–face Π_2 with ϵ-contiguity diagram Γ described above then we define the degree of contiguity of Π_1 to Π_2 to be $(\Pi_1, \Gamma, \Pi_2) = \frac{\|q_1\|}{\|\Pi_1\|}$ (or, if u_2 is a boundary subpath of Δ, the degree of contiguity of Π_1 to the boundary subpath u_2 to be $(\Pi_1, \Gamma, u_2) = \frac{\|q_1\|}{\|\Pi_1\|}$).

The next two lemmas provide the basic connection between the notions of small cancellation and diagrams over hyperbolic groups.

Lemma 3.6 ([Olsh93], Lemma 5.2). *(i) If the symmetized system $\mathcal{R}$ satisfies the $C(\epsilon,\mu,\rho)$–condition, then for any reduced diagram Δ and any ϵ-contiguity subdiagram Γ of a face Π_1 to another face Π_2 the following inequalities hold:*

$$\|q_1\| < \mu\,\|\partial\Pi_1\|\,,\ \ \|q_2\| < \mu\,\|\partial\Pi_2\|\,,$$

where $\partial(\Pi_1,\Gamma,\Pi_2) = p_1q_1p_2q_2$ for any reduced diagram Δ over G_1.

(ii) Suppose a diagram Δ has a pair of $\mathcal{R}$–faces Π_1,Π_2 and an ϵ-contiguity subdiagram Γ ($\partial\Gamma = p_1q_1p_2q_2$) such that

$$max\{(\Pi_1,\Gamma,\Pi_2),(\Pi_2,\Gamma,\Pi_1)\} \geq \mu.$$

Then Π_1,Π_2 are opposite with respect to each of the paths p_1,p_2.

Note that part 2 of the above lemma is an immediate corollary of small cancellation property. □

Lemma 3.7 ([OlOsSa], Lemma 4.6). *For any hyperbolic group G there exists $\mu_0 > 0$ such that for any $0 < \mu \leq \mu_0$ there are $\epsilon \geq 0$ and ρ (it is suffice to choose $\rho > \frac{10^6\epsilon}{\mu}$) with the following property:*

Let the symmetized system $\mathcal{R}$ satisfy the $C(\epsilon,\mu,\rho)$-condition and furthermore let Δ be a reduced disc diagram over G_1 whose boundary $\partial\Delta$ is decomposed into geodesic sections $q^1,\ldots,q^r$, where $1 \leq r \leq 12$. Then, provided Δ has an $\mathcal{R}$–face, there exists a reduced diagram Δ' equivalent to Δ, an $\mathcal{R}$–face Π in Δ and disjoint ϵ-contiguity subdiagrams $\Gamma_1,\ldots,\Gamma_r$ (some of them can be absent) of Π to $q^1,\ldots,q^r$ respectively such that

$$(\Pi,\Gamma_1,q_1) + \cdots + (\Pi,\Gamma_r,q_r) > 1 - 23\mu.$$

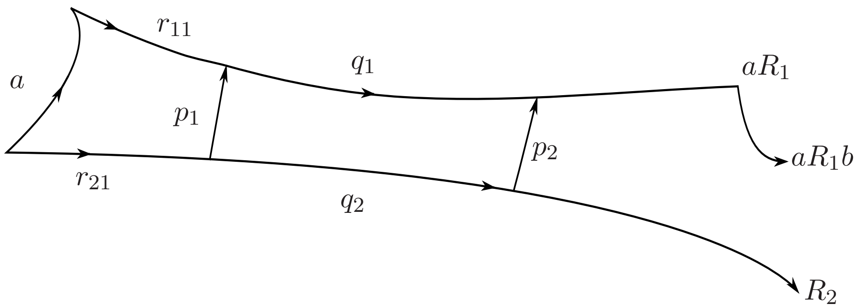
The following lemma is a special case of that in [Olsh93]:

Lemma 3.8. *([Olsh93], Lemmas 6.7, 7.4) Let G be a non-elementary hyperbolic group. There exists $\mu_0 > 0$ such that for any $0 < \mu \leq \mu_0$ there exists $\epsilon \geq 0$ such that for every $N > 0$ there exists $\rho > 0$ with the following property:*

if $\mathcal{R}$ is finite and satisfies $C(\epsilon,\mu,\rho)$ then G_1 is a non-elementary hyperbolic group and $W = 1$ in G_1 iff $W = 1$ in G for every word W with $\|W\| \leq N$.

Definition 3.9. *We say that a system $\mathcal{R}$ of geodesic words satisfies the $\tilde{C}(\epsilon,\mu,\rho$–condition if $\mathcal{R}$ is symmetrized, satisfies $C(\epsilon,\mu,\rho)$–condition and consists of words which represent cyclically reduced elements in G.*

Figure 3: $C(\epsilon, \mu, \rho)\ \ => \ \ C'(2\mu)$



4. Condition $C'(\mu)$ and connection to $C(\epsilon, \mu, \rho)$–condition

Remark 4.1. Suppose the system of geodesic words $\mathcal{R}$ satisfies $\tilde{C}(\epsilon, \mu, \rho$–condition, $\mu < 1/100$, $\epsilon \geq \epsilon_0 \geq 6\delta$, $\rho > \frac{500\delta}{\mu(1-8\mu)}$. Then $\mathcal{R}$ satisfies $C'(2\mu)$.

Proof. Take arbitrary words $R_1, R_2 \in \mathcal{R}$. We denote by M the minimum $min(|R_1|, |R_2|)$. To check the condition $C'(2\mu)$ we assume that $\langle aR_1b, R_2\rangle > 2\mu M$ for some $a, b \in G$ such that $|a|, |b| \leq 100\delta$.

We denote by W a geodesic equal to aR_1b, by ν a path R_2 and by γ a path aR_1b in the Cayley graph $Cay(G)$.

Consider vertices o_2 on ν and o_3 on the geodesic W at distance $[2\mu M]$ from identity vertex 1. By Remark 2.5 (part 1), we have that $\Phi(o_2) = \Phi(o_3)$ and (by part 2) $|o_2 - o_3| \leq 4\delta$. Now we may apply Lemma 2.1 (for $K = 100\delta$) to segments ${}_\gamma R_1, W$ and hence there exists a subsegment $[u, v]$ of W such that $|u - e| \leq 102\delta$, $|v - \gamma_+| \leq 102\delta$ and $[u, v]$ is within 6δ–neighborhood of ${}_\gamma R_1$. Vertex o_3 lies on $[u, v]$ because on one hand $|o_3 - e| = [2\mu M] > 2K + 20\delta$ and on the other hand

$$|o_3 - \gamma_+| \geq |R_1| - |a| - |b| - [2\mu M] \geq (1 - 3\mu)M > 2K + 20\delta.$$

We get that o_3 is within 6δ–neighborhood of some vertex o_1 on path ${}_\gamma R_1$.

We consider two subsegments $[e, o_2]$ and $[({}_\gamma a)_+, o_1]$ of ν and ${}_\gamma R_1$ respectively and apply Lemma 2.1 to get that there exists a subsegment q_2 of R_2 between e and o_2 such that

$$|q_2| \geq [2\mu M] - 200\delta - 4\delta > \frac{3}{2}\mu M + 20\delta$$

which is within 6δ–neighborhood from ${}_\gamma R_1$. Now define q_1 to be a subsegment of ${}_\gamma R_1$ with $|q_{1-} - q_{2-}|, |q_{1+} - q_{2+}| \leq 6\delta$.

We have that

$$|q_i| > \frac{3}{2}\mu min(|R_1|, |R_2|) \text{ for i=1,2.} \quad (7)$$

Define p_1 (p_2) to be a geodesic path between q_{2-}, q_{1-} (q_{1+}, q_{2+}), see Figure 3. To justify the Figure 3, we must show that $|(_\gamma a)_+ - (q_1)_-| < |(_\gamma a)_+ - (q_1)_+|$ (this inequality follows from [Olsh93] Lemma 1.10, but we include the argument here). By triangle inequality and definition of q_1, we have that

$$|(_\gamma a)_+ - (q_1)_-| \leq |a| + |p_1| + |e - (q_1)_-| \leq 100\delta + 6\delta + 102\delta = 208\delta;$$

on the other hand,

$$|(_\gamma a)_+ - (q_1)_+| \geq |e - (q_2)_+| - |p_2| - |a| = |e - (q_2)_-| + |q_2| - |p_2| - |a| \geq$$

$$102\delta + \frac{3}{2}\mu M + 20\delta - 100\delta - 6\delta > \mu M \geq 500\delta$$

and hence we got $|(_\gamma a)_+ - (q_1)_-| < |(_\gamma a)_+ - (q_1)_+|$, as desired.

We denote labels of q_i and p_i as Q_i and P_i respectively. Define four subpaths r_{ij}, $i, j \in \{1,2\}$ by equalities $_\gamma R_1 = r_{11}r_{12}$, $\nu = r_{21}r_{22}$ and $(r_{11})_+ = (p_1)_+$, $(r_{21})_+ = (p_1)_-$. Define words R_{ij}, Q', Q'' by equalities $lab(r_{ij}) = R_{ij}$, $R_{12}R_{11} \equiv Q_1Q'$, $R_{22}R_{21} \equiv Q_2Q''$. We have that $Q_2 = P_1Q_1P_2^{-1}$, $\|P_i\| \leq 6\delta$, and taking into account the inequality (7) we conclude by $\tilde{C}(\epsilon, \mu, \rho$–condition that $P_1R_{12}R_{11}P^{-1} = R_{22}R_{21}$, which in turn is equivalent to $(R_{21}P_1R_{11}^{-1})(R_{11}R_{12})(R_{11}P_1^{-1}R_{21}^{-1}) = R_{21}R_{22}$. It remains to observe that $a = (R_{21}P_1R_{11}^{-1})$ and so $aR_1a^{-1} = R_2$. □

Corollary 4.2. *Suppose $\mathcal{R}$ satisfies $\tilde{C}(\epsilon, \mu, \rho$–condition and $n \geq 1$,*

$$\prod_{k=1}^{n} U_kR_kU_k^{-1} = 1 \text{ in } G, \text{ where } U_kR_kU_k^{-1} \in \mathcal{X}. \quad (8)$$

Then (i) There exists a reducible pair $i < j$ in the sense of Definition 2.14 and

$$U_{i+1}R_{i+1}U_{i+1}^{-1} \dots U_{j-1}R_{j-1}U_{j-1}^{-1} = U_iR_iU_i^{-1} \dots U_jR_jU_j^{-1} \text{in } G. \quad (9)$$

(ii) For every reducible pair $i < j$ in (8), there exists a van-Kampen diagram Δ' over G with the boundary γ' labeled by the word $U_1R_1U_1^{-1} \dots U_nR_nU_n^{-1}$ and a subdiagram Γ in Δ' with boundary $p_1q_1p_2q_2$ such that q_1 is a subpath of $_{\gamma'}R_i$, q_2 is a subpath of $_{\gamma'}R_j$, $|p_i| \leq 11\delta + 3$ and $max(\frac{|q_1|}{|R_i|}, \frac{|q_2|}{|R_j|}) \geq 2\mu - \frac{10\delta+3}{\rho}$. The only vertices of paths p_i that are on the boundary of Δ are initial and terminal vertices $p_{i\pm}$.

(iii) Consider the diagram Δ' from part (ii) and let ν' be any of the four paths given by the formula $\nu' = {}_{\gamma'}(U_i^{\pm 1}) s_1 p_1^{-1} s_2 {}_{\gamma'}(U_j^{\pm 1})$, where s_1 is a subpath of ${}_{\gamma'}R_i$, s_2 is a subpath of ${}_{\gamma'}R_j$. Then

$$\phi(({}_{\gamma'}U_i)^{\pm 1} s_1 p_1^{-1} s_2 ({}_{\gamma'}U_j^{\pm 1})) = \prod_{k=i+d}^{j-c} U_k R_k U_k^{-1} \text{ in } G,$$

where c, d take values 0 or 1 depending on the path ν' and $(c,d) \neq (0,0)$. Moreover, depending on values c and d, the word $H \equiv \prod_{k=i+c}^{j-d} U_k R_k U_k^{-1}$ conjugates $U_i R_i U_i^{-1}$ to $U_j R_j^{\pm 1} U_j^{-1}$, namely:

$$H^{-1} U_i R_i U_i^{-1} H = U_j R_j^e U_j^{-1}, \text{ where } e \in \{\pm 1\}.$$

Proof. By Remark 4.1, $\tilde{C}(\epsilon, \mu, \rho$–condition implies the condition $C'(2\mu)$. The product $\prod_{k=1}^{n} U_k R_k U_k^{-1}$ equals to identity in G so by Lemma 2.15 it is not reduced in the sense of Definition 2.14. Hence there exists a reducible pair $i < j$ (in particular, we have that $|R_i| = |R_j|$) such that the approximation tree for a_i, b_i, a_j, b_j is of shape on Figure 1 and by Lemma 2.13 the corresponding geodesic word Y satisfies:

$$|Y| \geq 2\mu M + 10\delta + 3, \text{ where } M = |R_i|. \tag{10}$$

Lemma 2.13 also provides the equation (4) and thus (i) is proved.

Diagram Δ' over G with boundary γ' labeled by $\prod_{k=1}^{n} U_k R_k U_k^{-1}$ exists by van-Kampen Lemma. Consider the map $\phi' : \Delta' \longmapsto Cay(G)$. We denote $\phi'(\gamma')$ as γ'' (a path in $Cay(G)$ with label $\prod_{k=1}^{n} U_k R_k U_k^{-1}$). We adopt notations from the definition of a reducible pair $i < j$ and Figure 1. Consider a geodesic path α in $Cay(G)$ starting from a_i with label XYZ (hence it ends at b_i) and a geodesic up to $18\delta + 6$ path β in $Cay(G)$ starting from a_j with label $U^{-1}Y^{-1}V$ (it ends at b_j). By definition of X, Y, Z, U, V, we have $({}_{\alpha}Y)^{-1} =_{\beta} Y^{-1}$. From the fact that XYZ is geodesic, it follows from Remark 2.3 (ii) that there exists a subpath q_1' of ${}_{\gamma''}R_i$ such that:

$$\left|{}_{\alpha}Y_- - q'_{1-}\right|, \left|{}_{\alpha}Y_+ - q'_{1+}\right| \leq \delta, \tag{11}$$

which implies that:

$$\left|q_1'\right| \geq |Y| - 2\delta. \tag{12}$$

Similarly, we consider the path β geodesic up to $18\delta + 6$ and apply again Remark 2.3 (ii) to obtain that there exists a subpath q_2' of ${}_{\gamma''}R_j$ such that:

$$\left|{}_{\alpha}Y_- - q'_{2+}\right|, \left|{}_{\alpha}Y_+ - q'_{2-}\right| \leq (9\delta + 3) + \delta, \tag{13}$$

and hence :

$$\left|q_2'\right| \geq |Y| - 20\delta - 6. \tag{14}$$

The inequalities (11), (13) imply also that $\left|q_{1-}' - q_{2+}'\right|, \left|q_{1+}' - q_{2-}'\right| \leq 11\delta + 3$.

Consider subpaths q_1 of ${}_\gamma R_i$ and q_2 of ${}_\gamma R_j$ in the boundary $\partial\Delta'$ such that $\phi'(q_{i-}) = q_{i-}'$, $\phi'(q_{i+}) = q_{i+}'$. The Remark 3.4 implies that (after some elementary transformations) there exists a subdiagram Γ in Δ' with boundary $p_1q_1p_2q_2$, vertices of p_i are interior except for initial and terminal ones and $|p_i| \leq 11\delta + 3$. Equations (12), (14), (10) provide that:

$max(\frac{|q_1|}{|R_i|}, \frac{|q_2|}{|R_j|}) \geq \frac{|Y|-20\delta-6}{M} \geq \frac{2\mu M+10\delta+3-20\delta-6}{M} \geq 2\mu - \frac{10\delta+3}{M}$. Part (ii) is proved.

To justify part (iii) we look at each of the 4 options for the path ν'. For example, if $\nu' = ({}_{\gamma'}U_i)s_1p_1^{-1}s_2({}_{\gamma'}U_j^{-1})$ then ϕ' maps the vertex $\nu_- = ({}_{\gamma'}U_i)_-$ of Δ' to the vertex $g_{i-1} = \prod_{k=1}^{i-1} U_kR_kU_k^{-1}$ in $Cay(G)$, $\nu_+' = ({}_{\gamma'}U_j^{-1})_+$ to the vertex $g_j = \prod_{k=1}^{j} U_kR_kU_k^{-1}$ in $Cay(G)$. Hence $lab(\phi'(\nu')) = g_{i-1}^{-1}g_j = \prod_{k=i}^{j} U_kR_kU_k^{-1}$.

A direct computation using the relation (9) yields that for every possible value of c and d the word H conjugates $U_iR_iU_i^{-1}$ to $U_jR_j^{\pm 1}U_j^{-1}$. For example, $H \equiv U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1}$ conjugates $U_iR_iU_i^{-1}$ to $U_jR_j^{-1}U_j^{-1}$:

$$U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1}U_jR_j^{-1}U_j^{-1}(U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1})^{-1} =$$

$$U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_{j-1}R_{j-1}U_{j-1}^{-1}(U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1})^{-1} =$$

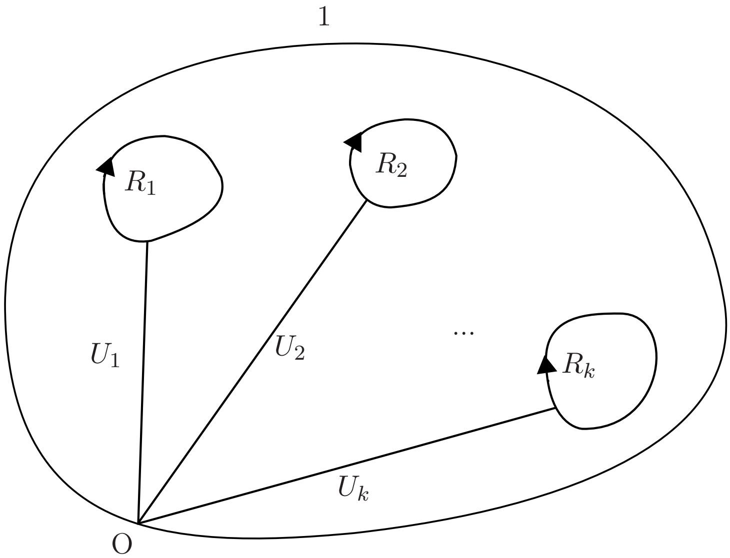
$$U_iR_iU_i^{-1}\dots U_jR_jU_j^{-1}(U_{i+1}R_{i+1}U_{i+1}^{-1}\dots U_jR_jU_j^{-1})^{-1} = U_iR_iU_i^{-1},$$

where the last inequality holds by (9). It remains to notice that by relation (9), in the word H the parameters $c = d = 0$ may be replaced by $c = d = 1$. □

Definition 4.3. *For every reducible pair $i < j$ consider the diagram Δ' from Corollary 4.2, identify each edge of ${}_{\gamma'}U_s$ with corresponding edge of ${}_{\gamma'}U_s^{-1}$ and fill in the $\mathcal{R}$-faces Π_s to get a van-Kampen diagram Δ over G_1 which has a $(11\delta + 3)$-contiguity subdiagram Γ such that $max\{(\Pi_i, \Pi_j), (\Pi_j, \Pi_i)\} \geq 2\mu - \frac{10\delta+3}{\rho}$. We will refer to a described diagram Δ as a standard diagram for relation (8). We denote the image of γ' in Δ by γ.*

By definition, the standard diagram is a spherical diagram, but for convenience we draw it on Figure 4 as a disc diagram with boundary label 1.

Figure 4: Standard Diagram



Remark 4.4. According to the identifications made in the definition of the standard diagram Δ, any of the four paths ν' in Δ' corresponds to a closed path in Δ with label $\nu = ({}_{\gamma}U_i)r_1p_1r_2({}_{\gamma}U_j^{-1})$, where r_i correspond to s_i. One can observe that different paths ν' have different images in Δ, but we will not use this fact later. Note that the subpaths $({}_{\gamma'}U_i)^{\pm 1}$ and $({}_{\gamma'}U_j)^{\pm 1}$ of ν' in Δ' correspond respectively to subpaths ${}_{\gamma}U_i$ and ${}_{\gamma}U_j$ of ν.

5. Generators of a free normal subgroup in G

In this section we assume that the set $\mathcal{R}$ satisfies $\tilde{C}(\epsilon, \mu, \rho$–condition, where the parameters ϵ, μ, ρ are chosen according to Lemma 3.7 and satisfy inequalities $\epsilon > \epsilon_0 = 19\delta + 3$, $\mu < 1/100$, $\rho > \frac{500\epsilon}{6\mu(1-8\mu)}$.

It is well known (see [Gro]2.2A) that a hyperbolic group contains only finitely many conjugacy classes of torsion elements. So, given a group G, we may choose the constant ρ to be larger then the length of shortest representative in each conjugacy class of torsion elements. Thus we will assume in the sequel that for values of ρ large enough:

Remark 5.1. The set $\mathcal{R}$ consists of elements of infinite order.

Definition 5.2. *We call a (reduced) diagram* Δ an octagon diagram *if* $\partial\Delta = l_1 j_1 \ldots l_4 j_4$*, where* l_i *are geodesic in* G*, and* $\|j_i\| \leq \epsilon$.

Definition 5.3. *Consider an octagon reduced diagram* Δ *with boundary* $\partial\Delta = l_1 j_1 \ldots l_4 j_4$ *and pick a number* $0 < \kappa < 1$*. We say that an arc* l_i

satisfies the condition $\mathcal{U}_\Delta(\kappa)$ *if for every diagram* Δ' *equivalent to* Δ *and every* $\mathcal{R}$*–face* Π *in* Δ' *such that there is a contiguity subdiagram* Γ *between* Π *and* l_i*, we have the inequality* $(\Pi, \Gamma, l_i) < \kappa$.

It is clear that if l_i has a subpath l which is a boundary arc of some subdiagram Δ_1 of Δ then l satisfies $\mathcal{U}_{\Delta_1}(\kappa)$ as well.

Lemma 5.4. *Let* Δ *be an arbitrary octagon diagram and* $\phi(l_1) = U \in \mathcal{U}$*, then (in notations of Definition 5.2)* l_1 *satisfies* $\mathcal{U}_\Delta(\frac{1}{2} + \frac{1}{5}\mu)$.

Proof. Note that by definition of ρ we have that $\frac{2\epsilon+34\delta}{\rho} < \frac{1}{5}\mu$. We suppose that there exists an octagon diagram Δ, with boundary arc l_1, $\phi(l_1) = U \in \mathcal{U}$. Assume that (after elementary transformations) there exists an $\mathcal{R}$–face Π in Δ and a corresponding subdiagram Γ between Π and l_1 with boundary $\partial(\Pi, \Gamma, l_1) = p_1 q_1 p_2 q_2$ such that $(\Pi, \Gamma, l_1) \geq \frac{1}{2} + \frac{2\epsilon+34\delta}{\rho}$.

Now we may apply Remark 3.4(ii) to the diagram Γ and conclude that (after elementary transformations) there exists a subdiagram Γ' of Γ with boundary $p'_1 q'_1 p'_2 q'_2$ such that q'_i are subpaths of q_i and:

$$\left|p'_i\right| \leq 6\delta, \; \left|q'_1\right| = |q_1| - 2\epsilon - 4\delta. \tag{15}$$

By definition of q'_1, we have $|q'_1| = |q_1| - 2\epsilon - 4\delta \geq \frac{1}{2}|\partial\Pi| + 30\delta$ and it's complement q'_3 ($\partial\Pi = q'_1 q'_3$) satisfies $|q'_3| \leq \frac{1}{2}|\partial\Pi| - 30\delta$. Thus the condition (i) of definition 2.10 is satisfied.

We define paths l', l'' such that $l_1 = l' q'_2 l''$. The equality $U = \phi(l_1) = \phi(l' p'_1 q'_1 p'_2 l'')$ holds in G, moreover, by inequalities (15), we have:

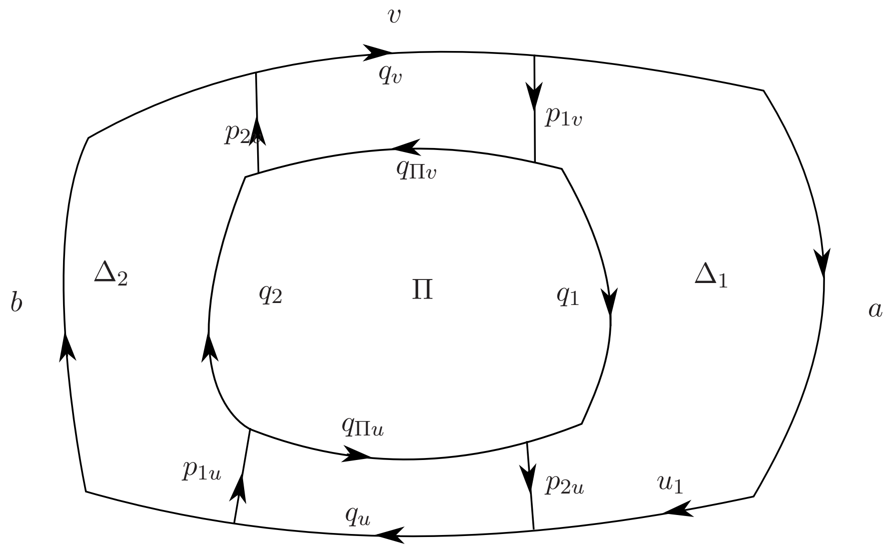
$$\left|l'\right| + \left|p'_1\right| + \left|q'_1\right| + \left|p'_2\right| + \left|l''\right| \leq \left|l'\right| + 2\left|p'_1\right| + \left|q'_2\right| + 2\left|p'_2\right| + \left|l''\right| \leq |l_1| + 4 \cdot 6\delta.$$

Hence the condition (ii) of definition 2.10 is checked for the factorization $\phi(l' p'_1)\phi(q'_1)\phi(p'_2 l'')$ of the word U.

By Definition 2.10, the word U does contain more then half of a relation and thus $U \notin \mathcal{U}$ contrary to our assumption. □

Definition 5.5. *Consider a reduced octagon diagram* Δ *with boundary* $l_1 j_1 \ldots l_4 j_4$. *Denote for simplicity of notation* $u = l_1$ *and* $v^{-1} = l_3$, $a = j_3 l_4 j_4$, $b = j_1 l_2 j_2$ *and define the base point of* Δ *to be* $o = (l_1)_-$. *Consider an* $\mathcal{R}$*–face* Π *and disjoint contiguity subdiagrams* Γ_u, Γ_v *of* Π *to boundary arcs* u, v*, define boundary arcs of* Γ_u, Γ_v *by* $\partial(\Pi, \Gamma_u, u) = p_{1u} q_{\Pi u} p_{2u} q_u$, $\partial(\Pi, \Gamma_v, v) = p_{1v} q_{\Pi v} p_{2v} q_v$ *and define* q_1, q_2 *by equality* $\partial\Pi = q_{\Pi v}^{-1} q_1 q_{\Pi u}^{-1} q_2$ *(see Figure 5). We say that a subdiagram* $\Delta_0 = \Delta_0(\Delta, \Pi)$ *with a boundary path* $p_{2u} q_u p_{1u} q_2 p_{2v} q_v p_{1v} q_1$ (u, v)–bond (through Π) *if both values* $(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)$ *are greater then* μ. *We define subdiagrams* $\Delta_1 = \Delta_1(\Delta, \Pi)$, $\Delta_2 = \Delta_2(\Delta, \Pi)$ *of* Δ *with boundaries* $u_1 p_{2u}^{-1} q_1 p_{1v}^{-1} v_1^{-1} a$ *and* $u_2 b v_2^{-1} p_{2v}^{-1} q_2^{-1} p_{1u}^{-1}$ *respectively, where* u_1 (v_1) *is an initial subpath of* u (v)

Figure 5: Bond Between u And v



and v_2 (u_2) is a terminal subpath of v (u) (recall that the orientation of the boundary is clockwise).

For an arbitrary reduced octagon diagram Δ, $\partial\Delta = l_1 j_1 \dots l_4 j_4$, where l_i are geodesic in G, $\|j_i\| \le \epsilon$, there exist a pair of (possibly empty) sets $V = \{\Pi_1, \dots \Pi_m\}$ of $\mathcal{R}$–faces and $\Sigma(\Delta) = \{\Gamma_{1,u}, \Gamma_{1,v}, \dots, \Gamma_{m,u}, \Gamma_{m,v}\}$ of disjoint ϵ-contiguity subdiagrams, where $\Gamma_{i,u}, \Gamma_{i,v}$ are contiguity subdiagrams such that $\Delta_0(\Pi_i) = \Pi_i \cup \Gamma_{iu} \cup \Gamma_{iv}$ is a (u, v)–bond. We call a pair $(V, \Sigma(\Delta))$ *a system of bonds between u and v.*

Remark 5.6. (i) It is clear that in a non-empty system of (u, v)–bonds $(V, \Sigma(\Delta))$ for a reduced diagram Δ there exists a unique face Π in V such that the associated (see definition 5.5) paths u_1 and v_1 are the longest. Moreover, any other face $\Pi' \in V$ belongs to $\Delta_1(\Pi)$.

(ii) For every face Π in V we have that

$$|u_1| \le |u| - (\Pi, \Gamma_u, u)\,|\partial\Pi| + 2\epsilon, \quad |v_1| \le |v| - (\Pi, \Gamma_v, v)\,|\partial\Pi| + 2\epsilon. \quad (16)$$

The following remark will allow us to extend systems of bonds of subdiagrams Δ_i to the diagram Δ.

Remark 5.7. Consider a reduced octagon diagram Δ over G_1 and assume that there is a (u, v)–bond $\Delta_0(\Pi) = \Pi \cup \Gamma_u \cup \Gamma_v$ in Δ satisfying $(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v) \ge \mu$ and two systems of (u_i, v_i)–bonds $(V_i, \Sigma(\Delta_i))$ in $\Delta_i = \Delta_i(\Pi, \Delta)$, $i = 1, 2$. Then the sets $V = V_1 \cup V_2 \cup \{\Pi\}$ and $\Sigma(\Delta) = \Sigma(\Delta_1) \cup \Sigma(\Delta_2) \cup \{\Gamma_u, \Gamma_v\}$ comprise the system of (u, v)–bonds $(V, \Sigma(\Delta))$ in Δ.

Lemma 5.8. *Let Δ be a reduced octagon diagram with at least one $\mathcal{R}$–face with boundary $\partial\Delta = aj_1uj_2bj_3v^{-1}j_4$, where u, v, a satisfy the condition $\mathcal{U}_\Delta(\frac{1}{2} + \frac{\mu}{5})$, b satisfies $\mathcal{U}_\Delta(\mu)$ and $|j_k| \leq \epsilon$ for every k.*

(i) Then Δ has a non-empty system of (u, a)–, (v, a)– or (u, v)–bonds.

(ii) Assume in addition that Δ does not have (u, a)– or (v, a)–bonds. Then, for the set V consisting of all $\mathcal{R}$–faces, there exists a system of (u, v)–bonds $(V, \Sigma(\Delta))$ such that for every $\mathcal{R}$–face Π in Δ there exist subdiagrams $\Gamma_u, \Gamma_v \in \Sigma(\Delta)$ satisfying:

$$(\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > 1 - 26\mu; \tag{17}$$

$$max[(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)] > \frac{1}{2} - 13\mu; \tag{18}$$

$$min[(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)] > \frac{1}{2} - 27\mu. \tag{19}$$

Proof. (i) On the one hand we may consider an $\mathcal{R}$–face Π satisfying Lemma 3.7 such that $(\Pi, \Gamma_a, a) + (\Pi, \Gamma_b, b) + (\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > (1 - 23\mu) - \frac{4 \cdot 3\epsilon}{|\partial\Pi|}$ (note that $(\Pi, \Gamma_{j_i}, j_i)\,|\partial\Pi| \leq 3\epsilon$ because $|j_i| \leq \epsilon$). Together with condition on b it means that

$$(\Pi, \Gamma_a, a) + (\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > (1 - 24\mu) - \frac{4 \cdot 3\epsilon}{|\partial\Pi|} \tag{20}$$

On the other hand each summand on the left-hand side of (21) is smaller then $\frac{1}{2} + \frac{\mu}{5}$. Hence at least two of them are larger then 12μ.

(ii) We continue the considerations in the proof of part (i). We cannot have $(\Pi, \Gamma_a, a) \geq \mu$ because at least one of the other summands in (20) is larger then 12μ and we would get a (u, a)– or (v, a)–bond involving a which is impossible. Hence we get that

$$(\Pi, \Gamma_u, u) + (\Pi, \Gamma_v, v) > (1 - 25\mu) - \frac{4 \cdot 3\epsilon}{|\partial\Pi|} \tag{21}$$

and so the inequality (17) holds for Π. The inequality

$$max[(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)] > \frac{1}{2} - \frac{25}{2}\mu - \frac{2 \cdot 3\epsilon}{|\partial\Pi|}$$

follows immediately since $\mu < 1/100$, while for

$$min[(\Pi, \Gamma_u, u), (\Pi, \Gamma_v, v)] > \frac{1}{2} - 26\frac{1}{5}\mu$$

it is enough to recall that $|q_u|, |q_v| < (\frac{1}{2} + \frac{1}{5}\mu)\,|\partial\Pi|$. We have proved the formulas (17)–(19) for the face Π satisfying Lemma 3.7, taking into account that (by definition of ρ): $\frac{4 \cdot 3\epsilon}{|\partial\Pi|} \leq \frac{4 \cdot 3\epsilon}{\rho} < \frac{1}{5}\mu$.

When $n = 1$, the diagram Δ has a single $\mathcal{R}$–face Π and we are done by the argument above.

We induct on a number n of $\mathcal{R}$-faces in the octagon diagram Δ with base $n = 1$. If $n > 1$ we consider subdiagrams $\Delta_i = \Delta_i(\Delta, \Pi)$ for the face Π (we follow notations of Definition 5.5 here). It is clear that diagrams Δ_i satisfy the induction assumption. Each has a number of $\mathcal{R}$–faces strictly less then n because neither contains the face Π, the arcs p_{iu}, p_{iv} on the boundary of Δ_i are not longer then ϵ. The boundary arcs q_i of Δ_i satisfy the condition $\mathcal{U}_{\Delta_i}(\mu)$ by Lemma 3.6 because they are boundary arcs of the $\mathcal{R}$–face Π in the reduced diagram Δ. As we mentioned before the proof of the lemma, conditions $\mathcal{U}_{\Delta_i}(\mu)$ for q_i imply that there are no bonds involving q_i in Δ_i. The induction assumption is now checked for Δ_i, hence there exist systems of (u_i, v_i)–bonds $(V_i, \Sigma(\Delta_i))$ in Δ_i satisfying the conclusion of the lemma. Finally we are in position to apply the Lemma 5.7 to Δ relative to the bond $\Delta_0(\Pi)$: we obtain a system of (u, v)–bonds $(V, \Sigma(\Delta))$ such that V contains all $\mathcal{R}$–faces and the set $\Sigma(\Delta)$ is comprised of $\Sigma(\Delta_i)$ for $i = 1, 2$ and Γ_u, Γ_v. The inequalities (17)–(19) hold for every $\mathcal{R}$–face in Δ except for the face Π by induction assumption, and for the face Π we have obtained them above. □

We denote words URU^{-1} by $A_{R,U}$. If u is a path in some diagram Δ, we write $A_{R,u}$ for $A_{R,\phi(u)}$.

Definition 5.9. *Define a weight of a word $A_{R,U}$ by $\psi(A_{R,U}) = |R| + 4\,|U|$.*

Lemma 5.10. *Let Δ be a reduced diagram over the group G_1 with boundary $uj_1aj_2v^{-1}$, where u, v, a satisfy the condition $\mathcal{U}_\Delta(\frac{1}{2} + \frac{\mu}{5})$, $|j_i| \leq \epsilon$ for $i = 1, 2$ and there are no (u, a)– or (v, a)–bonds. Then $\phi(uj_1aj_2v^{-1}) = \prod_{i=1}^{n} A_{R_j,U_j'}$ in G, where $max_{1\leq j\leq n}\psi(A_{R_j,U_j'}) < 4max(|u|, |v|)$.*

Proof. We proceed by induction on the number n of $\mathcal{R}$–faces in Δ. The conclusion of the lemma holds for $k = 0$ because $\phi(uj_1aj_2v^{-1}) = 1$ in G and there are no $A_{R,U}$'s.

Assume that the lemma is true for $n - 1$. Consider a face Π satisfying the Remark 5.6. By Lemma 5.8(ii), the $\mathcal{R}$–face Π of Δ is in the set V for some system of (u, v)–bonds $(V, \Sigma(\Delta))$, and inequalities (17)–(19) hold for Π. We recall the inequality (18) and assume that

$$(\Pi, \Gamma_u, u) > (\frac{1}{2} - 13\mu), \tag{22}$$

in the other case proof is the same.

By the choice of Π, we have that every other $\mathcal{R}$–face of Δ is in the subdiagram Δ_1 ($\Delta_i = \Delta_i(\Delta, \Pi)$) and the subdiagram Δ_2 is a diagram over G (we are using notations from Definition 5.5 and the reader can refer to Figure 5 in the sequel of the proof). We consider a system of (u, v)–bonds provided by Lemma 5.8. Denote a subdiagram of Δ consisting of Δ_2, Δ_0 by Δ'. It contains a single $\mathcal{R}$–face Π, so we get the following equations in the group G:

$$\phi(\partial_{u_{1+}}\Delta') = \phi(\partial_{u_{1+}}\Delta_0) = \phi(p_{2u}^{-1}(\partial_{p_{2u-}}\Pi)p_{2u}). \tag{23}$$

Now notice that paths $\partial_{u_-}\Delta$ and $u_1(\partial_{u_{1+}}\Delta')u_1^{-1}(\partial_{u_-}\Delta_1)$ coinside after the elimination of returns in the latter path, so their labels are equal in the free group generated by S. We get that

$$\phi(\partial_{u_-}\Delta) = \phi(u_1(\partial_{u_{1+}}\Delta')u_1^{-1}(\partial_{u_-}\Delta_1)) = \phi(u_1(\partial_{u_{1+}}\Delta')u_1^{-1})\phi(\partial_{u_-}\Delta_1), \tag{24}$$

and taking into account (23),

$$\phi(u_1p_{2u}^{-1})\phi(\partial_{(p_{2u})_-}\Pi)\phi(u_1p_{2u}^{-1})^{-1}\phi(\partial_{u_-}\Delta_1) = 1 \text{ in } G_1,$$

where the number of faces in the diagram Δ_1, bounded by the path $u_1p_{2u}^{-1}q_1^{-1}p_{1v}^{-1}v_1^{-1}$, is $n-1$. For convenience we denote $\phi(\partial_{(p_{2u})_-}\Pi)$ by R_1. By induction assumption, we have the following equality in G for the boundary of Δ':

$$\phi(u_1p_{2u}^{-1}q_1^{-1}p_{1v}v_1^{-1}) = \prod_{i=2}^{n} A_{R_j,u_j},$$

where for every $1 < j \le n$ we have $\psi(A_{R_j,u_j}) < 4max(|u_1|, |v_1|)$.

By Remark 5.6 part (ii), we have that $max(|u_1|, |v_1|) < max(|u|, |v|)$. By inequalities (16) and (22), we have $\left|u_1p_{2u}^{-1}\right| \le |u| - (\Pi, \Gamma_u, u)\,|\partial\Pi| + 2\epsilon + \epsilon < |u| - (\frac{1}{2} - 13\mu)\,|\partial\Pi| + 3\epsilon < |u| - \frac{1}{4}\,|\partial\Pi|$, hence

$$\psi(A_{R_1,u_1p_{2u}^{-1}}) = |\partial\Pi| + 4\left|u_1p_{2u}^{-1}\right| < |\partial\Pi| + 4\,|u| - |\partial\Pi| = 4\,|u|. \quad \square$$

Remark 5.11. Let Δ be a reduced octagon diagram with boundary $\partial\Delta = l_1j_1\dots l_4j_4$. Assume that $\phi(l_1)$ is a subword of some $R \in \mathcal{R}$ and $|l_1| \le \frac{1}{2}\,|R|$. Then l_1 satisfies $U_\Delta(\frac{1}{2} + \frac{\mu}{5})$.

Proof. Suppose on the contrary, there exists an $\mathcal{R}$–face Π and a contiguity subdiagram Γ such that $(\Pi, \Gamma, l_1) \ge \frac{1}{2} + \frac{\mu}{5}$, $\partial(\Pi, \Gamma, l_1) = p_1q_1p_2q_2$. Then, by $\tilde{C}(\epsilon, \mu, \rho$–condition, R and $\phi(\partial\Pi)$ are conjugate so $|\partial\Pi| = |R|$. Hence we get

$$\frac{1}{2}\,|\partial\Pi| \ge |l_1| \ge |q_1| - 2\epsilon \ge (\frac{1}{2} + \frac{\mu}{5})\,|\partial\Pi|,$$

which is a contradiction. $\square$

For technical reasons we introduce a notation

$$N_{R,U} = gp\langle A_{R',U'} \mid \psi(A_{R',U'}) < \psi(A_{R,U})\rangle.$$

We say that $A_{R',U'}$ *is equivalent* ($\approx$) *to* $A_{R,U}$ iff $\psi(A_{R',U'}) = \psi(A_{R,U})$ and there exists a word H in $N_{R,U}$ such that $HA_{R',U'}H^{-1} = A_{R,U}$ in G. To prove that the relation $\approx$ is a correctly defined equivalence it is enough to notice that $N_{R,U} = N_{R',U'}$ whenever $\psi(A_{R',U'}) = \psi(A_{R,U})$. It is clear that equivalence classes with respect to $\approx$ are finite.

Definition 5.12. *Let $\mathcal{A}$ be a maximal set of words $A_{R,U}$ where $R \in \mathcal{R}$, $U \in \mathcal{U}$ such that*

(i) $A_{R,U} \notin N_{R,U}$;

(ii) if $A_{R',U'} \approx A^{\pm 1}_{R,U}$, then at most one of them belongs to $\mathcal{A}$.

Lemma 5.13. *(i) Suppose that some geodesic word U contains more then half of a relation, then for every $R \in \mathcal{R}$ we have that $A_{R,U} \in \mathcal{N}_{R,U}$.*

(ii) If URU^{-1} is not geodesic up to 10δ then there exists a geodesic up to 10δ word $VR'V^{-1}$ such that $A_{R,U} = A_{R',V}$ in G and $\psi(A_{R,U}) > \psi(A_{R',V})$.

(iii) $\mathcal{A}$ is a subset of $\mathcal{X}$ from Lemma 2.11.

(iv) $\mathcal{A}$ generates $\mathcal{N}(\mathcal{R})$, moreover every $A_{R,U}$ is a product of elements of $\mathcal{A}^{\pm 1}$ with weights not larger then $\psi(A_{R,U})$.

Proof. Pick some word $A_{R,U}$.

(i) Assume that U contains more then half of a relation, then (using notations and statement of Remark 2.12(i)) we have

$$A_{R,U} = A_{r_1r_2,U_1}A_{R,U_1r_2^{-1}U_2}A^{-1}_{r_1r_2,U_1}, \text{ where } U = U_1r_1U_2,\ r_1r_2 \in \mathcal{R}, \quad (25)$$

and the following inequalities hold:

$$|r_1| + |U_1| + |U_2| \le |U| + 50\delta, \quad |r_1| \ge |r_2| + 60\delta. \quad (26)$$

It follows from 2.12(i) that $\psi(A_{R,U_1r_2^{-1}U_2}) < \psi(A_{R,U})$. Now we use inequalities (26) to estimate:

$$\psi(A_{r_1r_2,U_1}) = |r_1r_2| + 4\,|U_1| = |r_1| + |r_2| + 4\,|U_1| \le$$

$$2\,|r_1| + 4\,|U_1| = 2(|r_1| + |U_1|) + 2\,|U_1| \le$$

$$\le 2(|U| + 50\delta) + 2(|U| + 50\delta - |r_1|) \le 4\,|U| + 200\delta - \rho < 4\,|U|\,.$$

Hence $A_{R,U}$ is equal to the product (25) such that both $\psi(A_{r_1r_2,U_1})$ and $\psi(A_{R,U_1r_2^{-1}U_2})$ are strictly less then $\psi(A_{R,U})$ and we conclude that $A_{R,U} \in \mathcal{N}_{R,U}$. Contradiction with Definition 5.12. Hence, if $A_{R,U} \in \mathcal{A}$ then U does not contain more then half of a relation.

(ii) Suppose that $A_{R,U}$ is not geodesic up to 10δ. The Remark 2.12 (ii) implies that then there exists $R' \in \mathcal{R}$ and a geodesic word V such that $URU^{-1} = VR'V^{-1}$ in G. By the same remark, the word $VR'V^{-1}$ is geodesic up to 10δ and $|R| = |R'|$ and so $|U| > |V|$. Thus we have got inequality $\psi(A_{R,U}) > \psi(A_{R',V})$ contradicting the choice $A_{R,U} \in \mathcal{A}$ again.

(iii) Follows from (i) and (ii) by definition of $\mathcal{X}$ in Lemma 2.11.

(iv) By Lemma 2.11, if $g \in \mathcal{N}$ then $g = \prod_{s=1}^{n} U_s R_s U_s^{-1}$ for some $U_s R_s U_s^{-1} \in \mathcal{X}$. Hence it is enough to show that every $A_{R,U} \in \mathcal{X}$ is equal to a product of elements of $\mathcal{A}$. We proceed by induction on possible values of $k = \psi(*)$ on the set $\mathcal{X}$.

If $A_{R_0,1} \in \mathcal{X}$ has minimal weight $\psi(A_{R_0,1})$, we have that $\mathcal{N}_{R_0,1} = \{1\}$ and so $A_{R_0,1} \notin \mathcal{N}_{R_0,1}$. By maximality of the set $\mathcal{A}$, the exists a word $A_{R',U'} \in \mathcal{A}$ such that $A_{R_0,1} \approx A^{\pm 1}_{R',U'}$ which implies that $A_{R_0,1} = A^{\pm 1}_{R',U'}$ in G.

Now pick $A_{R,U} \in \mathcal{X}$ such that $\psi(A_{R,U}) = k$. There are two cases.

CASE 1. $A_{R,U} \in \mathcal{N}_{R,U}$. In this case $A_{R,U}$ is a product of words $A_{R',U'}$ such that $\psi(A_{R',U'}) < \psi(A_{R,U})$ and we are done by the induction assumption.

CASE 2. $A_{R,U} \notin \mathcal{N}_{R,U}$. Consider all words $A_{R',U'}$ such that $A_{R',U'} \approx A_{R,U}$. Clearly, $A_{R',U'} \notin \mathcal{N}_{R,U} = \mathcal{N}_{R',U'}$. By maximality of the set $\mathcal{A}$, there exists a word $A_{R',U'} \in \mathcal{A}$ and by Corollary 4.2 (iii) we have that there exists $H \in N_{R,U}$ such that $HA^{\pm 1}_{R',U'}H^{-1} = A_{R,U}$ in G. By induction assumption, H is a product of elements of $\mathcal{A}$ with weights smaller then $\psi(A_{R,U})$, while $\psi(A_{R,U}) = \psi(A_{R',U'})$. □

Lemma 5.14. *Let Δ be a reduced diagram over the group G_1 with boundary $upav^{-1}$ where $|p| \le \epsilon$, $\phi(u), \phi(v) \in \mathcal{U}$, $\phi(a)^{-1}A' \equiv R \in \mathcal{R}$ for some word A' and $|\phi(a)| \le \frac{1}{2}|R|$.*

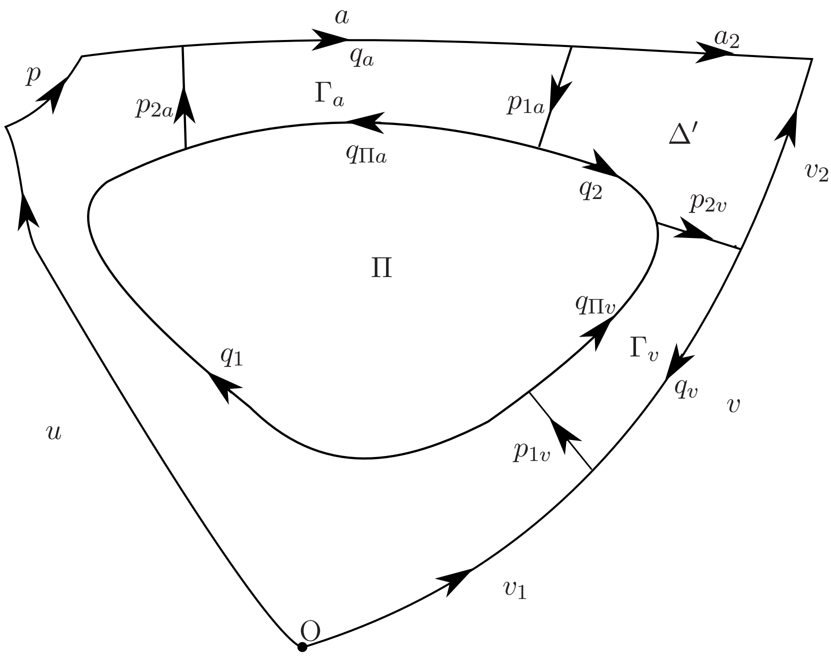
(i) Suppose that there exist an $\mathcal{R}$–face Π and contiguity subdiagrams Γ_a, Γ_v such that $(\Pi, \Gamma_a, a), (\Pi, \Gamma_v, v) \ge \mu$. Then $A_{R,v} \notin \mathcal{A}^{\pm 1}$.

(ii) Suppose that there exist an $\mathcal{R}$–face Π and disjoint contiguity subdiagrams Γ_a, Γ_u such that $(\Pi, \Gamma_a, a), (\Pi, \Gamma_u, u) \ge \mu$. In addition assume that $\phi(p)A'\phi(a)^{-1}\phi(p)^{-1} = R'$ in G for some $R' \in \mathcal{R}$. Then $A_{R',u} \notin \mathcal{A}^{\pm 1}$.

Proof. (i) We define arcs of Γ_a, Γ_v by equalities $\partial(\Pi, \Gamma_v, v) = p_{1v}q_{\Pi v}p_{2v}q_v$, $\partial(\Pi, \Gamma_a, a) = p_{1a}q_{\Pi a}p_{2a}q_a$ and define q_1, q_2 by equality $\partial\Pi = q_{\Pi v}^{-1}q_1q_{\Pi a}^{-1}q_2$. We also define v_1, v_2 by equality $v = v_1q_v^{-1}v_2$ (see Figure 6).

Consider a subdiagram Δ' with boundary $p_{2v}^{-1}q_2^{-1}p_{1a}^{-1}a_2v_2^{-1}$. Observe that q_2 satisfies $\mathcal{U}_{\Delta'}(\mu)$ by Lemma 3.6 (because it is a boundary subpath of the $\mathcal{R}$–face Π in the reduced diagram Δ), $|p_{1v}|, |p_{1a}| \le \epsilon$ and a_2, v_2 satisfy $\mathcal{U}_{\Delta'}(\frac{1}{2} + \frac{\mu}{5})$ (they are subpaths of a, v and a satisfies $\mathcal{U}_{\Delta}(\frac{1}{2} + \frac{\mu}{5})$ by Lemma 5.11). Choose $(a_2)_-$ as a base point of Δ. By Lemma 5.8, there exists a system of (a, v)–bonds $(V, \Sigma(\Delta'))$ such that V contains all $\mathcal{R}$–faces of Δ' and (assuming there are $\mathcal{R}$–faces in Δ'), by Remark 5.6, there exists a

Figure 6:



face Π' such that the diagram $\Delta_2(\Pi', \Delta')$ does not have $\mathcal{R}$–faces. The face Π' is in V so in order to simplify the notation we assume that $\Pi' = \Pi$ and Δ' itself is a diagram over G (i.e. it does not contain $\mathcal{R}$–faces).

Consider an $\mathcal{R}$–face $\bar{\Pi}$ disjoint from Δ and glue $\bar{\Pi}$ and Δ together along a. Define $\partial\bar{\Pi} = a^{-1}a'$ so that $\phi(a^{-1}a') \equiv R$. Since $(\Pi, \Gamma_a, a) \geq \mu$ we have that $\Pi, \bar{\Pi}$ comprise a pair of opposite faces with respect to p_{1a} hence

$$\phi((\partial_{(p_{1a})_+}\Pi)p_{1a}^{-1}(\partial_{(q_a)_+}\bar{\Pi})p_{1a}) = 1 \text{ in } G. \tag{27}$$

Now notice that $\phi(p_{1a}) = \phi(a_2 v_2^{-1} q_v p_{1v} q_{\Pi v} q_2^{-1})$ in the group G because it bounds the diagrams Δ' and Γ_v over G. We plug in the latter expression into the equation (27) and then conjugate by $\phi(p_{1v} q_{\Pi v} q_2^{-1})$ to obtain

$$\phi(p_{1v}[q_{\Pi v} q_2^{-1}(\partial_{(p_{2a})_+}\Pi) q_2 q_{\Pi v}^{-1}] p_{1v}^{-1} q_v^{-1} v_2 [a_2^{-1}(\partial_{(q_a)_+}\bar{\Pi}) a_2] v_2^{-1} q_v) = 1 \text{ in } G.$$

The paths in the square brackets are equal after elimination of returns to $\partial_{(p_{1v})_+}\Pi$ and $\partial_{v_+}\bar{\Pi}$ respectively. Denote $R' = \phi(\partial_{(p_{1v})_+}\Pi)$, recall that $R = \phi(\partial_{v_+}\bar{\Pi})$. Thus we have obtained that $A_{R', p_{1v}} A_{R, q_{\Pi}^{-1} v_2} = 1$ in G and, conjugating by v_1, we get:

$$A_{R', v_1 p_{1v}} A_{R, v} = 1 \text{ in } G. \tag{28}$$

But on the other hand we have that $|R| = |R'|$ (because they are labels of opposite $\mathcal{R}$–faces in Δ) and, using inequality (16),

$$|v_1 p_{1v}| \leq |v_1| + |p_{1v}| = |v| - |q_v| - |v_2| + |p_{1v}| \leq$$

$$\leq |v| - ((\Pi, \Gamma_v, v)\,|\partial\Pi| - 2\epsilon) + \epsilon < |v|\,.$$

Hence we get $\psi(A_{R', v_1 p_{1v}}) < \psi(A_{R,v})$ and so $A_{R,v} \notin \mathcal{A}^{\pm 1}$.

Proof of part (ii) repeats part (i) with obvious changes in notation. □

Recall that in the beginning of section 5 we chose constants ϵ, μ, ρ according to Lemmas 3.7, 3.8. Hence part (ii) of Theorem 1.3 follows immediately from aforementioned lemmas (and is due to Olshanskii [Olsh93]). We prove part (i) below:

Theorem 5.15. *The subgroup $\mathcal{N} = \mathcal{N}(\mathcal{R})$ is freely generated by the set $\mathcal{A}$.*

Proof. $\mathcal{A}$ generates $\mathcal{N}$ by lemma 5.13(iv).

We have to show that the set $\mathcal{A}$ generates $\mathcal{N}$ freely. We define a partial short-lex ordering on all words in alphabet $\mathcal{A}^{\pm 1}$. Let $W = A^{\epsilon_1}_{R_1,U_1} \ldots A^{\epsilon_k}_{R_k,U_k}$ ($\epsilon_i \in \pm 1$), $W' = \tilde{A}^{\epsilon'_1}_{R'_1,U'_1} \ldots \tilde{A}^{\epsilon_{k'}}_{R'_{k'},U'_{k'}}$, we say that $W \succ W'$ if either

(i) $k > k'$ or

(ii) length of W is equal to length of W' ($k = k'$) and there exists $m_0 \leq k$ such that $\psi(A_{R_m,U_m}) = \psi(\tilde{A}_{R'_m,U'_m})$ for any $m < m_0$ and $\psi(A_{R_{m_0},U_{m_0}}) > \psi(\tilde{A}_{R'_{m_0},U'_{m_0}})$.

Let $W(\mathcal{A}) \equiv A^{\epsilon_1}_{R_1,U_1} \ldots A^{\epsilon_n}_{R_n,U_n}$ be a nontrivial freely reduced word (in alphabet $\mathcal{A}$) such that $W = 1$ in G, assume that it is minimal with respect to the above ordering $\succ$. We are in position to apply Corollary 4.2 and consider the corresponding standard diagram Δ for the word W, a reducible pair of indexes $i < j$, the standard contiguity subdiagram Γ between Π_i and Π_j with $|p_1| < 11\delta + 3$. We apply Lemma 3.5 to faces Π_i, Π_j, path p_1 and vertices $o_1 = ({}_\gamma U_i)_+, o_2 = ({}_\gamma U_j)_+$. It provides the path $s_1 p_1 s_2$ in Δ such that $\phi(s_1 p_1 s_2) = P\phi(a)$ in G with $|P| \leq 11\delta + 3 + 8\delta$, $|a| \leq \frac{1}{2}\,|\partial\Pi_j|$, a is a subpath of $\partial\Pi_j$ and (using formula (6)) provides the equality $(P\phi(a))^{-1} R_i^{\epsilon_i} (P\phi(a)) R_j^{\epsilon_j} = 1$ in G or, equivalently,

$$P^{-1} R_i^{\epsilon_i} P[\phi(a) R_j^{\epsilon_j} \phi(a)^{-1}] = 1 \text{ in } G, \tag{29}$$

where the the word $[\phi(a) R_j^{\epsilon_j} \phi^{-1}(a)]$ is a cyclic conjugation of $R_j^{\epsilon_j}$ so $R_j^{\epsilon_j} \equiv \phi^{-1}(a) A'$ for some A'.

We have that the path ${}_\gamma U_i s_1 p_1 s_2 ({}_\gamma U_j)^{-1}$ is closed in the standard diagram Δ by Remark 4.4 and we have chosen $s_1 p_1 s_2$ so that

$$\phi({}_\gamma U_i s_1 p_1 s_2 {}_\gamma U_j^{-1}) = U_i P \phi(a) U_j^{-1}. \tag{30}$$

Consider a reduced diagram $\tilde{\Delta}$ with boundary upa_1v^{-1} such that $\phi(u) = U_i$, $\phi(p) = P$, $\phi(a_1) = A$, where $A = \phi(a)$, $a \in \Delta$, $\phi(v) = U_j$. We will show that in fact it satisfies conditions of Lemma 5.10. We first check conditions of Lemma 5.14: we have that paths u, v are in $\mathcal{U}$, thus they satisfy condition $\mathcal{U}_{\tilde{\Delta}}(\frac{1}{2} + \frac{\mu}{5})$ by Lemma 5.4 and so does the path a_1 by Lemma 5.11. We also have that $\phi(v)R_j^{\epsilon_j}\phi^{-1}(v) \in \mathcal{A}^{\pm 1}$ by definition of v and $R_i^{\epsilon_i} = PA'\phi^{-1}(a_1)P^{-1}$ by equation (29), so Lemma 5.14 provides us that there are no (u, a_1)– or (v, a_1)–bonds in $\tilde{\Delta}$. We have just checked the conditions of Lemma 5.10 for the diagram $\tilde{\Delta}$ and conclude that:

$$\phi(ua_1sv^{-1}) = \prod_{m=1}^{k} A_{R'_m,U'_m} \text{ in } G,$$

where $max_{1\leq m\leq k}\psi(A_{R'_m,U'_m}) < 4max(|u|, |v|)$.

The last relation together with (30) implies that $\phi({}_\gamma U_i s_1 p_1 s_{2\gamma} U_j^{-1})$ belongs to at least one of the groups $\mathcal{N}_{R_i,U_i}, \mathcal{N}_{R_j,U_j}$. By Corollary 4.2(iii), we have that $\phi({}_\gamma U_i s_1 p_1 s_{2\gamma} U_j^{-1}) = H$ in G (where $H \equiv \prod_{k=i+d}^{j-c} A^{\epsilon_k}_{R_k,U_k}$, $(c, d) \neq (0, 0)$, $c, d \in \{0, 1\}$) and that

$$H^{-1}A^{\epsilon_i}_{R_i,U_i}H = A^{e}_{R_j,U_j} \text{ in } G \text{ for some } e \in \{\pm 1\}. \tag{31}$$

Suppose that $A_{R_i,U_i} \succ A_{R_j,U_j}$, then both words H and A_{R_j,U_j} belong to $\mathcal{N}_{R_i,U_i}$. Hence $A_{R_i,U_i} \in \mathcal{N}_{R_i,U_i}$, contradiction.

It remains consider the case when $\psi(A_{R_i,U_i}) = \psi(A_{R_j,U_j})$. By equation (31), $A_{R_i,U_i} \approx A^{e}_{R_j,U_j}$ and since they are both in $\mathcal{A}$ we have that $U_i \equiv U_j$, $R_i \equiv R_j$. Thus we can glue together the paths u and v of the boundary of $\tilde{\Delta}$ and obtain a diagram with boundary pa_1 (we will also call it $\tilde{\Delta}$). For every $\mathcal{R}$–face Π in $\tilde{\Delta}$ we now have that $(\Pi, \Gamma_p, p) \leq 3\epsilon$ because $|p| \leq \epsilon$ and $(\Pi, \Gamma_{a_1}, a_1) \leq \frac{1}{2} + \frac{1}{5}\mu$ thus

$$(\Pi, \Gamma_{a_1}, a_1) + (\Pi, \Gamma_p, p) \leq \frac{1}{2} + \frac{\mu}{5} + 3\epsilon < 1 - 23\mu,$$

which contradicts Lemma 3.7. Hence there are no $\mathcal{R}$–faces in $\tilde{\Delta}$ and $H = \phi(pa_1) = 1$ in G. But the word $H \equiv \prod_{k=i+d}^{j-c} A_{R_k,U_k}$ is a subword of W which is strictly shorter then W so $W \succ H$ and $H = 1$ in G. By minimality of W, we have equality $H \equiv 1$ which can only happen if $i + 1 = j$ so $A^{\epsilon_i}_{R_i,U_i}A^{\epsilon_{i+1}}_{R_{i+1},U_{i+1}}$ is a subword of W, $U_i \equiv U_j$, $R_i \equiv R_j$ and by the relation (9) in G:

$$U_iR_i^{\epsilon_i}U_i^{-1}U_{i+1}R_{i+1}^{\epsilon_{i+1}}U_{i+1}^{-1} \equiv U_iR_i^{\epsilon_i}U_i^{-1}U_iR_i^{\epsilon_{i+1}}U_i^{-1} = 1,$$

which is equivalent to $R_i^{\epsilon_i+\epsilon_{i+1}} = 1$ in G and, taking into account the

Remark 5.1, we have that $\epsilon_i + \epsilon_{i+1} = 0$. Hence $A^{\epsilon_i}_{R_i,U_i} A^{\epsilon_{i+1}}_{R_{i+1},U_{i+1}} \equiv A^{\epsilon_i}_{R_i,U_i} A^{-\epsilon_i}_{R_i,U_i}$ is a subword of W. Contradiction with choice of W. □

Following [Olsh93], we call a pair of elements x, y of infinite order in G *non-commensurable* if x^k is not conjugate to y^s for any non-zero integers k, s. A group G is called *non-elementary* if it contains a finite index subgroup isomorphic to $\mathbb{Z}$.

In order to deduce Theorem 1.4 we will use the following remark.

Remark 5.16 ([Swe] Theorem 13)**.** (i) For every element x in a hyperbolic group G there exists $n > 0$ and a *straight word* Y_x (i.e. a word Y_x such that Y_x^s is geodesic for every s) such that Y_x is conjugate to x^n.

(ii) Given a set of geodesic words words $X_1, \dots, X_m$ we will denote by $\mathcal{R}_n = \mathcal{R}(X_1^{s_1}, \dots, X_m^{s_m}, n)$ a system of all cyclic permutations of $R_i^{\pm 1}$ where $R_i \equiv X_i^{s_i n}$. If $X_1, \dots, X_m$ are straight pairwise non-commensurable words in G, then for every $\mu > 0$, $\epsilon \geq \epsilon_0$ and $\rho > 0$ there exists a number $n > 0$ such that $\mathcal{R}_n$ satisfies $C(\epsilon, \mu, \rho)$–condition independent of a choice of non-zero integers $s_1, \dots, s_m$.

(iii) If Y is a straight word in G then for every integer m the word Y^m has a minimal length in it's conjugacy class.

Proof. Proof of part (ii) up to minor modifications repeats the proof of lemma 4.1 in [Olsh93] which states the same property for $m = 1$.

Part (iii). Assume that $Y^s = TZT^{-1}$ for some T and that $|Z| \leq |Y^s| - 1$ then for every k we have that

$$k\,|Z| + k \leq k(|Y^s| - 1) + k = k(|Y^s|) = \left|Y^{sk}\right| \leq 2\,|T| + \left|Z^k\right| \leq 2\,|T| + k\,|Z|,$$

which implies that $k \leq 2\,|T|$. Contradiction. □

Proof of Theorem 1.4. Let us first consider a set of pairwise non-commensurable elements $x_1, \dots, x_m$ of infinite order. By remark 5.16 (i), for each x_i there exists a straight word $\bar{Y}_{x_i}$ conjugate to $x_i^{n_i}$ for some $n_i > 0$. Define $n_0 = \prod_{1 \leq i \leq m} n_i$. Clearly words $Y_{x_1} \equiv \bar{Y}_{x_1}^{n_0}, \dots, Y_{x_m} \equiv \bar{Y}_{x_m}^{n_0}$ are pairwise non-commensurable and, by parts (ii) and (iii) remark 5.16, there exists an integer $K > 0$ such that the system $\mathcal{R}_K = \mathcal{R}(Y_1^{s_1}, \dots, Y_m^{s_m}, K)$ satisfies $\tilde{C}(\epsilon, \mu, \rho$–condition for any choice of positive $s_1, \dots, s_m$. By Theorem 1.3, the group $\mathcal{N}(\mathcal{R}_K)$ is free and the quotient $G/\mathcal{N}(\mathcal{R}_K)$ is non-elementary hyperbolic.

Now consider an arbitrary set of elements $x_1, \dots, x_m$ in G. If some of the elements x_i have finite orders $n_{i_1}, \dots, n_{i_q}$ we define $n_0 = n_{i_1} \dots n_{i_q}$ and replace the set $x_1, \dots, x_m$ with $x_1^{n_0}, \dots, x_m^{n_0}$ (which after deletion of identity elements contains only the elements of infinite order). Hence we can assume that all elements $x_1, \dots, x_m$ are of infinite order. For every pair x_i, x_j $(i < j)$ define a pair of nonzero integers k_{ij}, k_{ji} such that $x_i^{k_{ij}}$ is conjugate to $x_j^{k_{ji}}$ if x_i, x_j are commensurable and let $k_{ij} = k_{ji} = 1$ if the pair x_i, x_j is not commensurable. Define $K_0 = \prod_{1 \le i,j \le m} k_{ij}$ and let $K_0 = 1$ if $m = 1$. We show by induction on m that

there exists an integer N *such that* $\mathcal{N} = \mathcal{N}(x_1^{s_1 K_0 N}, \dots, x_m^{s_m K_0 N})$ *is free for any choice of integers* $s_1, \dots, s_m$.

We have showed that the statement holds if the elements $x_1, \dots, x_m$ are pairwise non-commensurable and in particular if $m = 1$. Hence, in order to prove the induction step, we may assume that (after reenumeration of x_i's) x_1 is commensurable to x_2. Using the normality of $\mathcal{N}$ and the fact that for every $x \in G$ a subgroup generated by x^a, x^b is the equal to the one generated by $x^{gcd(a,b)}$ we get that

$$\mathcal{N}(x_1^{s_1 K_0 N}, x_2^{s_2 K_0 N}, ..., x_m^{s_m K_0 N}) = \mathcal{N}(x_1^{k_{12} s_1 \frac{K_0}{k_{12}} N}, x_2^{s_2 K_0 N}, ...) =$$

$$\mathcal{N}(x_2^{k_{21} s_1 \frac{K_0}{k_{12}} N}, x_2^{s_2 K_0 N}, ...) = \mathcal{N}(x_2^{gcd(k_{21} s_1 \frac{K_0}{k_{12}}, s_2 K_0) N}, x_3^{s_3 K_0 N}, ..., x_m^{s_m K_0 N}).$$

Thus $\mathcal{N}$ is generated by $m-1$ elements and we may apply the induction assumption completing the proof of theorem 1.4. □

We recall the notions of an SQ-universal group and a CEP-subgroup. A group G is said to be *SQ–universal* if every countable group K embeds in a quotient of G. Let H be a subgroup of G, then H is said to have a *congruence extension property* (CEP) if for every subgroup K, $K \triangleleft H$ there exists a subgroup K_1, $K_1 \triangleleft G$, such that $K_1 \cap H = K$. It is easy to see that if the group G has a free infinitely generated CEP-subgroup then G is SQ–universal (see, for example, Proposition [Olsh95]).

Proof of Corollary 1.5. (i) If G is non-elementary, there exists a pair of non-commensurable straight words X_1, X_2 in G (see for example [Olsh93], Lemma 1.14). By Remark 5.16, there exists a number n such that $\mathcal{R} = \mathcal{R}(X_1, X_2, n)$ satisfies the small cancellation property $\tilde{C}(\epsilon, \mu, \rho$–condition for sufficiently small μ and hence $\mathcal{N}(\mathcal{R})$ is a free group by Theorem 1.4. The rank $\mathcal{N}(\mathcal{R})$ is greater then 1 because X_1, X_2 are non-commensurable.

(ii) It is a result of Olshanskii [Olsh95] that

(*) inside every non-elementary subgroup of G *there exists a free countably generated CEP-subgroup in* G (Theorem 4, [Olsh95]);

Consider a free normal subgroup $\mathcal{N}$ in G of rank greater then 1. There exists a free infinite rank CEP-subgroup N_1 in G, $N_1 < \mathcal{N}$ by (*). Hence for every countable group H there exists $M_1 \triangleleft N_1$ such that $H \cong N_1/M_1$. By congruence extension property, the (normal in G) subgroup $M = M_1^G$ satisfies $M \cap H = M_1$, so H embeds in G/M. Clearly $M = M_1^G$ is free (being a subgroup of a free group $\mathcal{N}$), and thus (ii) is proved. □

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