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RESEARCH ARTICLE

On the one-side equivalence of matrices with given canonical diagonal form

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Abstract. The simpler form of a matrix with canonical diagonal form $diag(1,\dots,1,\varphi,\dots,\varphi)$ obtained by the one-side transformation is determined.

Let R be an adequate ring [1] i.e. a commutative domain in which every finitely generated ideal is principal, and which further satisfies the following condition: for any $a, c \in R$ with $a \neq 0$, one can write $a = rs$ with $(r, c) = 1$ and $(s', c) \neq 1$ for any non unit divisor s' of s. Let A be an $n \times n$ matrix over R. It is known [1] that there exist invertible matrices P, Q, such that

$$PAQ = diag(\varphi_1, \dots, \varphi_n) = \Phi. \tag{1}$$

The matrix Φ is called the canonical diagonal form of the matrix A, $\varphi_i |\, \varphi_{i+1}$, $i = 1, \dots, n-1$. In solving of some matrix problems especially factorization of matrices [2,3], in description of all the Abelian subgroups [4], there emerges the necessity of finding all the non-associated matrices with canonical diagonal form given beforehand. Usual Hermite normal form does not approach to our purposes because it evaluates in the rough way and gives a possibility to describe non-associated matrices with set-up determinant only. That is why there emerges the necessity of building such form of matrix with respect to one sided transformation, giving a glance to which is enough to make a decision as for its canonical diagonal form. The equality (1) gives us a possibility to write matrix A in the following way $A = P^{-1}\Phi Q^{-1}$. Making changes in its right part we will have a new form

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$P^{-1}\Phi$. But this type of matrices is not a normal form of the matrix A as for the right side changes because the matrix P determined ambiguously.

By [2] the set $\mathbf{P_A}$ of all invertible matrices which satisfies equation (1) has the form $\mathbf{P_A} = G_\Phi P$, where

$$G_\Phi = \{H \in GL_n(R) \mid H\Phi = \Phi H_1, H_1 \in GL_n(R)\}.$$

This set is a multiplicative group and if $\det \Phi \neq 0$ consists of all invertible matrices of the form

$$H = \left\| \begin{matrix} h_{11} & h_{12} & \dots & h_{1.n-1} & h_{1n} \\ \frac{\varphi_2}{\varphi_1} h_{21} & h_{22} & \dots & h_{2.n-1} & h_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_1} h_{n1} & \frac{\varphi_n}{\varphi_2} h_{n2} & \dots & \frac{\varphi_n}{\varphi_{n-1}} h_{n.n-1} & h_{nn} \end{matrix} \right\|.$$

Thus, $\mathbf{P_A}$ is a left conjugacy class $GL_n(R)$ with respect to the group G_Φ. Therefore, in order that the matrix $P^{-1}\Phi$ be a normal form of the matrix A, with respect to the transformation from the right, it is necessary either to choose a representative in the class $G_\Phi P$ or, what is the same, indicate the normal form of the invertible matrices with respect to the action of the group G_Φ. The present paper is devoted to the investigation of this question.

Let $\Phi = E_t \oplus \varphi E_{n-t}$, $\Phi_* = \varphi E_t \oplus E_{n-t}$, $\varphi \neq 0$, $1 \leq t < n$, where E_t is the identity $t \times t$ matrix. In this case, the group G_Φ consists of all invertible matrices of the form

$$\left\| \begin{matrix} H_{11} & H_{12} \\ \varphi H_{21} & H_{22} \end{matrix} \right\|,$$

where H_{11} is a $t \times t$ matrix. A matrix is called primitive if the greatest common divisor of minor of maximal order is equal to 1. The matrix A is called left associate to the matrix B if $A = UB$, where $U \in GL_n(R)$. This fact will be denoted $A \overset{l}{\sim} B$.

Lemma 1. *Let*

$$B = \left\| \begin{matrix} B_1 \\ B_2 \\ B_3 \end{matrix} \right\|$$

be a primitive $n \times (n-k+1)$ *matrix,* $t < k < n$. *The matrices* B_1, B_3 *is* $t \times (n-k+1)$, $(n-k+1) \times (n-k+1)$ *matrices, respectively. Let*

$$\Phi_* B \overset{l}{\sim} \left\| \begin{matrix} 0 \\ 0 \\ B_3 \end{matrix} \right\|. \tag{2}$$

Then there exists a matrix $H \in G_\Phi$ such that

$$HB = \left\| \begin{matrix} B_1 \\ 0 \\ B_3. \end{matrix} \right\|.$$

Proof. Consider the matrix equation

$$XB_3 = \left\| \begin{matrix} \varphi B_1 \\ B_2 \end{matrix} \right\|. \tag{3}$$

The matrix

$$\Phi_* B = \left\| \begin{matrix} \varphi B_1 \\ B_2 \\ B_3 \end{matrix} \right\|$$

is extended matrix of equation (3). From (2) it follows that the invariant factors of the matrices $\Phi_* B, B_3$ are equal. By Theorem 2 from [3, p. 218] equation (3) has the solution $X = U = \left\| \begin{matrix} U_1 \\ U_2 \end{matrix} \right\|$, where U_1 is a $t \times (n-k+1)$ matrix and U_2 is a $(k-t-1) \times (n-k+1)$ matrix. Then

$$\left\| \begin{matrix} E_t & 0 & -U_1 \\ 0 & E_{k-t-1} & -U_2 \\ 0 & 0 & E_{n-k+1} \end{matrix} \right\| \left\| \begin{matrix} \varphi B_1 \\ B_2 \\ B_3 \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} E_{k-1} & -U \\ 0 & E_{n-k+1} \end{matrix} \right\| \left\| \begin{matrix} \varphi B_1 \\ B_2 \\ B_3 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ B_3 \end{matrix} \right\|.$$

This implies that

$$\underbrace{\left\| \begin{matrix} E_t & 0 & 0 \\ 0 & E_{k-t-1} & -U_2 \\ 0 & 0 & E_{n-k+1} \end{matrix} \right\|}_{H} \left\| \begin{matrix} B_1 \\ B_2 \\ B_3 \end{matrix} \right\| = \left\| \begin{matrix} B_1 \\ 0 \\ B_3 \end{matrix} \right\|.$$

Observing that $H \in G_\Phi$, we conclude the proof of the lemma. □

Lemma 2. *Let A be an $n \times m$ matrix and $H \in G_\Phi$. Then*

$$\Phi_* HA \overset{l}{\sim} \Phi_* A.$$

Proof. Since

$$H = \left\| \begin{matrix} H_{11} & H_{12} \\ \varphi H_{21} & H_{22} \end{matrix} \right\|,$$

where H_{11} is a $t\times t$ matrix we have

$$\Phi_* H = \left\| \begin{array}{cc} \varphi H_{11} & \varphi H_{12} \\ \varphi H_{21} & H_{22} \end{array} \right\| = \left\| \begin{array}{cc} H_{11} & \varphi H_{12} \\ H_{21} & H_{22} \end{array} \right\| \Phi_* = H_1 \Phi_*.$$

The matrix Φ_* is nonsingular, so that $\det H = \det H_1$ i.e. the matrix H_1 is invertible. Consequently,

$$\Phi_* H A = H_1 \Phi_* A \overset{l}{\sim} \Phi_* A.$$

□

Lemma 3. *Let*

$$B_k = \left\| \begin{array}{c|ccccc} b_{11} & b_{12} & b_{13} & \dots & b_{1.n-k} & b_{1.n-k+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{t1} & b_{t2} & b_{t3} & \dots & b_{t.n-k} & b_{t.n-k+1} \\ \hline b_{t+1.1} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{k1} & 0 & 0 & \dots & 0 & 0 \\ \hline b_{k+1.1} & \beta_{k+1} & 0 & & 0 & 0 \\ b_{k+2.1} & b_{k+2.2} & \beta_{k+2} & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \\ b_{n-1.1} & b_{n-1.2} & b_{n-1.3} & & \beta_{n-1} & 0 \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{n.n-k} & \beta_n \end{array} \right\| =$$

$$= \left\| \begin{array}{cc} B_{11} & B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|$$

be a primitive $n\times(n-k+1)$ matrix, $t<k<n$, and

$$\Phi_* B_k \overset{l}{\sim} \left\| \begin{array}{cc} 0 & 0 \\ D_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|, \tag{4}$$

where

$$D_{21} = \left\| \begin{array}{cccc} 0 & \dots & 0 & \beta_k \end{array} \right\|^T.$$

Then there exists a matrix $H \in G_\Phi$ such that

$$H B_k = \left\| \begin{array}{cc} B'_{11} & B'_{12} \\ D_{21} & 0 \\ B_{31} & B_{32} \end{array} \right\|. \tag{5}$$

Proof. Consider the equation

$$XB_{32} = \varphi B_{12}. \tag{6}$$

The equality

$$\Phi_* B_k = \left\| \begin{matrix} \varphi B_{11} & \varphi B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{matrix} \right\|,$$

is valid. From (4) we conclude that

$$\left\| \begin{matrix} \varphi B_{12} \\ 0 \\ B_{32} \end{matrix} \right\| \stackrel{l}{\sim} \left\| \begin{matrix} 0 \\ 0 \\ B_{32} \end{matrix} \right\|.$$

This implies that the invariant factors of the matrices B_{32}, $\left\| \begin{matrix} \varphi B_{12} \\ B_{32} \end{matrix} \right\|$ are equal. By Theorem 2 from [3, p. 218], equation (6) has the solution $X = U_{13}$. Thus, the equality

$$\left\| \begin{matrix} E_t & 0 & -U_{13} \\ 0 & E_{k-t} & 0 \\ 0 & 0 & E_{n-k} \end{matrix} \right\| \left\| \begin{matrix} \varphi B_{11} & \varphi B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{matrix} \right\| = \left\| \begin{matrix} B'_{11} & 0 \\ B_{21} & 0 \\ B_{31} & B_{32} \end{matrix} \right\|,$$

holds, where

$$B'_{11} = \left\| \begin{matrix} b'_{11} & \dots & b'_{t1} \end{matrix} \right\|^T.$$

By Lemma 2,

$$\left\| \begin{matrix} B'_{11} & B'_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \end{matrix} \right\| \stackrel{l}{\sim} \left\| \begin{matrix} 0 & 0 \\ D_{21} & 0 \\ B_{31} & B_{32} \end{matrix} \right\|,$$

so that

$$(b'_{11}, \dots, b'_{t1}, b_{t+1.1}, \dots, b_{k1}) = \beta_k.$$

According to property 6 from [5], there exist $v_1, \dots, v_k$ such that

$$v_1 b'_{11} + \dots + v_t b'_{t1} + v_{t+1} b_{t+1.1} + \dots + v_k b_{k1} = \beta_k,$$

and

$$(v_k, \varphi) = 1.$$

Let us complement the primitive row $\left\| \begin{matrix} v_1 & \dots & v_k \end{matrix} \right\|$ to an invertible matrix V_k in which this row is the last. Consider the invertible matrix

$$\left\| \begin{matrix} V_k & 0 \\ 0 & E_{n-k} \end{matrix} \right\| \left\| \begin{matrix} E_t & 0 & -U_{13} \\ 0 & E_{k-t} & 0 \\ 0 & 0 & E_{n-k} \end{matrix} \right\| = \left\| \begin{matrix} V_k & U \\ 0 & E_{n-k} \end{matrix} \right\| = V.$$

Taking into account that $\| \, v_1 \; \dots \; v_k \; u_{k+1} \; \dots \; u_n \, \|$ is the k-th row of this matrix, we obtain

$$\| \, v_1 \; \dots \; v_k \; u_{k+1} \; \dots \; u_n \, \| \Phi_* B_k = \| \, \beta_k \; 0 \; \dots \; 0 \, \|,$$

i.e.,

$$\| \, \varphi v_1 \; \dots \; \varphi v_t \; v_{t+1} \; \dots \; v_k \; u_{k+1} \; \dots \; u_n \, \| B_k = \\ = \| \, \beta_k \; 0 \; \dots \; 0 \, \|.$$

Since

$$(v_1, \dots, v_k) = 1, (v_k, \varphi) = 1,$$

we have

$$(\varphi v_1, \dots, \varphi v_t, v_{t+1}, \dots, v_k) = 1.$$

It means that the matrix

$$F_k = \left\| \begin{matrix} \varphi v_1 & \dots & \varphi v_t & v_{t+1} & \dots & v_k & u_{k+1} & \dots & u_n \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & & \ddots & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & & 1 \end{matrix} \right\|$$

is primitive. By property 2 from [5], the matrix F_k can be complemented to an invertible matrix $H_k = \left\| \begin{matrix} * \\ F_k \end{matrix} \right\|$ which belongs to G_Φ. Then

$$H_k B_k = \left\| \begin{matrix} b'_{11} & b'_{12} & \dots & b'_{1.n-k+1} \\ \dots & \dots & \dots & \dots \\ b'_{t1} & b'_{t2} & \dots & b'_{t.n-k+1} \\ \hline b'_{t+1.1} & b'_{t+1.2} & \dots & b'_{t+1.n-k+1} \\ \dots & \dots & \dots & \dots \\ b'_{k-1.1} & b'_{k-1.2} & \dots & b'_{k-1.n-k+1} \\ \hline \beta_k & 0 & & 0 \\ b_{k+1.1} & \beta_{k+1} & & 0 \\ & & \ddots & \\ b_{n1} & b_{n2} & & \beta_n \end{matrix} \right\| = \left\| \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \right\|.$$

By Lemma 2

$$\Phi_* H_k B_k \overset{l}{\sim} \left\| \begin{matrix} 0 & 0 \\ D_{21} & 0 \\ B_{31} & B_{32} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ A_3 \end{matrix} \right\|.$$

According to Lemma 1, the group G_Φ contain a matrix H'_k such that

$$H'_k H_k \left\| \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \right\| = \left\| \begin{matrix} A_1 \\ 0 \\ A_3 \end{matrix} \right\|,$$

which has form (5). The proof is complete. $\square$

Let us denote by $K(f)$ the set of representatives of the conjugate classes of $R/Rf, f \in R$.

Theorem 1. *Let* $B = \|b_{ij}\|_1^n = \left\| \begin{matrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{matrix} \right\|$ *be an ivertible matrix, where* B_{11} *is a* $t \times t$ *matrix and*

$$\Phi_* B \overset{l}{\sim} \left\| \begin{matrix} \beta_1 & 0 & & 0 \\ * & \beta_2 & & 0 \\ & & \ddots & \\ * & * & & \beta_n \end{matrix} \right\| \tag{7}$$

is the left Hermite normal form of the matrix $\Phi_* B$. *Then the group* G_Φ *contains a matrix* H *such that*

$$HB = \left\| \begin{matrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{matrix} \right\|, \tag{8}$$

where

$$C_{22} = \left\| \begin{matrix} \beta_{t+1} & 0 & & 0 \\ c_{t+2.t+1} & \beta_{t+2} & & 0 \\ \vdots & & \ddots & \\ c_{n.t+1} & c_{n.t+2} & & \beta_n \end{matrix} \right\|,$$

$c_{ij} \in K(\beta_j)$, $i = t+2, t+3, \ldots, n, j = t+1, t+2, \ldots, n-1$. *The elements* c_{ij} *are uniquely determined and do not depend on the choice of the matrix* H.

Proof. Using (6) we obtain

$$\Phi_* \left\| \begin{matrix} b_{1n} & b_{2n} & \ldots & b_{nn} \end{matrix} \right\|^T \sim \left\| \begin{matrix} 0 & \ldots & 0 & \beta_n \end{matrix} \right\|^T.$$

By Theorem 2 from [6], there exists a matrix $H_n \in G_\Phi$ such that

$$H_n \left\| \begin{matrix} b_{1n} & b_{2n} & \ldots & b_{nn} \end{matrix} \right\|^T = \left\| \begin{matrix} b'_{1n} & \ldots & b'_{n-1.n} & \beta_n \end{matrix} \right\|^T.$$

According to Lemma 2

$$\Phi_* \left\| \begin{matrix} b'_{1n} & \ldots & b'_{n-1.n} & \beta_n \end{matrix} \right\|^T =$$

$$= \left\| \begin{matrix} \varphi b'_{1n} & \ldots & \varphi b'_{tn} & b'_{t+1.n} & \ldots & b'_{n-1.n} & \beta_n \end{matrix} \right\|^T \sim$$

$$\sim \left\| \begin{matrix} 0 & \ldots & 0 & \beta_n \end{matrix} \right\|^T.$$

Therefore $b'_{in} = \beta_n d_i$, $i = t+1, t+2, \ldots, n-1$. Then

$$\left(E_{n-t} \oplus \left\| \begin{matrix} 1 & 0 & & 0 & -d_{t+1} \\ 0 & 1 & & 0 & -d_{t+2} \\ & & \ddots & & \vdots \\ 0 & 0 & & 1 & -d_{n-1} \\ 0 & 0 & & 0 & 1 \end{matrix} \right\| \right) H_n B = B_n.$$

Using Lemmas 2 and 3 in consequently to the last two columns of the matrix B_n, to the last three columns of the derived matrix and so fors we get $H_{t+1} \in G_\Phi$ such that

$$H_{t+1}B = \left\| \begin{matrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{matrix} \right\|,$$

where

$$D_{22} = \left\| \begin{matrix} \beta_{t+1} & 0 & & 0 \\ d_{t+2.t+1} & \beta_{t+2} & & 0 \\ \vdots & & \ddots & \\ d_{n.t+1} & d_{n.t+2} & & \beta_n \end{matrix} \right\|.$$

There exists a lower unitriangular matrix U such that

$$UD_{22} = \left\| \begin{matrix} \beta_{t+1} & 0 & & 0 \\ c_{t+2.t+1} & \beta_{t+2} & & 0 \\ \vdots & & \ddots & \\ c_{n.t+1} & c_{n.t+2} & & \beta_n \end{matrix} \right\|$$

is the left Hermite normal form of the matrix D_{22}, i.e., $c_{ij} \in K(\beta_j)$, $i = t+2, t+3, \ldots, n, j = t+1, t+2, \ldots, n-1$. The the matrix $(E_t \oplus U)H_{t+1}B$ has the form (8) and $(E_t \oplus U)H_{t+1}B \in G_\Phi$.

We will show the uniqueness of the elements c_{ij}. Let $H_1 \in G_\Phi$ and

$$H_1 B = \left\| \begin{matrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{matrix} \right\|,$$

where

$$C'_{22} = \left\| \begin{matrix} \beta_{t+1} & 0 & & 0 \\ c'_{t+2.t+1} & \beta_{t+2} & & 0 \\ \vdots & & \ddots & \\ c'_{n.t+1} & c'_{n.t+2} & & \beta_n \end{matrix} \right\|,$$

$c'_{ij} \in K(\beta_j)$, $i = t+2, t+3, \ldots, n, j = t+1, t+2, \ldots, n-1$. Concerning the proof of Lemma 3 we conclude that there exists an upper unitriangular matrix U such that

$$U\Phi_* \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\| = \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\|. \tag{9}$$

Since

$$H \left\| \begin{array}{c} B_{12} \\ B_{22} \end{array} \right\| = \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\|$$

and

$$H_1 \left\| \begin{array}{c} B_{12} \\ B_{22} \end{array} \right\| = \left\| \begin{array}{c} C'_{12} \\ C'_{22} \end{array} \right\|,$$

we obtain

$$\left\| \begin{array}{c} C'_{12} \\ C'_{22} \end{array} \right\| = H_2 \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\|, H_2 = H_1 H^{-1} \in G_\Phi.$$

Rewrite (9) us

$$U\Phi_* H_2^{-1} H_2 \left\| \begin{array}{c} C_{12} \\ C_{22} \end{array} \right\| = \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\|.$$

By Lemma 2

$$\Phi_* H_2^{-1} = H_3 \Phi_*, H_3 \in GL_n(R).$$

Thus,

$$UH_3\Phi_* \left\| \begin{array}{c} C'_{12} \\ C'_{22} \end{array} \right\| = \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\|.$$

Let

$$(UH_3)^{-1} = \left\| \begin{array}{cc} V_{11} & V_{12} \\ V_{21} & V_{22} \end{array} \right\|,$$

where V_{11} is a $t \times t$ matrix. Then

$$\left\| \begin{array}{cc} V_{11} & V_{12} \\ V_{21} & V_{22} \end{array} \right\| \left\| \begin{array}{c} 0 \\ C_{22} \end{array} \right\| = \left\| \begin{array}{c} \varphi C'_{12} \\ C'_{22} \end{array} \right\|,$$

i.e., $V_{22}C_{22} = C'_{22}$. Since $|C_{22}| = |C'_{22}|$, the matrix V_{22} is invertible. Hence, the matrices C_{22}, C'_{22} are left assosiated. Therefore their left Hermite normal forms are equal. Remark that the matrices C_{22}, C'_{22} are precisely the left Hermite form. This finished the proof. □

Theorem 2. *Let $A = P^{-1}\Phi Q^{-1}$, where $\Phi = E_t \oplus \varphi E_{n-t}$. Then there exists an invertible matrix U such that*

$$AU = V^{-1}\Phi,$$

where V is the matrix of the form (8).

Proof. By Theorem 1, there exists $H \in G_\Phi$ such that the matrix $HP = V$ has the form (8). Then

$$A = P^{-1}\Phi Q^{-1} = (HP)^{-1}(H\Phi)Q^{-1} = V^{-1}\Phi H_1 Q^{-1}.$$

Since the matrix H_1 is invertible, $U = QH_1^{-1}$ is desired matrix. □

References

[1] Helmer O. *The elementary divisor theorem for certain rings without chain condition*, Bull. Amer. Math. Soc., 1943, Vol.**49**, pp. 225 - 236.

[2] Shchedryk V.P. *Structure and properties of matrix divisors over commutative elementary divisor domain* Math. studii, 1998, Vol.**10**(2), pp. 115-120.

[3] Kazimirskij P.S. *Decomposition of matrix polynomials into factors* Kyiv, Naukova dumka, 1981. - 224 p.

[4] Bhowmik G., Ramare O. *Algebra of matrix arithmetic* Journal of Algebra, 1998, Vol.**210**, pp.194 - 215.

[5] Shchedryk V.P. *Some determinant properties of primitive matrices over Bezout B-domain* Algebra and Discrete Mathematics, 2005, №2, pp. 46-57.

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