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RESEARCH ARTICLE

Length of the inverse symmetric semigroup

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Abstract. The length of the lattice of subsemigroups of the inverse symmetric semigroup $\mathcal{IS}_n$ is calculated.

1. Introduction

In the theory of inverse semigroups the inverse symmetric semigroup $\mathcal{IS}(M)$ plays a role similar to the role of the symmetric group in group theory. Especially interesting is the case $|M| = n < \infty$, since apart from specific semigroup problems, a lot of combinatorial problems arise there. A great number of papers, and even specialized monographs (see [1], [2] and literature cited there) are dedicated to the study of the semigroup $\mathcal{IS}_n$.

An essential question in the study of any semigroup is about the structure of the lattice of its subsemigroups. Although such lattices were studied actively (see e.g., [3]), not very much is known about the structure of lattices of subsemigroups of particular semigroups. The main reason is that lattices of subsemigroups have quite a complex structure. Even for monogenic semigroups this question is nontrivial. For groups, it is hard too: the paper [4] uses the classification of finite simple groups to calculate the length of the lattice of subgroups of the symmetric group S_n.

In the case of $\mathcal{IS}_n$ a little is known as well. Though the lattice of two-sided ideals of $\mathcal{IS}_n$ is quite simple (a chain of length n), the structure of the lattice of left (right) ideals is considerably more complex (see [2,

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Chapter 4.3]). Clearly, additional difficulties arise when considering the lattice of all subsemigroups.

Recall that the length $l(S)$ of a semigroup S is defined as the length of its lattice of subsemigroups, i.e., the maximal integer n, for which there exists a strictly increasing chain of subsemigroups

$$\varnothing \neq H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = S. \tag{1}$$

The objective of the present paper is the calculation of the length of the inverse symmetric semigroup $\mathcal{IS}_n$ (Theorem 8). As an auxiliary result, a description of maximal subsemigroups of a finite Brandt semigroup is obtained (Theorem 6).

In this paper only finite semigroups and groups are considered.

2. The main Lemma

Lemma 1. *For every ideal I of a finite semigroup S the following equality holds*

$$l(S) = l(I) + l(S/I)\,, \tag{2}$$

where S/I denotes the Rees quotient modulo the ideal I.

Proof. Let S be a finite semigroup and I an ideal of S. Assume that (1) gives a longest chain of subsemigroups in S. First we prove that for every extension $H_k \subset H_{k+1}$ the set $H_{k+1} \setminus H_k$ either is a subset of the ideal I or is disjoint with it. Indeed, if this is not the case, we would have the strict inclusions

$$H_k \subset H_k \cup (H_{k+1} \cap I) \subset H_{k+1}\,. \tag{3}$$

The subsemigroup $H_{k+1} \cap I$ is an ideal of H_{k+1}; therefore, the set $H_k \cup (H_{k+1} \cap I)$, being a union of an ideal and a subsemigroup, would be a subsemigroup too. Then (3) would contradict the maximality of chain (1).

Extend chain (1) by the empty subsemigroup $H_{-1} = \varnothing$ and transform the chain

$$H_{-1} = \varnothing \subset H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = S \tag{4}$$

in the following way: if this chain contains a fragment $H_{k-1} \subset H_k \subset H_{k+1}$, where $H_k \setminus H_{k-1} \nsubseteq I$ and $H_{k+1} \setminus H_k \subseteq I$, replace it by the fragment

$$H_{k-1} \subset H_{k-1} \cup (H_{k+1} \cap I) \subset H_{k+1}$$

(strictness of the inclusions is obvious, and, similarly to the above, one shows that the set $H_{k-1} \cup (H_{k+1} \cap I)$ is a subsemigroup). By a finite number of such transformations we get the chain

$$H_{-1} = \varnothing \subset H'_0 \subset H'_1 \subset H'_2 \subset \cdots \subset H'_{n-1} \subset H_n = S\,, \tag{5}$$

in which all extensions by the elements of ideal I come first, followed by extensions by elements of $S \setminus I$. From this property and maximality of chain (5) it follows that for some m equality $H'_m = I$ must hold. Thus, from (5) we get the chain

$$\varnothing \subset H'_0 \subset H'_1 \subset H'_2 \subset \cdots \subset H'_m$$

of subsemigroups of I and the chain

$$0 = I/I = H'_m/I \subset H'_{m+1}/I \subset \cdots \subset H'_{n-1}/I \subset H_n/I = S/I$$

of subsemigroups of S/I. From the definition of Rees quotient it follows that in the last chain all extensions are strict too. This gives us the inequality $l(S) \leq l(I) + l(S/I)$.

The opposite inequality follows from the following fact: if

$$\varnothing \neq H_0 \subset H_1 \subset \cdots \subset H_p = I \quad \text{and} \quad \varnothing \neq K_0 \subset K_1 \subset \cdots \subset K_q = S/I$$

are chains of subsemigroups for I and S/I correspondingly, and $\pi : S \to S/I$ is the canonical epimorphism, then

$$\varnothing \neq H_0 \subset H_1 \subset \cdots \subset H_p \subseteq \pi^{-1}(K_0) \subset \pi^{-1}(K_1) \subset \cdots \subset \pi^{-1}(K_q) = S$$

is a chain of subsemigroups for S. □

3. The length of the Brandt semigroup

Recall that the Brandt semigroup $B(n, G)$ over a group G is the semigroup of all matrices of dimension n, all entries of which are zero, except for at most one entry which is supposed to be an element of G. The matrix, in which an element $g \in G$ is in k-th row and l-th column, is denoted by $(g)_{kl}$. The zero matrix is the zero in $B(n, G)$ and the multiplication of nonzero elements from $B(n, G)$ is the usual matrix multiplication: $(g)_{kl} \cdot (h)_{pq}$ is equal to $(gh)_{kq}$ if $l = p$, and to 0 otherwise.

We will use the following notation: e is the unit of the group G; for an arbitrary subset $H \subseteq G$ denote by $(H)_{kl}$ the set $\{(h)_{kl} \mid h \in H\}$; for an arbitrary subset $S \subseteq B(n, G)$ denote $S_{kl} = S \cap (G)_{kl}$.

For an arbitrary subgroup $H \leq G$ and elements $g_1 = e$, g_2, ..., g_n of G define $S(H; g_2, \ldots, g_n)$ by

$$S(H; g_2, \ldots, g_n) = \Big(\bigcup_{k,l} (g_k^{-1} H g_l)_{kl}\Big) \cup \{0\}\,. \tag{6}$$

Lemma 2. *For an arbitrary subgroup $H \leq G$ and elements $g_1 = e$, g_2, ..., g_n of G the set $S(H; g_2, \ldots, g_n)$ is a subsemigroup of the semigroup $B(n, G)$.*

Proof. That $S(H; g_2, \ldots, g_n)$ is closed under multiplication is verified by a direct calculation. □

Lemma 3. *Let S be a subsemigroup of $B(n, G)$ such that for every k and l $S \cap (G)_{kl} \neq \varnothing$. Then S is equal to $S(H; g_2, \ldots, g_n)$ for some $g_2, \ldots, g_n$.*

Proof. Let S be a subsemigroup of $B(n, G)$ such that $S_{kl} \neq \varnothing$ for arbitrary k and l. In every S_{kl} choose an element $s_{kl} = (g_{kl})_{kl}$. Obviously, S_{11} is a subgroup in $B(n, G)$; thus, S_{11} is equal to $(H)_{11}$, where H is some subgroup of G. In particular, we can take $g_{11} = e$. From the inequalities

$$s_{lk} \cdot S_{kk} \cdot s_{kl} \subseteq S_{ll}, \qquad S_{kk} \cdot s_{kl} \subseteq S_{kl}, \qquad S_{kl} \cdot s_{lk} \subseteq S_{kk} \tag{7}$$

it follows that all of the sets S_{kl} are equinumerous. Hence, inclusions (7) are equalities.

Since $s_{1k}s_{k1} \in S_{11}$, then $g_{1k}g_{k1} \in H$. From the equality $S_{k1} = s_{k1} \cdot S_{11}$ it follows that, replacing g_{k1} by $g_{k1} \cdot (g_{1k}g_{k1})^{-1}$, we can assume $g_{k1} = g_{1k}^{-1}$. Then, from the equality $S_{kl} = s_{k1} \cdot S_{11} \cdot s_{1l}$ it follows that $S_{kl} = (g_{1k}^{-1} H g_{1l})_{kl}$ and the subsemigroup S is equal to $S(H; g_2, \ldots, g_n)$, where $g_i = g_{1i}$, $2 \leq i \leq n$. □

Corollary 4. *The semigroup $S(H; g_2, \ldots, g_n)$ is isomorphic to the semigroup $B(n, H)$.*

Proof. The map $(g_k^{-1} h g_l)_{kl} \mapsto (h)_{kl}$ is an isomorphism between $S(H; g_2, \ldots, g_n)$ and $B(n, H)$. □

For arbitrary subsets $K, L \subseteq \{1, 2, \ldots, n\}$ denote

$$S(K, L) = B(n, G) \setminus \bigcup_{i \in K, j \in L} (G)_{ij}\,. \tag{8}$$

Lemma 5. *If sets K and L form a covering of the set $\{1, 2, \ldots, n\}$, then $S(K, L)$ is a subsemigroup of the semigroup $B(n, G)$.*

Proof. If $S(K, L)$ is not a subsemigroup, then there exist non-zero elements $(g)_{pq}, (h)_{rt} \in S(K, L)$ such that $(g)_{pq} \cdot (h)_{rt} \notin S(K, L)$. But then $p \in K$, $q \notin L$, $t \in L$, $r \notin K$ and $q = r$, which is impossible. □

Theorem 6. *Let $n > 1$ and G be a finite group. A subsemigroup S of a Brandt semigroup $B(n, G)$ is maximal if and only if either it is equal to the subsemigroup $S(H; g_2, \ldots, g_n)$, where H is a maximal subgroup of the group G, or to the subsemigroup $S(K, L)$, where sets K and L form a partition of the set $\{1, 2, \ldots, n\}$.*

Proof. Let S be a subsemigroup of $B(n,G)$. There are two possible cases.

I. For arbitrary k and l we have $S_{kl} \neq \varnothing$. Then, by Lemma 3, S is equal to some $S(H; g_2, \dots, g_n)$. From the statement that all sets S_{kl} are equinumerous it follows that $S(H; g_2, \dots, g_n)$ is a maximal subsemigroup of $B(n,G)$ if and only if H is a maximal subgroup of G.

II. There exist k and l such that $S_{kl} = \varnothing$. Since from $S_{ij} \neq \varnothing$ and $S_{ji} \neq \varnothing$ it follows that $S_{ii} \supseteq S_{ij} \cdot S_{ji} \neq \varnothing$, we can assume that $k \neq l$. Denote

$$K = \{p \mid S_{kp} \neq \varnothing\} \cup \{k\}, \qquad L = \{q \mid S_{ql} \neq \varnothing\} \cup \{l\}\,.$$

Sets K and L do not intersect. Indeed, if $i \in K \cap L$, then for arbitrary $(g)_{ki} \in S_{ki}$ and $(h)_{il} \in S_{il}$ we have $(gh)_{kl} \in S_{kl}$, which contradicts to the condition $S_{kl} = \varnothing$.

Note, that for arbitrary $i \in K, j \in L$ we have $S_{ij} = \varnothing$. Indeed, for S_{kj} and S_{il} this follows from the definition of the sets K and L. If $i \in K \setminus \{k\}$, $j \in L \setminus \{l\}$, this follows from the inclusion $S_{ki} \cdot S_{ij} \cdot S_{jl} \subseteq S_{kl}$.

Show that if S is a maximal subsemigroup, then $K \cup L = \{1, \dots, n\}$. Indeed, let $m \in \{1, \dots, n\} \setminus (K \cup L)$. Assume that there exists $p \in K$ such that $S_{pm} \neq \varnothing$. Then S is a proper subset of the subsemigroup $S_1 = \langle S, (G)_{km} \rangle$. On the other hand, from the inclusion $S_{pm} \cdot S_{mq} \subseteq S_{pq}$ it follows that $S_{mq} = \varnothing$ for all $q \in L$. Thus, S_1 does not contain elements of $(G)_{kl}$, i.e. it is a proper subsemigroup of $B(n,G)$. If for all $p \in K$ we have $S_{pm} = \varnothing$, then by the same argument it can be proved that S is a proper subsemigroup of $\langle S, (G)_{ml} \rangle$.

Therefore, if S is a maximal subsemigroup, then K and L form a partition of the set $\{1, 2, \dots, n\}$. Since S is a subset of $S(K,L)$, it must be equal to the last one, i.e. $S = S(K,L)$.

It is left to show that, if K and L form a partition, then the subsemigroup $S(K,L)$ is maximal. Indeed, $S(K,L)$ cannot be a subset of any other subsemigroup of this kind. Thus, if $S^* \supsetneqq S(K,L)$, then $S^*_{kl} \neq \varnothing$ for arbitrary k and l, and by Lemma 3, S^* is equal to some subsemigroup $S(H; g_2, \dots, g_n)$. But $S^*_{11} = (G)_{11}$, so $H = G$ and $S(H; g_2, \dots, g_n) = B(n,G)$. □

We denote by G^0 the group G with a zero adjoint.

Theorem 7. *For a finite group G*

$$l(B(n,G)) = n \cdot l(G^0) + \frac{n(n-1)}{2}|G| + n - 1. \tag{9}$$

Proof. We use induction on the parameter n. Since $B(1,G) \simeq G^0$, then for $n = 1$ equality (9) is true.

Now, assume that equality (9) is proved for all semigroups $B(m,G)$, where $m < n$. From Theorem 6 it follows that $l(B(n,G))$ is equal to $1 + l(S(K,L))$ for some partition $K \cup L = \{1,2,\dots,n\}$, or to $1 + l(S(H;g_2,\dots,g_n))$ for some maximal subgroup $H < G$.

Let us determine $l(S(K,L))$ for $|K| = p$, $|L| = q$, $p+q = n$. To this end, consider three subsets of $S(K,L)$:

$$S_K = \{0\} \cup \bigcup_{i,j\in K} (G)_{ij}, \quad S_L = \{0\} \cup \bigcup_{i,j\in L} (G)_{ij}, \quad I = \{0\} \cup \bigcup_{i\in L, j\in K} (G)_{ij}.$$

It is obvious that each of these subsets is a subsemigroup; in particular, $S_K \simeq B(p,G)$, $S_L \simeq B(q,G)$, and I is a subsemigroup with zero multiplication. Moreover, $S(K,L) = S_K \cup S_L \cup I$. It is easy to check that I is an ideal in $S(K,L)$, and the Rees quotient $S(K,L)/I$ is isomorphic to $S_K \cup S_L$. In turn, in the semigroup $S_K \cup S_L$ both subsemigroup S_K and S_L are ideals with the following property: Rees quotient modulo S_K is isomorphic to S_L and, vice versa, Rees quotient modulo S_L is isomorphic to S_K. Therefore, by Lemma 1

$$l(S(K,L)) = l(I) + l(S_K \cup S_L) = l(I) + l(S_K) + l(S_L)\,. \tag{10}$$

Since for a semigroup T with zero multiplication we have $l(T) = |T| - 1$, by the inductive hypothesis we get:

$$l(S(K,L)) = pq|G| + \Big(p \cdot l(G^0) + \frac{p(p-1)}{2}|G| + p - 1\Big) +$$
$$+\Big(q \cdot l(G^0) + \frac{q(q-1)}{2}|G| + q - 1\Big) = n \cdot l(G^0) + \frac{n(n-1)}{2}|G| + n - 2.$$

In particular, $l(S(K,L))$ does not depend on the choice of K and L.

To complete the proof of the Theorem we need to show that

$$l(B(n,G)) = l(S(K,L)) + 1\,. \tag{11}$$

We will do this by induction on the order of the group G. Equality (11) is obvious if $|G| = 1$, because in this case every maximal subsemigroup of $B(n,G)$ is isomorphic to $S(K,L)$. Assume now that (11) holds for all groups of order t. Let $t < |G| \le 2t$ and H be a maximal subgroup of G, then $|H| \le |G|/2 \le t$. Since $S(H;g_2,\dots,g_n) \simeq B(n,H)$, then by the inductive hypothesis

$$l(S(H;g_2,\dots,g_n)) = n \cdot l(H^0) + \frac{n(n-1)}{2}|H| + n - 1\,,$$

what clearly is less than $l(S(K,L))$. This completes the proof of the equality (11). □

4. Length of $\mathcal{IS}_n$

Theorem 8. *The length of the inverse symmetric semigroup $\mathcal{IS}_n$ is equal to*

$$l(\mathcal{IS}_n) = \sum_{k=1}^{n} \left[\binom{n}{k} \left(\left\lceil \frac{3k}{2} \right\rceil - b(k) + 1 \right) + \frac{\binom{n}{k}\left(\binom{n}{k} - 1\right)}{2} \cdot k! - 1 \right],$$

where we denote by $\lceil x \rceil$ the least integer, which is not less than x, and by $b(k)$ the number of nonzero digits in the binary expansion of k.

Proof. It is known that semigroup $\mathcal{IS}_n$ has $n+1$ ideals [2, Chapter 4], which form the chain

$$\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = \mathcal{IS}_n\,.$$

For every k, $1 \le k \le n$, the Rees quotient I_k/I_{k-1} is isomorphic to the Brandt semigroup $B\left(\binom{n}{k}, S_k\right)$, where S_k is the symmetric group of degree k. Thus, from Lemma 1 and Theorem 7 it follows that the length of semigroup $\mathcal{IS}_n$ is equal to

$$\begin{aligned} l(\mathcal{IS}_n) &= \sum_{k=1}^{n} l(I_k/I_{k-1}) = \sum_{k=1}^{n} l\left(B\left(\binom{n}{k}, S_k \right) \right) \\ &= \sum_{k=1}^{n} \left[\binom{n}{k} \cdot l(S_k^0) + \frac{\binom{n}{k}\left(\binom{n}{k} - 1\right)}{2} |S_k| + \binom{n}{k} - 1 \right]. \end{aligned} \tag{12}$$

In the paper [4] it is proved that $l(S_k) = \left\lceil \frac{3k}{2} \right\rceil - b(k) - 1$. Therefore, $l(S_k^0) = l(S_k) + 1 = \left\lceil \frac{3k}{2} \right\rceil - b(k)$. Moreover, $|S_k| = k!$. Putting these values into (12), we get the statement of the Theorem. □

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