Algebra and Discrete Mathematics
Volume **20** (2015). Number 1, pp. 13–31 RESEARCH ARTICLE

Lie and Jordan structures of differentially semiprime rings

Orest D. Artemovych and Maria P. Lukashenko

Communicated by A. P. Petravchuk

Abstract. Properties of Lie and Jordan rings (denoted respectively by R^L and R^J) associated with an associative ring R are discussed. Results on connections between the differentially simplicity (respectively primeness, semiprimeness) of R, R^L and R^J are obtained.

1. Introduction

Throughout here, R is an associative ring (with respect to the addition "+" and the multiplication " $\cdot$ ") with an identity, $\operatorname{Der} R$ is the set of all derivations in R. On the set R we consider two operations: the Lie multiplication "$[-,-]$" and the Jordan multiplication "$(-,-)$" defined by the rules

$$[a, b] = a \cdot b - b \cdot a$$

and

$$(a, b) = a \cdot b + b \cdot a$$

for any $a, b \in R$. Then

$$R^L = (R, +, [-,-])$$

is a Lie ring and

$$R^J = (R, +, (-,-))$$

2010 MSC: Primary 16W25, 16N60; Secondary 17B60, 17C50.
Key words and phrases: Derivation, semiprime ring, Lie ring.

is a Jordan ring (see [13] and [14]) associated with the associative ring R. Recall that an additive subgroup A of R is called:

- *a Lie ideal* of R if
$$[a, r] \in A,$$
- *a Jordan ideal* of R if
$$(a, r) \in A$$

for all $a \in A$ and $r \in R$. Obviously, A is a Lie (respectively Jordan) ideal of R if and only if A^L (respectively A^J) is an ideal of R^L (respectively R^J).

In all that follows Δ will be any subset of $\operatorname{Der} R$ (in particular, $\Delta = \{0\}$) and $\delta \in \operatorname{Der} R$. A subset K of R is called Δ-*stable* if $d(a) \in K$ for all $d \in \Delta$ and $a \in K$. An ideal I of a (Lie, Jordan or associative) ring A is said to be a Δ-*ideal* if I is Δ-stable. A (Lie, Jordan or associative) ring A is said to be:

- *simple* (respectively Δ-*simple*) if there no two-sided ideals (respectively Δ-ideals) other 0 or A,
- *prime* (respectively Δ-*prime*) if, for all two-sided ideals (respectively Δ-ideals) K, S of A, the condition $KS = 0$ implies that $K = 0$ or $S = 0$ (if $\Delta = \{\delta\}$ and A is Δ-prime, then we say that A is δ-*prime*),
- *semiprime* (respectively Δ-*semiprime*) if, for any two-sided ideal (respectively Δ-ideal) K of A, the condition $K^2 = 0$ implies that $K = 0$,
- *primary* if, for any two-sided ideals K, S of A, the condition $KS = 0$ implies that $K = 0$ or S is nilpotent.

Every non-commutative Δ-simple ring is Δ-prime and every Δ-prime ring is Δ-semiprime. We say that R is $\mathbb{Z}$-*torsion-free* if, for any $r \in R$ and integers n, the condition $nr = 0$ holds if and only if $r = 0$. If the implication
$$2r = 0 \Rightarrow r = 0$$
is true for any $r \in R$, then R is said to be 2-*torsion-free*. Let
$$F_p(R) = \{a \in R \mid a \text{ has an additive order } p^k \\ \text{for some non-negative} k = k(a)\}$$
be the p-part of R, where p is a prime. Then $F_p(R)$ is a Δ-ideal of R. If R is Δ-semiprime, then
$$pF_p(R) = 0.$$

In particular, in a Δ-prime ring R it holds $F_p(R) = 0$ (and so the characteristic $\operatorname{char} R = 0$) or $F_p(R) = R$ (and therefore $\operatorname{char} R = p$). Obviously that the additive group R^+ of a Δ-prime ring R is torsion-free if and only if $\operatorname{char} R = 0$. Recall that a ring R is said to be *of bounded index* m, if m is the least positive integer such that $x^m = 0$ for all nilpotent elements $x \in R$. We say that a ring R satisfies *the condition* (X) if one of the following holds:

(1) R or $R/\mathbb{P}(R)$ is $\mathbb{Z}$-torsion-free, where $\mathbb{P}(R)$ is the prime radical of R,

(2) R is of bounded index m such that an additive order of every nonzero torsion element of R, if any, is strictly larger than m.

As noted in [16, p.283], a $\mathbb{Z}$-torsion-free δ-prime ring is semiprime. In this way we prove the following

Proposition 1. *For a ring R the following hold:*

(1) *if R is a Δ-semiprime ring with the condition (X), then it is semiprime,*

(2) *if R is both semiprime (respectively satisfies the condition (X)) and Δ-prime, then R is prime.*

Relations between properties of an associative ring R, a Lie ring R^L and a Jordan ring R^J was studied by I.N. Herstein and his students (see [7, 8, 11] and bibliography in [9] and [5]); he has obtained, for a ring R of characteristic different from 2, that the simplicity of R implies the simplicity of a Jordan ring R^J [7, Theorem 1], and also that every Lie ideal of a simple Lie ring R is contained in the center $Z(R)$ [7, Theorem 3]. K. McCrimmon [20, Theorem 4] has proved that R is a simple algebra if and only if R^J is a simple Jordan algebra. Our result is the following

Theorem 1. *For a 2-torsion-free ring R the following statements are true:*

(1) *R is a Δ-simple ring if and only if R^J is a Δ-simple Jordan ring,*

(2) *R is a Δ-prime ring if and only if R^J is a Δ-prime Jordan ring,*

(3) *R is a Δ-semiprime ring if and only if R^J is a Δ-semiprime Jordan ring.*

Let us $d \in \Delta$. Since $C(R)$ and $\operatorname{ann} C(R)$ are Δ-ideals, the rule

$$\overline{d} : R/\operatorname{ann} C(R) \ni r + \operatorname{ann} C(R) \mapsto d(r) + \operatorname{ann} C(R) \in R/\operatorname{ann} C(R)$$

determines a derivation $\overline{d}$ of the quotient ring $R/\operatorname{ann} C(R)$. Then

$$\overline{\Delta} = \{\overline{d} \mid d \in \Delta\} \subseteq \operatorname{Der}(R/\operatorname{ann} C(R)).$$

Inasmuch $d(Z(R)) \subseteq Z(R)$, the rule

$$\widehat{d} : R^L/Z(R) \ni r + Z(R) \mapsto d(r) + Z(R) \in R^L/Z(R)$$

determines a derivation $\widehat{d}$ of the Lie ring $R^L/Z(R)$. Then

$$\widehat{\Delta} = \{\widehat{d} \mid d \in \Delta\} \subseteq \operatorname{Der}(R^L/Z(R)).$$

Since the center $Z(R)$ is a nonzero Lie ideal of an associative ring R with an identity, a Lie ring R^L is not Δ-simple. Our next result is the following

Theorem 2. *Let R be a 2-torsion-free ring. Then the following are true:*

(1) *if the quotient ring $R^L/Z(R)$ is a $\widehat{\Delta}$-simple Lie ring, then R is non-commutative and $R/\operatorname{ann} C(R)$ is a $\overline{\Delta}$-simple ring,*

(2) *if R is a Δ-simple ring, then $R^L/Z(R)$ is a $\widehat{\Delta}$-simple Lie ring or R is commutative,*

(3) *if $R^L/Z(R)$ is a $\widehat{\Delta}$-semiprime Lie ring, then R is non-commutative and the quotient ring $R/\operatorname{ann} C(R)$ is a $\overline{\Delta}$-semiprime ring,*

(4) *if R is a Δ-semiprime ring, then $R^L/Z(R)$ is a $\widehat{\Delta}$-semiprime Lie ring or R is commutative,*

(5) *if $R^L/Z(R)$ is a $\widehat{\Delta}$-prime Lie ring, then R is non-commutative and $R/\operatorname{ann} C(R)$ is a $\overline{\Delta}$-prime ring,*

(6) *if R is a Δ-prime ring, then $R^L/Z(R)$ is a $\widehat{\Delta}$-prime Lie ring or R is commutative.*

Throughout, let $Z(R)$ denote the center of R, $[A, B]$ (respectively (A, B)) an additive subgroup of R generated by all commutators $[a, b]$ (respectively (a, b)), where $a \in A$ and $b \in B$, $C(R)$ the commutator ideal of R, $N(R)$ the set of nilpotent elements in R, $\operatorname{char} R$ the characteristic of R, $\operatorname{ann}_l I = \{a \in R \mid aI = 0\}$ the left annihilator of I in R, $\operatorname{ann}_r I = \{a \in R \mid Ia = 0\}$ the right annihilator of I in R, $\operatorname{ann} I = (\operatorname{ann}_r I) \cap (\operatorname{ann}_l I)$, $C_R(I) = \{a \in R \mid ai = ia \text{ for all } i \in I\}$ the centralizer of I in R and $\partial_a(x) = [a, x]$ for $a, x \in R$.

All other definitions and facts are standard and it can be found in [10], [17] and [19].

2. Differentially prime right Goldie rings

Let agree that

$$d^0 = \mathrm{id}_R$$

is the identity endomorphism for $d \in \Delta$.

Lemma 1. *The following conditions are equivalent:*

(1) *R is a Δ-semiprime ring,*

(2) *for any Δ-ideals A, B of R the implication*

$$AB = 0 \Rightarrow A \cap B = 0$$

is true,

(3) *if $a \in R$ is such that*

$$aR\delta_1^{m_1} \dots \delta_k^{m_k}(a) = 0$$

for any integers $k \geqslant 1$, $m_i \geqslant 0$ and derivations $\delta_i \in \Delta$ $(i = 1, \dots, k)$, then $a = 0$.

Proof. A simple modification of Proposition 2 from [17, §3.2]. □

Lemma 2. *The following conditions are equivalent:*

(1) *R is a Δ-prime ring,*

(2) *a left annihilator $\mathrm{ann}_l\, I$ of a left Δ-ideal I of R is zero,*

(3) *a right annihilator $\mathrm{ann}_r\, I$ of a right Δ-ideal I of R is zero,*

(4) *if $a, b \in R$ are such that*

$$aR\delta_1^{m_1} \dots \delta_k^{m_k}(b) = 0$$

for any integers $k \geqslant 1$, $m_j \geqslant 0$ and derivations $\delta_j \in \Delta$ $(j = 1, \dots, k)$, then $a = 0$ or $b = 0$.

Proof. A simple consequence of Lemma 2.1.1 from [10]. □

If I is an ideal of a ring R, then

$$\mathcal{C}_R(I) = \{x \in R \mid x + I \text{ is regular in the quotient ring } R/I\}$$

(see [19, Chapter 2, §1]). The next lemma extends Proposition 1 of [15].

Lemma 3. *Let R be a right Goldie ring and $\delta \in \operatorname{Der} R$. If R is δ-prime, then:*

(*a*) *the set $N = N(R)$ of nilpotent elements of R is its prime radical,*

(*b*) $\bigcap_{i=1}^{k} \delta^{-1}(N) = 0$ *for some integer k,*

(*c*) $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.

Proof. From Theorem 2.2 of [16] (see the part $(ii) \Rightarrow (iii)$ of its proof), we obtain (a) and (b). By Proposition 4.1.3 of [19], $\mathcal{C}_R(0) \subseteq \mathcal{C}_R(N)$. By the same argument as in [16, p.284], we can obtain that $\mathcal{C}_R(0) = \mathcal{C}_R(N)$. □

Corollary 1. *If R is a commutative δ-prime Goldie ring and $\delta \in \operatorname{Der} R$, then $N(R)$ contains all zero-divisors of R.*

By Corollary 1.4 of [6], if I is a δ-prime ideal of a right Noetherian ring R and R/I has characteristic 0, then I is prime. The following lemma is an extension of Lemma 2.5 from [6].

Lemma 4. *Let R be a 2-torsion-free commutative Goldie ring and $\delta \in \operatorname{Der} R$. If R is δ-prime, then it is an integral domain.*

Proof. Assume that $a \in \operatorname{ann} N(R)$, $b \in N(R)$ and $r \in R$. Then

$$\begin{aligned} 0 &= \delta^2(arb) = \delta(\delta(a)rb + a\delta(r)b + ar\delta(b)) \\ &= \delta^2(a)rb + 2\delta(a)\delta(r)b + 2\delta(a)r\delta(b) + a\delta^2(r)b + 2a\delta(r)\delta(b) + ar\delta^2(b) \end{aligned}$$

and so

$$2\delta(a)R\delta(b) \subseteq N(R).$$

This means that $\delta(a) \in N(R)$ or $\delta(b) \in N(R)$. Hence $N(R)$ is δ-stable. By Lemma 3, $N(R)$ is a ideal and therefore $N(R) = 0$. By Lemma 1.2 of [4], R is prime and consequently it is an integral domain. □

Proof of Proposition 1.

(1) By Proposition 1.3 of [6] and Theorem 1 of [1], the prime radical $\mathbb{P}(R)$ is a Δ-ideal and so $\mathbb{P}(R) = 0$ is zero.

(2) Since $\mathbb{P}(R) = 0$, R is prime by Lemma 1.2 from [4]. □

By Theorem 4 of [22], a Δ-simple ring R of characteristic 0 is prime. Since every non-commutative Δ-simple ring is Δ-prime, in view of Proposition 1 we obtain the following

Corollary 2. *Let R be a semiprime ring (respectively a ring R satisfy the condition (X)). If R is Δ-simple, then it is prime.*

3. Differential analogues of Herstein's results

For the proof of Theorem 2 we need the next results. In the proofs below we use the same consideration, as in [12, Chapter 1, §1], and present them here in order to have the paper more self-contained. Let agree that everywhere in this section $k \geqslant 1$ and $m_i \geqslant 0$ are integers $(i = 1, \dots, k)$.

Lemma 5. *Let R be a Δ-semiprime ring, A and B its Δ-ideals. Then the following statements hold:*

- (i) *if $AB = 0$, then $BA = 0$.*
- (ii) $\operatorname{ann}_l A = \operatorname{ann}_r A$.
- (iii) $A \cap \operatorname{ann}_r A = 0$.

Proof. (i) Indeed, BA is a Δ-ideal and $(BA)^2 = 0$ and so $BA = 0$.

(ii) We denote $(\operatorname{ann}_r A)A$ by X. Since X is a Δ-ideal and $X^2 = 0$, we deduce that $X = 0$. This means that

$$\operatorname{ann}_r A \subseteq \operatorname{ann}_l A.$$

The inverse inclusion we can prove similarly.

(iii) Since $A \cap \operatorname{ann}_r A$ is a nilpotent Δ-ideal, the assertion holds.

□

Henceforth

$$X_a = \{[\delta_1^{m_1} \dots \delta_k^{m_k}(a), x] \mid x \in R,\ \delta_i \in \Delta,\ m_i \geqslant 0 \\ \text{and } k \geqslant 1 \text{ are integers} \quad (i = 1, \dots, k)\}.$$

It is clear that $[a, x] \in X_a$.

Lemma 6. *Let R be a Δ-semiprime ring and $a \in R$. Then the following statements hold:*

- (i) *if*
$$a[\delta_1^{m_1} \dots \delta_k^{m_k}(a), R] = 0$$
for any integers $k \geqslant 1$, $m_i \geqslant 0$ and derivations $\delta_i \in \Delta$ $(i = 1, \dots, k)$, then $a \in Z(R)$,
- (ii) *if I is a right Δ-ideal of R, then $Z(I) \subseteq Z(R)$,*
- (iii) *if I is a commutative right Δ-ideal of R and I is nonzero, then $I \subseteq Z(R)$. If, moreover, R is Δ-prime, then it is commutative.*

Proof. (i) Let $x, y \in R$ and $d, \delta \in \Delta$. Since

$$[b, xy] = [b, x]y + x[b, y] \tag{3.1}$$

for any $b \in X_a$ and $a[b, xy] = 0$, we conclude that $ax[b, y] = 0$. This gives that $ayx[b, y] = 0$ and $yax[b, y] = 0$ and consequently

$$(R[a, y]R)^2 = 0. \tag{3.2}$$

In addition,

$$0 = d(a[b, x]) = d(a)[b, x].$$

Multiplying (3.1) by $d(a)$ on left we get $d(a)x[b, y] = 0$. Moreover,

$$0 = \delta(ax[d(b), y]) = \delta(a)x[d(b), y]$$

and, by the similar argument, we obtain that

$$\delta_1^{m_1} \ldots \delta_k^{m_k}(a)x[\delta_1^{m_1} \ldots \delta_k^{m_k}(a), y] = 0$$

for any integers $k \geqslant 1$, $m_i \geqslant 0$ and derivations $\delta_i \in \Delta$ $(i = 1, \ldots, k)$. As in the proof of the condition (3.2), we deduce that

$$(R[\delta_1^{m_1} \ldots \delta_k^{m_k}(a), y]R)^2 = 0.$$

Then

$$I = \sum_{k=1}^{\infty} \sum_{\substack{\delta_1 \ldots \delta_k \in \Delta \\ y \in R}} R[\delta_1^{m_1} \ldots \delta_k^{m_k}(a), y]R$$

is a sum of nilpotent ideals and therefore it is a nil ideal. Since I is a Δ-ideal, we conclude that $I = 0$ and, as a consequence, $a \in Z(R)$.

(ii) Let $a \in Z(I)$ and $y \in R$. Then, for $\delta_1, \ldots, \delta_k \in \Delta$, we have

$$\delta_1^{m_1} \ldots \delta_k^{m_k}(a) \in Z(I)$$

and $ay \in I$. This gives that

$$a(\delta_1^{m_1} \ldots \delta_k^{m_k}(a)y) = \delta_1^{m_1} \ldots \delta_k^{m_k}(a)(ay) = a(y\delta_1^{m_1} \ldots \delta_k^{m_k}(a)),$$

and thus

$$a[\delta_1^{m_1} \ldots \delta_k^{m_k}(a), y] = 0.$$

By (i), $a \in Z(R)$ is central.

(iii) By (ii), $I \subseteq Z(R)$. Assume that R is Δ-prime, $u, v \in R$ and $a \in I$. Then $au \in I$ and so $au \in Z(R)$. Since

$$a(uv) = (au)v = v(au) = (va)u = a(vu),$$

we see that

$$[u, v] \in \operatorname{ann}_r I.$$

By Lemma 2(3), $[u, v] = 0$ and hence R is commutative. □

Lemma 7. *Let R be a Δ-prime ring and $a \in R$. If $a \in C_R(I)$ for some nonzero right Δ-ideal I of R, then $a \in Z(R)$.*

Proof. Let us $y \in R$ and $b \in I$. Then $by \in I$ and so $bay = a(by) = bya$. This yields that

$$I[a, y] = 0 = [a, y]I.$$

By Lemma 2(3), $[a, y] = 0$. Hence $a \in Z(R)$. □

Lemma 8. *The left annihilator $\operatorname{ann}_l(X_a)$ is a left Δ-ideal of R.*

Proof. Immediate from the definition. □

Lemma 9. *If R is a Δ-semiprime ring, then $C_R([R, R]) \subseteq Z(R)$.*

Proof. Let us $a \in C_R([R, R]), d, \delta \in \Delta$ and $x, y \in R$. Putting x for a and $xd(a)$ for xy in (3.1) we obtain

$$[x, xd(a)] = [x, x]d(a) + x[x, d(a)]$$

and, as a consequence, $[a, x[x, d(a)]] = 0$ and $[a, x][x, d(a)] = 0$. Then, by the same reasons as in the proof of Lemma 6(i), we obtain that $[a, x] \in \operatorname{ann}_l(X_a)$ and $A = \operatorname{ann}_l(X_a)$ is a Δ-ideal. Then

$$[\delta(a), x][d(a), x] = \delta([a, x][d(a), x]) = 0.$$

Since $A \cap \operatorname{ann}_l A = 0$, we deduce that is a nilpotent Δ-ideal and so $a \in Z(R)$. □

Lemma 10. *Let R be a 2-torsion-free Δ-semiprime ring. If $a \in R$ commutes with all elements of X_a, then $a \in Z(R)$.*

Proof. Let $r, x, y \in R$ and $d \in \Delta$. It is clear that $\partial_a^2(x) = 0$. From $\partial_a^2(xy) = 0$ it follows that

$$2\partial_a(x)\partial_a(y) = 0$$

and so $\partial_a(x)\partial_a(y) = 0$. Since

$$0 = \partial_a(x)\partial_a(rx) = \partial_a(x)\partial_a(r)x + \partial_a(x)r\partial_a(x) = \partial_a(x)r\partial_a(x),$$

we deduce that $\partial_a(x)R\partial_a(x) = 0$ and $(\partial_a(x)R)^2 = 0$. Moreover, $a[b, x] = [b, x]a$ for any $[b, x] \in X_a$ and therefore

$$d(a)[b, x] + a[d(b), x] + a[b, d(x)] = [b, x]d(a) + [d(b), x]a + [b, d(x)]a.$$

From this it holds that

$$d(a)[b, x] = [b, x]d(a).$$

This means that $C_R(X_a)$ is Δ-stable and $(\partial_{d(a)}(x)R)^2 = 0$. As a consequence,

$$I = \sum_{k=1}^{\infty} \sum_{\substack{x\in R\\ m_k\geqslant 0\\ \delta_1,\dots,\delta_k\in\Delta}} \partial_{\delta_1^{m_1}\dots\delta_k^{m_k}(a)}(x)R$$

is a sum of nilpotent ideals and so I is a nil ideal. Since I is a Δ-ideal, we deduce that $I = 0$. Hence $a \in Z(R)$. $\square$

The next lemma is an extension of Lemma 1 from [11] in the differential case.

Lemma 11. *Let R be a 2-torsion-free Δ-semiprime ring, T its Lie Δ-ideal. If $[T, T] \subseteq Z(R)$, then $T \subseteq Z(R)$.*

Proof. Let $x \in R$ and $t \in T$.

1) If $[T, T] = 0$, then $[t, x] \in T$ and so $[t, [t, x]] = 0$. By Lemma 10, $T \subseteq Z(R)$.

2) Now assume that $0 \neq [a, b] \in [T, T]$ for some $a, b \in T$. Then

$$\partial_a(b) \in Z(R) \text{ and } \partial_a^2(R) \subseteq Z(R).$$

Moreover, we have that

$$\begin{aligned} Z(R) \ni \partial_a^2(bx) &= \partial_a(\partial_a(b)x + b\partial_a(x)) \\ &= \partial_a^2(b)x + 2\partial_a(b)\partial_a(x) + b\partial_a^2(x) \\ &= 2\partial_a(b)\partial_a(x) + b\partial_a^2(x) \end{aligned}$$

and hence

$$[2\partial_a(b)\partial_a(x) + b\partial_a^2(x), b] = 0.$$

Then

$$\begin{aligned} 0 &= 2\partial_b(\partial_a(b))\partial_a(x) + 2\partial_a(b)\partial_b(\partial_a(x)) + \partial_b(b)\partial_a^2(x) + b\partial_b(\partial_a^2(x)) \\ &= 2\partial_a(b)\partial_b(\partial_a(x)) \end{aligned} \tag{3.3}$$

and

$$\partial_a(ba) = \partial_a(b)a + b\partial_a(a) = \partial_a(b)a.$$

Replacing ba for x in (3.3) we have

$$0 = 2\partial_a(b)\partial_b(\partial_a(b)a) = 2\partial_a(b)(\partial_b(\partial_a(b)) + \partial_a(b)\partial_b(a)) = -2\partial_a(b)^3$$

and thus $\partial_a(b)^3 = 0$. Then $R\partial_a(b)$ is a nilpotent ideal in R and, as a consequence,

$$\sum_{a,b\in T} R\partial_a(b)$$

is a nonzero nil Δ-ideal, a contradiction. □

Lemma 12. *If U is a Lie Δ-ideal of a ring R and $I(U) = \{u \in R \mid uR \subseteq U\}$, then $I(U)$ is the largest Δ-ideal of R such that $I(U) \subseteq U$.*

Proof. Let $u, v \in I(U)$, $x, y \in R$ and $\delta \in \Delta$. Clearly that $I(U)$ is an additive subgroup of R, $I(U) \subseteq U$ and $(ux)y = u(xy) \in (ux)R = u(xR) \subseteq uR \subseteq U$ that is $ux \in I(U)$. From

$$u(xy) - (yu)x = (ux)y - y(ux) = [ux, y] \in U$$

(and so $(yu)x \in U$) it holds that $yu \in I(U)$. Hence U is a two-sided ideal of R. Moreover,

$$\delta(u)x + u\delta(x) = \delta(ux) \in \delta(U) \subseteq U$$

and $u\delta(x) \in uR \subseteq U$. Therefore $\delta(u)x \in U$. This means that $I(U)$ is a Δ-ideal of R. If A is a Δ-ideal of R that is contained in U, then $AR \subseteq A \subseteq U$ and hence $A \subseteq I(U)$. □

Lemma 13. *Let U be a Lie Δ-ideal of R. If U is an associative subring of R, then $[U, U] = 0$ or U contains a nonzero Δ-ideal of R.*

Proof. Assume that $x \in R$ and $[U,U] \neq 0$. Then $[u,v] \neq 0$ for some $u, v \in U$ and

$$[u, vx] = u(vx) - (vx)u = (uv - vu)x + v(ux - xu).$$

Since $[u,x], [u,vx] \in U$ and $v[u,x] \in U$, we deduce that $[u,v]x \in U$. This means that $[u,v] \in I(U)$. In view of Lemma 12, $I(U)$ is a nonzero Δ-ideal of R that is contained in U. □

Proposition 2. *If U is a Lie Δ-ideal of R, then $[U,U] = 0$ or there exists a nonzero Δ-ideal I_U of R such that $[I_U, R] \subseteq U$.*

Proof. By Lemma 3 of [7],

$$T(U) = \{t \in R \mid [t,R] \subseteq U\}$$

is both a Lie ideal and an associative subring of R and $U \subseteq T(U)$. Moreover, for $\delta \in \Delta$, we have

$$[\delta(t), R] + [t, \delta(R)] = \delta([t,R]) \subseteq \delta(U) \subseteq U$$

and so $[\delta(t), R] \subseteq U$. Hence $T(U)$ is Δ-stable. If $[U,U] \neq 0$, then, by Lemmas 12 and 13,

$$I_U = I(T(U)) \subseteq T(U)$$

is a nonzero Δ-ideal of R such that $[I_U, R] \subseteq U$. □

Lemma 14. *Let U be a Lie Δ-ideal of a ring R. If $[U,U] = 0$, then the centralizer $C_R(U)$ is a Lie Δ-ideal and an associative subring of R.*

Proof. Is immediately. □

We extend Theorem 1.3 of [9] in the following

Proposition 3. *Let R be a Δ-simple ring of characteristic 2. If U is a Lie Δ-ideal of R, then one of the following holds:*

(1) $[R,R] \subseteq U$,

(2) $U \subseteq Z(R)$,

(3) *R contains a subfield P such that $U \subseteq P$ and $[P,R] \subseteq P$.*

Proof. If $[U,U] \neq 0$, then $[R,R] \subseteq U$ by Proposition 2. Therefore we assume that $[U,U] = 0$. By Lemma 14, $C_R(U)$ is a Lie Δ-ideal and an associative subring of R such that $U \subseteq C_R(U)$.

a) If $C_R(U)$ is non-commutative, then $C_R(U) = R$ by Lemma 13. Hence $U \subseteq Z(R)$.

b) Now assume that the centralizer $C_R(U)$ is commutative. If $c \in C_R(U)$ and $x \in R$, then

$$c^2 \in C_R(U) \text{ and } [c^2, x] = [[c,x],x] = 2c[c,x] = 0.$$

This gives that $c^2 \in Z(R)$. By Theorem 2 of [22], $Z(R)$ is a field. As a consequence, c^2 (and so c) is invertible in $C_R(U)$. Hence $C_R(U)$ is a field. □

Corollary 3. *Let R be a Δ-simple ring. If U is a Lie Δ-ideal of R, then one of the following holds:*

(1) $[R,R] \subseteq U$,

(2) $U \subseteq Z(R)$,

(3) $\operatorname{char} R = 2$ *and R contains a subfield P such that $U \subseteq P$ and* $[P,R] \subseteq P$.

4. Jordan properties

Lemma 15. *Let R be a Δ-simple ring of characteristic $\neq 2$, U its proper Jordan Δ-ideal and $a \in U$. If $[a,R] \subseteq U$, then $a = 0$.*

Proof. Let us $x, y \in R$. Since $[a,x] \in U$ and $(a,x) \in U$, we obtain that $2ax \in U$ and, as a consequence, $ax \in U$ and $(ax, y) \in U$. Moreover, from $axy \in U$ it follows that $yax \in U$. This means that $RaR \subseteq U$. Since $d(a) \in U$ for any $d \in \Delta$, in view of [21, Lemma 1.1] we obtain that

$$\sum_{k=1}^{\infty} \sum_{\substack{\delta_1,\ldots,\delta_k \in \Delta \\ (m_1,\ldots,m_k) \in \mathbb{N}^k}} R\delta_1^{m_1} \ldots \delta_k^{m_k}(a)R$$

is a proper Δ-ideal of R that is contained in U. Hence $a = 0$. □

Remark 1. Let R be a 2-torsion-free ring, U its Jordan Δ-ideal. If Δ contains all inner derivations of R, then U is an ideal of R.

In fact, we have

$$2xa = [a, x] + (a, x) \in U$$

for any $a, b, x \in U$ and so $xa \in U$. By the same argument, we can conclude that $ax \in U$.

Proof of Theorem 1.

(1) ($\Leftarrow$) If A is a nonzero proper Δ-ideal of a ring R, then A^J is a nonzero proper Δ-ideal of R^J, a contradiction.

($\Rightarrow$) Let U be a proper Jordan Δ-ideal of R, $a, b \in U$ and $x \in R$. By Lemma 1 of [7], $[(a, b), x] \in U$, and, by Lemma 15, we see that

$$(a, b) = 0. \tag{4.4}$$

In particular, $2a^2 = 0$ and, as a consequence, $a^2 = 0$ and $2axa = (a, (a, x)) = 0$. It follows that $axa = 0$. Since

$$0 = (a + b)x(a + b) = axb + bxa$$

and

$$0 = (b, (a, x)) = b(ax + xa) + (ax + xa)b = bax + bxa + axb + xba,$$

we deduce that $bax + xab = 0$. But $ab = -ba$ and so $bax - xba = 0$. This means that $ba \in Z(R)$. Then $(RabR)^2 = 0$. Since

$$I = \sum_{k=1}^{\infty} \sum_{\substack{a,b \in U,\ \delta_1,\ldots,\delta_k \in \Delta \\ (m_1,\ldots,m_k) \in \mathbb{N}^k}} Ra\delta_1^{m_1} \ldots \delta_k^{m_k}(b)R$$

is a Δ-ideal of R that is a sum of nilpotent ideals, we obtain that $I = 0$. Therefore

$$0 = (b, x)a = (bx + xb)a = bxa + xba = 2bxa.$$

We conclude that $URU = 0$. From $(RUR)^2 = 0$ and $\delta(RUR) \subseteq RUR$ for any $\delta \in \Delta$ it holds that $U = 0$.

(2) ($\Leftarrow$) If A, B are Δ-ideals of R such that $AB = 0$, then $(BA)^2 = 0$ and so BA is a Jordan ideal of R satisfying the condition

$$(BA, BA) = 0.$$

Thus the condition (4.4) is true for $U = BA$. As in the proof of the part (1), we obtain that $BA = 0$. Then A^J, B^J are Δ-ideals of a Jordan ring R^J such that

$$(A^J, B^J) = 0.$$

Hence $A = 0$ or $B = 0$.

($\Rightarrow$) Let $a_1, a_2 \in A$ and $x, y \in R$. Suppose that R^J is not Δ-prime and therefore there exist nonzero Jordan Δ-ideals A, B of R such that

$$(A, B) = 0.$$

By the same reasons as above, we conclude that $A \cap B = 0$. Then, by Lemma 1 of [7], we have $[(a_1, a_2), x] \in A$ and hence

$$[(a_1, a_2), x] \pm ((a_1, a_2), x) \in A.$$

Therefore $x(a_1, a_2)y \in A$. Thus R contains Δ-ideals $R(A, A)R \subseteq A$ and $R(B, B)R \subseteq B$ such that

$$R(A, A)R(B, B)R \subseteq A \cap B = 0.$$

Hence $(A, A) = 0$ or $(B, B) = 0$ and this leads to a contradiction.

(3) ($\Leftarrow$) If A is a nonzero Δ-ideal of R such that $A^2 = 0$, then A^J is a nonzero Δ-ideal of the Jordan ring R^J such that

$$(A^J, A^J) = 0,$$

a contradiction.

($\Rightarrow$) Suppose that R has a nonzero Jordan Δ-ideal U such that

$$(U, U) = 0.$$

Then the condition (4.4) is true for any $a, b \in U$. As in the proof of the part (1), we obtain that $U = 0$.

$\square$

If R is a ring, then on the set R we can to define a left Jordan multiplication "$\langle -, - \rangle$" by the rule

$$\langle a, b \rangle = 2ab$$

for any $a, b \in R$. Then the equalities

$$\langle\langle\langle a, a \rangle, b \rangle, a \rangle = \langle\langle a, a \rangle, \langle b, a \rangle\rangle \quad \text{and} \quad \langle\langle a, b \rangle, a \rangle = \langle a, \langle b, a \rangle\rangle$$

are true and hence

$$R^{lJ} = (R, +, \langle -, - \rangle)$$

is a non-commutative Jordan ring (which is called *a left Jordan ring associated with an associative ring* R). It is clear that, for commutative ring R, we have

$$R^J = R^{lJ}.$$

If A is an additive subgroup of R that $\langle a, r\rangle, \langle r, a\rangle \in A$ for any $a \in A$ and $r \in R$, then A is called *an ideal* of R^{lJ}. If $\delta \in \Delta$ and $a, b \in R$, then

$$\delta(\langle a, b\rangle) = \delta(2ab) = 2\delta(a)b + 2a\delta(b) = \langle \delta(a), b\rangle + \langle a, \delta(b)\rangle$$

and therefore $\delta \in \mathrm{Der}(R^{lJ})$. By the other hand, if $\delta \in \mathrm{Der}(R^{lJ})$, then

$$2\delta(ab) = \delta(\langle a, b\rangle) = \langle \delta(a), b\rangle + \langle a, \delta(b)\rangle = 2(\delta(a)b + a\delta(b)).$$

If R is a 2-torsion-free ring, then $\delta \in \mathrm{Der}\, R$. Similarly, as in Theorem 1, we can prove the following

Proposition 4. *For a 2-torsion-free ring R the following conditions are true:*

(1) *R is a Δ-simple ring if and only if R^{lJ} is a Δ-simple Jordan ring,*

(2) *R is a Δ-prime ring if and only if R^{lJ} is a Δ-prime Jordan ring,*

(3) *R is a Δ-semiprime ring if and only if R^{lJ} is a Δ-semiprime Jordan ring.*

5. Proofs

The next lemma in the prime case is contained in [18, Lemma 7].

Lemma 16 ([2, Lemma 1.7])**.** *Let R be a ring. If $[[R, R], [R, R]] = 0$, then the commutator ideal $C(R)$ is nil.*

Corollary 4. *If R is a non-commutative Δ-semiprime ring, then $[R, R]$ is non-commutative.*

Proof of Theorem 2.

(1) It is clear that a ring R is non-commutative. If A is a nonzero proper Δ-ideal of R, then A^L is a nonzero proper Δ-ideal of R^L. Therefore $A \subseteq Z(R)$ and, as a consequence, $A \cdot C(R) = 0$.

(2) Suppose that a Δ-simple ring R is non-commutative and U is its nonzero proper Lie Δ-ideal. By Proposition 2, $[U, U] = 0$. Then, by Lemma 11, $U \subseteq Z(R)$. Hence the quotient ring $R^L/Z(R)$ is $\widehat{\Delta}$-simple.

(3) Let A be a nonzero Δ-ideal of R such that $A^2 = 0$. Then A^L is a nonzero Δ-ideal of a Lie ring R^L and, moreover,

$$[A^L, A^L] = 0.$$

By Lemma 11, $A \subseteq Z(R)$ and hence $A \cdot C(R) = 0$.

(4) Suppose that R is non-commutative. Let A be a nonzero Lie Δ-ideal of R such that $[A, A] = 0$. Then, by Lemma 11, $A \subseteq Z(R)$ and, as a consequence, the Lie ring $R^L/Z(R)$ is $\widehat{\Delta}$-semiprime.

(5) Let A, B be nonzero Δ-ideals of R such that $AB = 0$. Obviously, $[A, B] \subseteq Z(R)$. Then $A \subseteq Z(R)$ or $B \subseteq Z(R)$.

(6) Assume that R is non-commutative and A, B are nonzero Lie Δ-ideals of R such that

$$[A, B] = 0.$$

Then $A \cap B \subseteq Z(R)$. Since $A \cap B \subseteq \operatorname{ann} C(R)$ in a Δ-prime ring R, we have that the intersection $A \cap B = 0$ is zero. If $T(A) = R$ (see proof of Proposition 2), then $[R, R] \subseteq A$ and $B \subseteq C_R([R, R])$. By Lemma 9, $B \subseteq Z(R)$. So we assume that $T(A) \neq R$. If $[T(A), T(A)] = 0$, then $[A, A] = 0$ and, by Lemma 11, $A \subseteq Z(R)$. Suppose that $[T(A), T(A)] \neq 0$. By Lemma 13, $T(A)$ contains a nonzero Δ-ideal I of R. Since

$$[I, B] \subseteq A \cap B = 0,$$

we conclude that $B \subseteq Z(R)$ by Lemma 7. □

The map

$$\partial_a : R \ni x \mapsto [a, x] \in R$$

is called *an inner derivation* of a ring R induced by $a \in R$. The set $\operatorname{IDer} R$ of all inner derivations of R is a Lie ring. Every prime Lie ring is primary Lie.

Lemma 17. *There is the Lie ring isomorphism*

$$\operatorname{IDer} R \ni \partial_a \mapsto a + Z(R) \in R^L/Z(R).$$

Proof. Evident. □

Corollary 5. *Let R be a ring. Then the following statements hold:*

(1) IDer R *is a simple Lie ring if and only if* $R^L/Z(R)$ *is a simple Lie ring,*

(2) IDer R *is a prime Lie ring if and only if* $R^L/Z(R)$ *is a prime Lie ring,*

(3) IDer R *is a semiprime Lie ring if and only if* $R^L/Z(R)$ *is a semiprime Lie ring,*

(4) IDer R *is a primary Lie ring if and only if* $R^L/Z(R)$ *is a primary Lie ring.*

References

[1] K. I. Beidar, A. V. Mikhal′ev, *Ortogonal completeness and minimal prime ideals* (in Russian), Trudy Sem. Petrovski, **10**, 1984, pp.227-234.

[2] H. E. Bell, A. A. Klein, *Combinatorial commutativity and finiteness conditions for rings*, Comm. Algebra, **29**, 2001, pp.2935-2943.

[3] J. Bergen, I. N. Herstein, E. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra, **71**, 1981, pp.259-267.

[4] J. Bergen, S. Montgomery, D. S. Passmann, *Radicals of crossed products of enveloping algebras*, Israel J. Math., **59**, 1987, pp.167-184.

[5] M. Brešar, M. A. Chebotar and W.S. Martindale, 3rd, *Functional identities*, Birkh′ auser Verlag, Basel Boston Berlin, 2000.

[6] K. R. Goodearl, R. B. Warfield, Jr., *Primitivity in differential operator rings*, Math. Z., **180**, 1982, pp.503-523.

[7] I. N. Herstein, *On the Lie and Jordan rings of a simple associative ring*, Amer. J. Math., **77**, 1955, pp.279-285.

[8] I. N. Herstein, *The Lie ring of a simple associative ring*, Duke Math. J., **22**, 1955, pp.471-476.

[9] I. N. Herstein, *Topics in Ring Theory*, The University of Chicago Press, Chicago London, 1965.

[10] I. N. Herstein, *Noncommutative rings*, The Mathematical Association of America, J. Wiley and Sons, 1968.

[11] I. N. Herstein, *On the Lie structure of an associative ring*, J. Algebra, **14**, 1970, pp.561-571.

[12] I. N. Herstein, *Rings with involution*, The University of Chicago Press, Chicago London 1976.

[13] N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

[14] N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., V. 39, Providence, R.I., 1968.

[15] C. R. Jordan, D. A. Jordan, *The Lie structure of a commutative ring with derivation*, J. London Math. Soc.(2), **18**, 1978, pp.39-49.

[16] D. A. Jordan, *Noetherian Ore extensions and Jacobson rings*, J. London Math. Soc. (2), **10**, 1975, pp.281-291.

[17] J. Lambek, *Lectures on rings and modules*, Blaisdell Publ. Co. A division of Ginn and Co. Waltham Mass. Toronto London, 1966.

[18] W. S. Martindale, 3rd, *Lie isomorphisms of prime rings*, Trans. Amer. Math. Soc., **142**, 1969, pp.437-455.

[19] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian rings.* With the cooperation of L. W. Small. Revised edition. Grad. Stud. in Math., 30. Amer. Math. Soc., Providence, RI, 2001.

[20] K. McCrimmon, *On Herstein's theorem relating Jordan and associative algebras*, J. Algebra, **13**, 1969, pp.382-392.

[21] A. Nowicki, *The Lie structure of a commutative ring with a derivation*, Arch. Math., **45**, 1985, pp.328-335.

[22] E. C. Posner, *Differentiably simple rings*, Proc. Amer. Math. Soc., **11**, 1960, pp.337-343.

[23] I. I. Zuev, *Lie ideals of associative rings* (in Russian), Uspehi Mat. Nauk, **18**, 1963, pp.155–158.

Contact information

O. D. Artemovych Institute of Mathematics
Cracow University of Technology
ul. Warszawska 24
Cracow 31-155 POLAND
E-Mail(s): `artemo@usk.pk.edu.pl`

M. P. Lukashenko Faculty of Mathematics and Informatics
PreCarpathian National University of Vasyl Stefanyk
Shevchenko St 57
Ivano-Frankivsk 76025 UKRAINE
E-Mail(s): `bilochka.90@mail.ru`

Received by the editors: 22.01.2015
and in final form 22.03.2015.