UDC 512.44

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ON GROUPS FACTORIZED BY FINITELY MANY SUBGROUPS*

ПРО ГРУПИ, ФАКТОРИЗОВАНІ СКІНЧЕННИМ ЧИСЛОМ ПІДГРУП

We prove that every group factorizable into a product of finitely many pairwise permutable central-by-finite minimax subgroups is a soluble-by-finite group.

Розвивається спектральна теорія та теорія розсіяння для одного класу самоспряжених матричних диференціальних операторів змішаного порядку.

1. Introduction. In 1986, M. J. Tomkinson [1] proved that if a group $G = A_1, \ldots, A_n$ is factorized by finitely many pairwise permutable Abelian minimax subgroups $A_1, \ldots, A_n$, then G is soluble. The aim of this short paper is to obtain a generalization of this result providing a positive answer to the following question suggested in [2] (Question 17):

Let the group $G = A_1, \ldots, A_n$ be the product of finitely many pairwise permutable minimax central-by-finite subgroups $A_1, \ldots, A_n$. Is G soluble-by-finite?

This problem should of course be seen in relation with the well-known Chernikov theorem [3] stating the almost solubility of an arbitrary group factorized by two central-by-finite subgroups.

Theorem A. *Let the group $G = A_1, \ldots, A_n$ be the product of finitely many pairwise permutable minimax central-by-finite subgroups $A_1, \ldots, A_n$. Then G is a soluble-by-finite group.*

The famous Kegel – Wielandt theorem on the solubility of a finite product of pairwise permutable nilpotent subgroups proves that the result of Tomkinson is a special case of Theorem A. Note also that the conditions of Theorem A cannot be weakened under assumption that the subgroups $A_1, \ldots, A_n$ have finite Prüfer rank, even if they are Abelian (see [2], Prop. 7.6.3). Since a soluble-by-finite product of polycyclic-by-finite subgroups is likewise polycyclic-by-finite (see [2], Theorem 4.4.2), Theorem A has the following consequence.

***Corollary* A_1.** *Let the group $G = A_1, \ldots, A_n$ be the product of finitely many pairwise permutable finitely generated central-by-finite subgroups $A_1, \ldots, A_n$. Then G is polycyclic-by-finite.*

On this subject, we also prove the following related result:

Theorem B. *Let the soluble-by-finite group $G = A_1, \ldots, A_n$ be the product of finitely many pairwise permutable cyclic-by-finite subgroups $A_1, \ldots, A_n$. Then G is Abelian-by-finite.*

* This work was done while the author was visiting the Ukrainian Academy of Sciences in Kiev. He is grateful to the Institute of Mathematics for its warm hospitality.

Note that for the case where the subgroups $A_1, \ldots, A_n$ are cyclic, a corresponding result was proved in [4].

As a consequence of Theorem A and Theorem B, we have, of course, the following:

Corollary $\mathbf{B}_1$. *Let the group* $G = A_1, \ldots, A_n$ *be the product of finitely many pairwise permutable (central cyclic)-by-finite subgroups* $A_1, \ldots, A_n$. *Then* G *is Abelian-by-finite.*

Note that Corollary B_1 can be applied, in particular, to products of finitely many pairwise permutable finite-by-cyclic subgroups.

Corollary $\mathbf{B}_2$. *Let the group* $G = AB$ *be the product of two (central cyclic)-by-finite subgroups* A *and* B. *Then* G *is metacyclic-finite.*

Our notation is mostly standard and can be found in [5]. Recall, in particular, that a soluble-by-finite group G is *minimax* if it has a series of finite length whose factors either are finite or infinite cyclic or quasicyclic of type p^∞ for some prime p. The number $m(G)$ of infinite factors in such a series is an invariant called the *minimax rank* of G.

2. Proofs. In order to prove Theorem A, we need the following already known lemmas. The second of them was proved in [6] in a more general situation.

Lemma 1 (see [7], Corollary 1). *Let* G *be a soluble-by-finite minimax group, and let* A *and* B *be subgroups of* G *such that* $AB = BA$. *If* A_1 *is a subgroup of finite index of* A, *there exists a subgroup of finite index* A_2 *in* A_1 *such that* $A_2B = BA_2$.

Lemma 2 (see [6], Theorem B). *Let the infinite metabelian group* $G = AB$ *be the product of two central-by-finite subgroups* A *and* B. *If* G *is minimax, then at least one of the subgroups* A *and* B *contains an infinite normal subgroup of* G.

Proof of Theorem A. Assume that the theorem is false, and among the counter examples with a minimal numbers of factors choose one $G = A_1 \ldots A_n$ such that the sum $\sum_{i=1}^{n} m(A_i)$ is minimal. The above quoted result of Chernikov [3] yields that $n \geq 3$. Set $A_1 \ldots A_{n-2} = A$, $A_{n-1} = B$, and $A_n = C$. By assumptions, the subgroups AB, BC, and AC are soluble-by-finite, and, hence, also minimax (see [8, 9] or [10]). By Lemma 1, there exist central subgroups of finite index B_1 of B and C_1 of C such that $B_1 C_1 = C_1 B_1$. The same lemma yields the existence of subgroups of finite index B_2 of B_1 and C_2 of C_1 such that $AB_2 = B_2A$ and $AC_2 = C_2A$. Clearly, A has infinite index in G, so that BC is infinite, and then $B_1 C_1$ is an infinite metabelian group. By Lemma 2, there exists an infinite normal subgroup N of $B_1 C_1$ which is contained either in B_1 or in C_1. In particular, we have either $N \leq Z(B)$ or $N \leq Z(C)$. If follows that either AB or AC is contained in $X = N_G(N^A)$. Suppose that $B \leq X$, so that $X = ABC_0$, where $C_0 = X \cap C$. Moreover, $N^A = N^{AB_2} = N^{AC_2}$, so that subgroup $\langle B_2, C_2 \rangle$ is contained in X, and so also $\langle A, B_2, C_2 \rangle \leq X$. It follows that X is of finite index in G (see [2], Lemma 1.2.5). Since N is infinite, and is contained either in B or in C, the minimal assumption yields that the factor X/N^A is soluble-by-finite. On the other hand, N^A is contained either in AB or in AC, so that it is soluble-by-finite. Therefore, G is also soluble-by-finite, and this contradiction proves the theorem.

Proof of Theorem B. The group G is polycyclic-by-finite by a famous result of Lennox – Roseblade and Zaitsev (see [2], Theorem 4.4.2). By induction on n, we can suppose that the subgroup $A = \langle A_1, \ldots, A_{n-1} \rangle$ is Abelian-by-finite. Let U be an Abelian subgroup of finite index of A, and consider cyclic subgroups of finite index $B_1, \ldots, B_n$ of $A_1, \ldots, A_n$, respectively. Set now $U_i = U \cap B_i$ for all $i \leq n-1$. Application of Lemma 1 yields that, for every $i = 1, \ldots, n-1$, there exist subgroups of finite index X_i of U_i and Y_i of B_n such that $X_i Y_i = Y_i X_i$. It follows now from Proposition C of [4] that there exist subgroups of finite index C_i of X_i and C_i^* of Y_i such that $\langle C_i^*, C_i \rangle$ is Abelian. Then $C_n = \bigcap_{i=1}^{n-1} C_i^*$ is a subgroup of finite index of B_n, and $\langle C_1, \ldots, C_n \rangle$ is an Abelian subgroup of finite index of G (see [2], Lemma 1.2.5). Therefore, G is Abelian-by-finite.

***Proof of Corollary* $\mathbf{B}_2$.** The group G is Abelian-by-finite by Corollary B_1, and so it contains an Abelian subgroup of finite index U. Let A_1 and B_1 be cyclic subgroups of finite index of A and B, respectively. Then also $\langle A_1 \cap U, B_1 \cap U \rangle$ has finite index in G (see [2], Lemma 1.2.5). On the other hand, the Abelian group $\langle A_1 \cap U, B_1 \cap U \rangle$ is obviously metacyclic and, hence, G is metacyclic-by-finite.

The author is grateful to Professor Ya. P. Sysak for his useful suggestions.

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Received 19.08.97