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On subgroups which cover or avoid chief factors of a finite group

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Dedicated to Professor Leonid A. Kurdachenko on the occasion of his 60th birthday

Abstract. A classical topic of research in Finite Group Theory is the following:

> What is the influence on the structure of the group of the fact that all members of some relevant family of subgoups enjoy a given embedding property?

The cover-avoidance property is a subgroup embedding property that has recovered much attention in the last few years. In this survey article we present a number of results showing that its influence goes much further from the classical supersolubility.

1. Introduction

In this survey article all groups are assumed to be finite. If a subgroup A of a group G has the property that either $HA = KA$ or $A \cap H = A \cap K$ for every chief factor H/K of G, then A is said to have the cover-avoidance

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property in G and is called a CAP-subgroup of G. This subgroup embedding property has afforded the attention of many authors. Some of them were interested in discovering some distinguished families of CAP-subgroups, mainly in the soluble universe, while others discovered some characterisations of soluble and supersoluble groups, or their corresponding local versions, in terms of CAP-property of the members of some relevant familes of subgroups. The reader may consult [2, Chapter 4], [3], [4], [15] and [28].

The purpose of this survey is to present some recent results about how the CAP-property of some distinguished subgroups of a group influences in its structure. In this context, the natural starting point is to think on groups in which every subgroup has the CAP-property. These groups are exactly the supersoluble ones. Recall that if p is a prime, a group G is said to be p-supersoluble if chief factor of G is cyclic of order p or a p'-group. The group G is supersoluble if it is p-supersoluble for all primes p, i.e., every chief factor of G is a cyclic group of prime order.

Our study confirms that in many cases the cover and avoidance property of some restricted families of subgroups gives rise to structural results involving saturated formations containing the class of all supersoluble groups. Recall that a class of groups $\mathfrak{F}$ is said to be a formation if $\mathfrak{F}$ is closed under taking epimorphic images and subdirect products. $\mathfrak{F}$ is saturated if is closed under taking Frattini extensions. It is well-known that the class of all supersoluble groups is a saturated formation

Three of the most popular families in the CAP context are the ones of maximal, 2-maximal and minimal subgroups of the Sylow subgroups. The following family includes them as particular cases: Let p be a prime. For a group G and a p-subgroup D of G, we define:

(1) If p is odd, then $\mathcal{F}_D(G) = \{H \leq G \colon |H| = |D|\}$

(2) If $p = 2$, then $\mathcal{F}_D(G) = \{H \leq G \colon |H| \in \{|D|, 2|D|\}\}$

2. Strong CAP-subgroups

Unfortunately, the cover-avoidance property is not inherited in intermediate subgroups. This means that if A is a CAP-subgroup of G and A is a subgroup of B, then A is not, in general, a CAP-subgroup of B.

Example 2.1. ([3, Example 1.3])

Let $G = \mathrm{Sym}(6)$ be the symmetric group of degree 6. Consider the following subgroups of G: $A = \langle (123), (12)(34), (14)(23) \rangle \cong \mathrm{Alt}(4)$, $C = \langle (56) \rangle$, $P = \langle (12)(34), (14)(23), (56) \rangle \cong C_2 \times C_2 \times C_2$ and $H =$

$\langle(12)(34)(56)\rangle$. Since $C \cap A = 1$ and $C \leq C_G(A)$, take $K = A \times C$. Then, $G = HN$, for $N = \mathrm{Alt}(6)$ and H is a CAP-subgroup of G.

However, P/C is a chief factor of K and $C < HC = \langle(12)(34), (56)\rangle < P$. Then, H is not a CAP-subgroup of K.

Definition 2.2. *Let A be a subgroup of a group G. Then we say that A is a* strong CAP-subgroup of G *if A is a CAP-subgroup of any subgroup of G containing A.*

Our main result of this section anaylises the impact of the strong CAP-property of some subgroups of the generalised Fitting subgroup on the structure of the group.

Theorem 2.3. *([4, Theorem A]) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of the generalised Fitting subgroup $F^*(E)$ of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups $H \in \mathcal{F}_D(P)$ are strong* CAP*-subgroups of G. Then $G \in \mathfrak{F}$.*

For the saturated formation of all supersoluble groups we have:

Corollary 2.4. *([4, Corollary 2]) A group G is supersoluble if and only if every non-cyclic Sylow subgroup P of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group) are strong* CAP*-subgroups of G.*

3. Partial CAP-subgroups

Another subgroup embedding property in the CAP orbit is the following.

Definition 3.1. *Let A be a subgroup of a group G. Then, we say that A has the* partial cover and avoidance property *or A is a* partial CAP-subgroup *(or* semi CAP-subgroup*) of G if there exists a chief series Γ_A of G such that A either covers or avoids each factor of Γ_A.*

This embedding property was first studied by Y. Fan, X. Guo and K. P. Shum in [11]. They called these subgroups *semi* CAP*-subgroups.* This is the name used in the subsequent papers [14], [15], [17], [7] and [28]. Nevertheless, since in this case the prefix "semi" does not refer to any half, we think that the name *partial* CAP*-subgroup* is more descriptive.

It is clear that CAP-subgroups are partial CAP-subgroups, but the converse does not hold in general.

Example 3.2. Consider as in Example 2.1 the symmetric group of degree 6, $G = \mathrm{Sym}(6)$. The subgroup H is a partial CAP-subgroup of K but H is not a CAP-subgroup of K.

In the chief series of K:

$$1 < V < P < K$$

$H \cap V = 1$, i.e. H avoids $V/1$, $P = HV$, i.e. H covers P/V and H trivially avoids K/P, since $H \leq P$.

The partial cover and avoidance property is also an extension of the c-normality introduced by Wang in [27]. However it is rather easy to construct examples showing that the partial cover and avoidance property does not imply c-normality.

Our study of the partial CAP-property tries to answer the following question:

Let G be a group. If all members of the family $\mathcal{F}$ of subgroups of G are partial CAP-*subgroups of G, then what is the structure of G?.*

The family $\mathcal{F}$ will be one of the following.

(1) $\mathcal{F}$: The meet-irreducible subgroups of G with order divisible by p.

(2) $\mathcal{F}$: The meet-irreducible subgroups with order a multiple of p of each maximal subgroup of G.

(3) $\mathcal{F}$: The Sylow p-subgroups of G.

(4) $\mathcal{F}$: The maximal subgroups of the Sylow p-subgroups of G.

(5) $\mathcal{F}$: The 2-maximal subgroups of the Sylow p-subgroups of G.

3.1. On meet-irreducible subgroups.

The following definition play a central role in this subsection.

Definition 3.3. *(D. L. Johnson. [24]) Let G be a group. A subgroup H is said to be* meet-irreducible in G *if whenever $H = X_1 \cap \ldots \cap X_n$, for some subgroups $X_1, \ldots, X_n$ of G, then $H = X_i$ for some i.*

This is clearly equivalent to say that H is a proper subgroup of the intersection of all subgroups of G which properly contain H.

A word on terminology: D. L. Johnson names these subgroups *primitive* and observes that they are also called “meet-irreducible.” We prefer this last name to avoid any confusion with the classical primitive groups.

The following result gives a characterisation of the p-supersoluble groups in terms of families of meet-irreducible subgroups whose order is divisible by p.

Theorem 3.4. *([3, Theorem 3.10]) Let G be a group and p a prime dividing the order of G. The following conditions are pairwise equivalent:*

(1) *G is a p-supersoluble group.*

(2) *Every meet-irreducible subgroup of G with order divisible by p is a partial* CAP*-subgroup of G.*

(3) *Every meet-irreducible subgroup with order a multiple of p of each maximal subgroup of G is a partial* CAP*-subgroup of G.*

3.2. On Sylow p-subgroups and its maximal subgroups.

Next we fix a prime p and analyse the structure of a group in which the members of the families (3), (4) and (5) above are partial CAP-subgroups.

Y. Fan,X. Guo and K. P. Shum proved in [11] the following facts:

- The p-solubility of a group can be characterised by the partial CAP-property of the Sylow p-subgroups.
- The supersolubility of a group is characterised by the partial CAP-property of the maximal subgroups of the Sylow subgroup of the group.

The method we used here is a local one. This means that it is generalised in a form referring to a prime. The reason for choosing this local method is to discover new situations. Roughly speaking, the global hypothesis, referring to all primes, force the solubility and some non-soluble cases do not appear. This also leads us to the interesting question of how the global properties can be obtained as the conjunction of the local ones for all primes.

Theorem 3.5. *([3, Theorem 3.2]) Let p be a prime dividing the order of a group G. Then, all maximal subgroups of every Sylow p-subgroup of G are partial* CAP*-subgroups of G if and only if*

(1) *either G is a group whose Sylow p-subgroups are cyclic groups of order p*

(2) *or G is a p-supersoluble group.*

Corollary 3.6. *([11]) Suppose that all maximal subgroups of every Sylow subgroup of a group G are partial* CAP*-subgroups of G. Then G is a supersoluble group.*

Theorem 3.7. *([3, Corollary 3.7]) Let G be a finite group and p a prime dividing the order of G. If every maximal subgroup of every Sylow p-subgroup of $F_p^*(G)$ is a partial* CAP*-subgroup of G, then either*

(1) *G is a p-supersoluble group, or*

(2) *$F_p^*(G)/\mathrm{O}_{p'}(G)$ is isomorphic to a non-abelian simple group whose Sylow p-subgroups are cyclic groups of order p.*

Here, $F_p^*(G)$ denotes the characteristic subgroup

$$\mathrm{F}_p^*(G) = \bigcap\{HC_G(H/K) : H/K \text{ is a chief factor of } G \text{ and } p \text{ divides } |H/K|\}.$$

3.3. The next aim

Our next aim is the characterisation of the class of all groups G enjoying the following local property:

> (†) *Every 2-maximal subgroup of every Sylow p-subgroup of G is a partial* CAP*-subgroup of G.*

Definition 3.8. *Given a group G, a subgroup K of G is called a* second maximal subgroup*, or* 2-maximal subgroup*, if there exists a maximal subgroup M of G such that $K \leq M$ and K is maximal in M.*

If G is a p-group (p a prime) of order p^n, then the 2-maximal subgroups of G are all subgroups of order p^{n-2}.

The idea of obtaining information on the structure of a group in which some 2-maximal subgroups are embedded in some particular way has produced many results. Let us emumerate some of them.

(1) B. Huppert [21] prove the following:

If every 2-maximal subgroup of a group G is normal in G, then G is supersoluble, and if moreover the order of G is divisible by at least three distinct primes, then G is nilpotent.

(2) A. Mann [25] obtained information on the structure of all groups whose 2-maximal subgroups are subnormal.

(3) R. K. Agrawal [1] obtained Huppert's result under hypothesis of permutability.

(4) Li Shirong [26] obtained the structure of all groups whose 2-maximal subgroups are TI-subgroups. (H is a TI-subgroup of G, if for every $g \in G$, we have $H \cap H^g = 1$.)

(5) Guo, Shum and Skiba [16] analysed some conditions of supersolubility and nilpotency of groups by means of the X-semiper-mutability of either 2-maximal subgroups or maximal subgroups.

In our study we must have some important facts in mind.

- The trivial case: If the Sylow p-subgroups have order p^2, then the family of all 2-maximal subgroups of the Sylow p-subgroups is composed of just the trivial subgroup. Trivially these groups verify property (†). Hence, what we will call "the trivial case" will be the characterisation of all groups whose Sylow p-subgroups have order p^2.
- By the Brauer-Suzuki Theorem, if the Sylow 2-subgroups of a group G are isomorphic to Q_8, then $Z(G/\operatorname{O}_{2'}(G))$ has even order. Hence every group whose Sylow 2-subgroups are isomorphic to Q_8 has property (†). We shall include these groups in our characterisation.
- The p-soluble case: The class of all p-supersoluble groups is composed of groups with property (†). Thus, our interest will be to characterise all p-soluble groups with property (†) which are not p-supersoluble. This is the most complicated part of the question.

The trivial case.

Theorem 3.9. *([5]) Let G be a group with Sylow p-subgroups of order p^2. Let us denote $\overline{G} = G/\operatorname{O}_{p'}(G) \quad S = Soc(\overline{G}) \quad F = \operatorname{O}^{p'}(\overline{G})$. Then we have one of the following.*

(1) *Either S is the direct product of two distinct minimal normal subgroups of $\overline{G}$, say N_1 and N_2.*

In this case, $F = S$ and N_1 and N_2 are simple groups with cyclic Sylow p-subgroups of order p.

(2) *Or S is a chief factor of G.*

In this case we have one of the following.

(a) *$S \cong C_p$; in this case G is p-supersoluble and $F \cong C_{p^2}$.*

(b) *$\overline{G}$ is a primitive group of type 1; in this case G is p-soluble and $F = S \cong C_p \times C_p$.*

(c) *$\overline{G}$ is a primitive group of type 2 and $F = S$; in this case,*

- *either $S \cong T \times T$, T non-abelian simple with Sylow p-subgroups of order p,*
- *or S non-abelian simple with Sylow p-subgroups of order p^2.*

(d) *$\overline{G}$ is a primitive group of type 2 and $S < F$: $\overline{G}$ is almost-simple*

- *S is non-abelian simple with Sylow p-subgroups of order p and*
- *$\overline{G}/S$ is soluble with Sylow p-subgroups of order p.*

The most difficult case to deal to rule out is when $\mathrm{Soc}(G) = \mathrm{F}(G)$ is a cyclic group of order p and $\mathrm{F}^*(G)/\mathrm{F}(G)$ a chief factor of G isomorphic to a non-abelian simple group with Sylow p-subgroups of order p.

The p-soluble case.

Theorem 3.10. *([5]) Let G be a p-soluble group. Then, G has property (†) if and only if one of the following holds.*

(1) *G is p-supersoluble.*

(2) *G is a group such that if P is a Sylow p-subgroup of G and Q is a 2-maximal subgroup of P, then*

$$\Phi(G/\mathrm{O}_{p'}(G)) \le Q\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G);$$

- *if $\Phi(G/\mathrm{O}_{p'}(G)) = Q\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$, then every chief series of the group G has exactly one complemented p-chief factor; moreover, this p-chief factor has order p^2;*
- *if $\Phi(G/\mathrm{O}_{p'}(G)) < Q\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$, then all complemented p-chief factors of G are G-isomorphic to a 2-dimensional irreducible G-module V which is not an absolutely irreducible G-module.*

Why does it appear the absolutely irreducible module?

In a minimal counterexample, we have $\mathrm{O}_{p'}(G) = \Phi(G) = 1$ and $\mathrm{F}(G)$ is the Sylow p-subgroup of G. In other words, we have to analyse $\mathrm{F}(G)$. Now $\mathrm{F}(G)$ is a homogeneous G-module over $\mathrm{GF}(p)$ and if V is an irreducible submodule of $\mathrm{F}(G)$ and V is absolutely irreducible, then number of irreducible submodules, or, equivalently the number of minimal normal subgroups of G is quite short and we do not have enough chief series to fulfill the condition by which every 2-maximal subgroup of $\mathrm{F}(G)$ is a partial CAP-subgroup of G.

The Theorem.

We bring the article to a close with the characterisation of groups enjoying property (†).

Theorem 3.11. *([5]) Let G be a group and $P \in \mathrm{Syl}_p(G)$. Assume that p^2 divides $|G|$. Then G has property (†) if and only if G satisfies one of the following conditions:*

(1) *G is p-supersoluble;*

(2) *G is p-soluble and if Q is 2-maximal in P, then*

$$\Phi(G/\mathrm{O}_{p'}(G)) \leq Q\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G);$$

- *if $\Phi(G/\mathrm{O}_{p'}(G)) = Q\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$, then every chief series of the group G has exactly one complemented p-chief factor; moreover, this p-chief factor has order p^2;*
- *if $\Phi(G/\mathrm{O}_{p'}(G)) < Q\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$, then all complemented p-chief factors of G are G-isomorphic to a 2-dimensional irreducible G-module V which is not an absolutely irreducible G-module.*

(3) *G is non p-soluble and $|P| = p^2$;*

(4) *$p = 2$ and G is non 2-soluble, and P is isomorphic to Q_8.*

References

[1] R. K. Agrawal, The influence on a finite group of its permutable subgroups. *Canad. Math. Bull.* **17** (1974) 159–165.

[2] A. Ballester-Bolinches and L. M. Ezquerro, *Classes of Finite groups*, Springer, Dordrecht, 2006.

[3] A. Ballester-Bolinches, Luis M. Ezquerro and Alexander N. Skiba, Local embeddings of some families of subgroups of finite groups, *Acta Math. Sinica, English Series* **25(7)** (2009) 869–882.

[4] A. Ballester-Bolinches, Luis M. Ezquerro and Alexander N. Skiba, Subgroups of finite groups with a strong cover-avoidance property. *Bull. of the Austral. Math. Soc.* **79** (2009) 499-506.

[5] A. Ballester-Bolinches, Luis M. Ezquerro and Alexander N. Skiba, On second maximal subgroups of Sylow subgroups of finite groups. Submitted.

[6] R. W. Carter and T. O. Hawkes, The $\mathfrak{F}$-normalizers of a finite soluble group *J. Algebra* **5** (1967) 175–201.

[7] Y.Cui and and X. Guo. Influence of the Semi Cover-Avoiding property onf the Structure of Finite Groups . *International J. Algebra* **2** 2008, 395-402.

[8] K. Doerk and T. Hawkes, *Finite Soluble Groups.* Walter de Gruyter, Berlin-New York, 1992.

[9] L. M. Ezquerro, A contribution to the theory of finite supersoluble groups, *Rend. Sem. Mat. Univ. Padova* **89** (1993) 161–170.

[10] W. Gaschütz. Prefrattinigruppen. *Arch. Math.* **13**, 1962, 418-426.

[11] Y. Fan, X. Guo and K. P. Shum. Remarks on two generalizations of normality of subgroups. *Chinese J. Contemp. Math.* **27** 2006, 139-146.

[12] R. L. Griess and P. Schmid, The Frattini module. *Arch. Math.* **30** (1978) 256–266.

[13] X. Guo and K. P. Shum, Cover-avoidance properties and the structure of finite groups. *J. Pure and Applied Algebra* **181** (2003) 297–308.

[14] X. Guo, Junxin Wang and K. P. Shum. On Semi-Cover-Avoiding Maximal subgroups and Solvability of Finite Groups. *Comm. Algebra* **34**, 2006, 3235-3244.

[15] X. Guo, P. Guo and K. P. Shum. On semi cover-avoiding subgroups of finite groups. J. Pure and Applied Algebra **209**, 2007, 151-158.

[16] W. Guo, K. P. Shum and A. N. Skiba, X-semipermutable subgroups of finite groups. *J. Algebra* **315** (2007), no. 1, 31–41.

[17] X. Y. Guo and L. L. Wang. On finite groups with some semi cover-avoiding subgroups. *Acta Math. Sin. (Engl. Ser.)* **23**, 2007, 1689-1696

[18] P. Hall, A note on soluble groups *J. London Math. Soc.* **3** (1928) 98–105.

[19] P. Hall, On the system normalizers of a soluble group. *Proc. London Math. Soc.(2)* **43** (1937) 507–528.

[20] B. Hartley. On Fischer's dualization of formation theory. *Proc. London Math. Soc.(3)* **19** 1969, 193–207.

[21] B. Huppert, Normalteiler und maximale Untergruppen endlicher Gruppen. *Math. Z.* **60** (1954) 409–434.

[22] B. Huppert, *Endliche Gruppen I.* Die Grundlehren der Mathematischen Wissenschaften 134 Springer-Verlag, Berlin-New York 1967.

[23] B. Huppert and N. Blackburn, *Finite Groups III.* Grundlehren der Mathematischen Wissenschaften 243. Springer-Verlag, Berlin-New York, 1982.

[24] D. L. Johnson. A note on supersoluble groups. *Can. J. Math.* **23**, 1971, 562-564.

[25] A. Mann, Finite groups whose n-maximal subgroups are subnormal. *Trans. Amer. Math. Soc.* **132** (1968) 395–409.

[26] Li Shirong, Finite non-nilpotent groups all of whose second maximal subgroups are TI-groups. *Math. Proc. R. Ir. Acad.* **100A** (2000) 65–71.

[27] Y. M. Wang. C-normality of groups and its properties. *J. Algebra* **180**, 1996, 954-965.

[28] L. Wang and G. Chen. Some properties of finite groups with some (semi-p-)cover-avoiding subgroups. *J. Pure and Applied Algebra* **213**, 2009, 686–689.

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