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Classification of irreducible non-dense modules for $A_2^{(2)}$

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Abstract. We obtain a classification of the supports of irreducible $A_2^{(2)}$-modules. In particular, we get a classification of all non-dense irreducible $A_2^{(2)}$-modules with at least one finite-dimensional weight subspace.

Introduction

Let $\mathfrak{g}$ be an affine Kac-Moody algebra with Cartan subalgebra $\mathfrak{h}$, root system Δ and center $\mathbb{C}c$. A $\mathfrak{g}$-module V is called a *weight* if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, $V_\lambda = \{v \in V \mid hv = \lambda(h)\, v \text{ for all } h \in \mathfrak{h}^*\}$. If V is an irreducible weight $\mathfrak{g}$-module then c acts on V as a scalar, called *level* of V. For a weight $\mathfrak{g}$-module V, the support is the set $supp\,(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}$. The root lattice Q is the free abelian group over Δ. If V is irreducible then $supp\,(V) \subset \lambda + Q$ for some $\lambda \in \mathfrak{h}^*$. An irreducible weight $\mathfrak{g}$-module V is called *non-dense*, if $supp\,(V) \subsetneq \lambda + Q$,

This work contains the classification of irreducible non-dense modules for the Kac-Moody algebra $A_2^{(2)}$ with at least one finite-dimensional weight subspace. The classification of non-dense irreducible $A_1^{(1)}$-modules with a finite-dimensional weight subspace has been done by V. Futorny

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[5]. The classification problem is also solved for all affine Kac-Moody algebras for non-zero level modules with all finite-dimensional weight subspaces (V. Futorny and A. Tsylke [4]). In these cases an irreducible module is either a quotient of a classical Verma module, or of a generalized Verma module, or of a loop module (induced from a Heisenberg subalgebra). That this will hold for irreducible non-dense modules of any affine Kac-Moody algebras has been conjectured by V. Futorny [5]. With this work we confirm the conjecture for non-dense irreducible $A_2^{(2)}$-modules with a finite-dimensional weight subspace.

We also obtain a classification of all possible supports for irreducible $A_2^{(2)}$-modules. The proof is elementary and involves only the combinatorics of the root system employing heavily the assumption of a „hole" in the weight lattice $\lambda + Q$, precisely the condition of non-density. This will always result in the „upper", „lower" or the „right" half of the weight lattice $\lambda + Q$ having all (or all but one) zero weight spaces (up to equivalence under the affine Weyl group). Upper and right half refer to the two non-equivalent classes of partitions. It is well known that these are the only ones [5].

If we omit the requirement of a finite-dimensional weight subspace then we do not get a complete classification. In this case we have a classification upto the classification of irreducible graded (with respect to the natural $\mathbb{Z}$-grading) modules over the Heisenberg subalgebra with non-zero level and all infinite-dimensional components. Nevertheless the classification of all supports provides a characterization of irreducible $A_2^{(2)}$-modules.

The proof is structured in form of a binary tree where each leaf corresponds to the construction of a so-called *primitive* element. This by definition is a vector v with the following property: Let $\mathcal{P}$ be a parabolic subalgebra with Levi decomposition $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_+$. If we take $\mathcal{P}$ the corresponding parabolics of a classical Verma module, a generalized Verma, or a loop module then v is annihilated by one of the corresponding $\mathcal{P}_+$ (here $\mathcal{P}$ is just a Borel subalgebra in the case of a classical Verma module). This primitive vector thus generates an irreducible quotient of a classical Verma module, a generalized Verma module or a loop module respectively [1, 2].

The paper is structured as follows:

In section 2 we review the realization of the twisted Kac-Moody algebra $A_2^{(2)}$ and the construction of its root system. Section 3 and 4 gives the definition of generalized Verma modules and loop modules, respectively. In section 5 the category $\tilde{\mathcal{O}}$ for not necessarily finite-dimensional weight modules is introduced following V. Chari [8] and V. Futorny [3]. In section 6 we proof the main result and section 7 states the classification of supports for irreducible $A_2^{(2)}$-modules.

1. Preliminaries

Let $A_2^{(2)}$ be the the Kac-Moody algebra defined by generators and relations due to the generalized Cartan matrix $(A_{ij})_{i,j=0,1} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$. Let $\Pi = \{\alpha_0, \alpha_1\}$ and $\Pi^\vee = \{h_0, h_1\}$ be linear independent subsets of the 2-dimensional vector space $\mathfrak{h}^*$ and its dual $\mathfrak{h}$ respectively, such that $\alpha_j(h_i) = A_{ij}$. Now $A_2^{(2)}$ is generated by e_0, e_1, f_0, f_1 due to the relations

$$\begin{aligned} [e_i f_i] &= \delta_{ij} h_i \\ [he_i] &= \alpha_i(h)\, e_i \\ [hf_i] &= -\alpha_i(h)\, f_i,\ h \in \mathfrak{h}, i = 0, 1 \end{aligned} \tag{1}$$

As $dim\,\mathfrak{h}^* = dim\,\mathfrak{h} = 2n - rk\,A = 3$ there are elements δ and d completing Π and $\Pi^\vee$ to be bases of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively. Furthermore $A_2^{(2)}$ permits a nontrivial 1-dimensional ideal spanned by the central element $c = 2h_0 + h_1$. One can define non-degenerate symmetric invariant bilinear $\mathbb{C}$-valued form $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{h}$ which can be uniquely extended to a bilinear form $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{g}$. The standard invariant form on $A_2^{(2)}$ is given by

$$\langle h_0, h_0 \rangle = 2,\ \langle h_0, h_1 \rangle = -2,\ \langle h_0, d \rangle = \frac{1}{2},\ \langle h_1, h_1 \rangle = 2,$$

all other brackets vanishing.

Realization. Let $\mathfrak{g}^0$ a simple Lie algebra. Let σ be a non-twisted graph automorphism of the Dynkin graph of simple roots Δ. σ is also an automorphism of $\mathfrak{g}^0$ by $\sigma : \mathfrak{g}^0_\beta \mapsto \mathfrak{g}^0_{\sigma(\beta)}$, $\beta \in \Delta$. When σ has order 2, then $\mathfrak{g}^0$ decomposes as a module as the set of fix points of σ and the eigenelements to the eigenvalue -1

$$\mathfrak{g}^0 = \left(\mathfrak{g}^0\right)^\sigma \oplus \left(\mathfrak{g}^0\right)_{-1}.$$

The example $\sigma(E_{\alpha+\beta}) = \sigma([E_\alpha E_\beta]) = [E_\beta E_\alpha] = -[E_\alpha E_\beta]$ illustrates, how the eigenvalue -1 occurs.

Let $\mathfrak{g}^0 = A_2$ and $\hat{\mathfrak{L}}\left(\mathfrak{g}^0\right) = \mathbb{C}\left[t, t^{-1}\right] \otimes \mathfrak{g}^0 \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the (extended) loop algebra with extended Dynkin graph

Define $\delta \in \mathfrak{h}^*$ by $\delta \mid_{\mathfrak{h}^0 \oplus \mathbb{C}c} = 0$ and $\delta(d) = 1$. Denote by $E_1 = E_\alpha$, $E_2 = E_\beta$, $F_1 = F_\alpha$, $F_2 = F_\beta$ the Chevalley generators of $\mathfrak{g}^0$. Then $\hat{\pi}^0 = \{\alpha, \beta, \delta\}$ is a basis for the root system $\hat{\Delta}^0$ of $\hat{\mathfrak{L}}\left(\mathfrak{g}^0\right)$. Denote $\theta = \alpha + \beta$, $\alpha_0 = \delta - \theta$. The σ-orbits on Δ are given by a high and a low

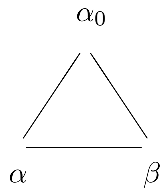

Figure 1: Extended Dynkin graph of A_2.

2-element orbit $(\alpha_0+\alpha, \alpha_0+\beta)$ and (α, β), respectively. The fixpoints are $\left(\hat{\Delta}^0\right)^\sigma = \Delta\left(\pi^\sigma\right) = \mathbb{Z}\pi^\sigma \cap \hat{\Delta}^0$ with respect to the basis $\pi^\sigma = \{\theta, \delta\}$.

The twisted graph automorphism τ of this loop algebra is defined by the maps $t^k \otimes E_1 \mapsto (-1)^k t^k \otimes E_2$, $t^k \otimes E_2 \mapsto (-1)^k t^k \otimes E_1$ and $t^k \otimes E_\theta$ to $(-1)^{k+1} t^k \otimes E_\theta$. The generators of $\left(\mathfrak{g}^0\right)^\sigma$ are given by

$$E_1+E_2, \qquad F_1+F_2, \qquad H_\theta, \qquad H_1+H_2,$$

where $H_\theta = [E_\theta F_\theta]$. And the generators of $\left(\mathfrak{g}^0\right)_{-1}$ are given by

$$E_1-E_2, \qquad F_1-F_2, \qquad E_\theta, \qquad F_\theta, \qquad H_1-H_2.$$

$\mathfrak{g} = A_2^{(2)}$ is realized as the fixed point set $\hat{\mathfrak{L}}\left(\mathfrak{g}^0\right)^\tau$. Consider therefore the bracket in $\hat{\mathfrak{L}}\left(\mathfrak{g}^0\right) = A_2^{(1)}$, given by

$$\begin{aligned}&\left[t^k \otimes a + \lambda c + \mu d, t^l \otimes a' + \lambda' c + \mu' d\right]\\ &= t^{k+l} \otimes [a,b] + t^l \otimes l\mu a' - t^k \otimes k\mu' a + k\delta_{k+l,0}\left\langle a, a'\right\rangle c\end{aligned}$$

$a, a' \in \mathfrak{g}^0$, $\lambda, \lambda', \mu, \mu' \in \mathbb{C}$, $k, l \in \mathbb{Z}$. The weight spaces with respect to $\hat{\mathfrak{h}}$ are defined as $V_\lambda = \left\{v \in V \mid h \cdot v = \lambda(h)\, v \text{ for all } h \in \hat{\mathfrak{h}}\right\}$. Eventually, the all one-dimensional weight spaces of $\mathfrak{g}\left(\tilde{A}_2\right)^\tau$ are generated by

$$\begin{aligned}
e_{2k\delta}^{(1)} &= t^{2k} \otimes (H_1+H_2)\\
e_{2k\delta}^{(2)} &= t^{2k} \otimes H_\theta + c\\
e_{(2k+1)\delta} &= t^{2k+1} \otimes (H_1-H_2)\\
e_{\alpha_1+2k\delta} &= t^{2k} \otimes (E_1+E_2)\\
e_{\alpha_1+(2k+1)\delta} &= t^{2k+1} \otimes (E_1-E_2)\\
e_{2\alpha_1+(2k+1)\delta} &= t^{2k+1} \otimes E_\theta\\
e_{-\alpha_1+k\delta} &= t^{2k} \otimes (F_1+F_2)\\
e_{-\alpha_1+(2k+1)\delta} &= t^{2k+1} \otimes (F_1-F_2)\,.
\end{aligned}$$

$$e_{-2\alpha_1+(2k+1)\delta} = t^{2k+1} \otimes F_\theta.$$

This gives us the complete root system. The set of simple roots are the disjoint union of short real roots $\Delta^{re,s}$, long real roots $\Delta^{re,l}$ and imaginary roots Δ^{im}, given by $\{\pm\alpha_1 + \mathbb{Z}\delta\}$, $\{\pm 2\alpha_1 + (2\mathbb{Z}+1)\,\delta\}$ and $\{k\delta \mid k \in \mathbb{Z}\setminus\{0\}\}$ respectively.

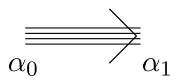

Figure 2: Dynkin graph of $A_2^{(2)}$

The (affine) Weyl group of $\mathfrak{g}$ is an affine reflection group generated by $W = \langle t_\theta, s\rangle$, fulfilling the relations $s^2 = 1$, $st_\theta s^{-1} = t_{s(\theta)} = t_{-\theta}$ and $t_\theta^k = t_{k\theta}$, $k \in \mathbb{Z}\setminus\{0\}$, where $s = s_1$ is the fundamental reflection at α_1, acting on the root lattice $Q(\pi)$, $\pi = \{\alpha_1, \delta - \alpha_1\}$ by

$$\begin{aligned} s\,(m\alpha_1 + n\delta) &= -\,m\alpha_1 + n\delta, \\ t_\theta^k\,(m\alpha_1 + n\delta) &= m\alpha_1 + (n-k)\,\delta,\ m,k,n \in \mathbb{Z}. \end{aligned}$$

Lemma 1.1 (Relations)**.** *The commutators are given by*

$$(i) \qquad \left[e_{k\delta}^{(1)}, e_{m\delta}^{(1)}\right] = 2k\delta_{k+m,0}c$$

$$(ii) \qquad \left[e_{k\delta}^{(1)}, e_{\pm\alpha+m\delta}\right] = \pm e_{\pm\alpha+(k+m)\delta}$$

$$(iii) \qquad \left[e_{2k\delta}^{(2)}, e_{\pm\alpha+m\delta}\right] = \pm e_{\pm\alpha+(2k+m)\delta}$$

$$(iv) \qquad \left[e_{2k\delta}^{(2)}, e_{2m\delta}^{(2)}\right] = 4k\delta_{k+m,0}c$$

$$(v) \qquad \left[e_{m\delta}^{(1)}, e_{2k\delta}^{(2)}\right] = 2k\delta_{2k+m,0}c$$

$$(vi) \qquad [e_{\alpha+k\delta}, e_{-\alpha+m\delta}] = \begin{cases} \left(e_{0\cdot\delta}^{(1)} + 2kc\right) & \text{if } m = -k \\ e_{(k+m)\delta}^{(1)} & \text{if } m \neq -k \end{cases}$$

$$(vii) \qquad [e_{\alpha+k\delta}, e_{\alpha+m\delta}] = \begin{cases} e_{2\alpha+(k+m)\delta} & \text{if } k \text{ even and } m \text{ odd} \\ 0 & \text{if } k+m \text{ even} \end{cases}$$

$$(viii) \qquad [e_{2\alpha+k\delta}, e_{-2\alpha+m\delta}] = \begin{cases} \left(e_{0\cdot\delta}^{(2)} + 2kc\right) & \text{if } m = -k \text{ odd} \\ e_{(k+m)\delta}^{(2)} & \text{if } m \neq -k \text{ both odd} \end{cases}$$

$$(ix) \qquad \left[e_{2k\delta}^{(1,2)}, e_{\pm 2\alpha+m\delta}\right] = \pm 2e_{2\alpha+(m+2k)\delta}$$

$$(x) \qquad \left[e_{k\delta}^{(1)}, e_{\pm 2\alpha+m\delta}\right] = 0$$

$$(xi) \qquad \left[e_{\pm\alpha+k\delta}, e_{\mp 2\alpha+(2l+1)\delta}\right] = -e_{\mp\alpha+(2l+k+1)\delta}$$

$$(xii) \qquad \left[e_{\pm\alpha+k\delta} + \mu d, e_{\kappa\alpha+l\delta}\right] = l\mu e_{\kappa\alpha+l\delta} + \frac{1}{2}k\delta_{k+l,0}c, \;\; \kappa = \pm 0, 1, 2$$

Proof. Compute for example (iii):

$$\begin{aligned}\left[e_{2k\delta}^{(2)}, e_{\pm\alpha+m\delta}\right] &= \left[\left[e_{2\alpha+(2k-i)\delta}, e_{-2\alpha+i\delta}\right], e_{\alpha+m\delta}\right] \text{ for an odd } i \\ &= \left[e_{2\alpha+(2k-i)\delta}, \left[e_{-2\alpha+i\delta}, e_{\alpha+m\delta}\right]\right] \\ &= \left[e_{2\alpha+(2k-i)\delta}, e_{-\alpha+(i+m)\delta}\right] = e_{\alpha+(2k+m)\delta}.\end{aligned}$$

□

Thus $e_{2k\delta}^{(1)} = e_{2k\delta}^{(2)} = e_{2k\delta}$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is generated by $\{e_\delta, e_{-\delta}, e_\alpha, e_{-\alpha}\}$.

2. Generalized Verma modules

Fix $\alpha = \alpha_1 \in \Delta^{re}$ and denote $\mathfrak{g}_{\alpha+k\delta} = t^k \otimes \mathfrak{g}_\alpha$, $k \in \mathbb{Z}$ and $\mathfrak{g}_{n\delta} = t^n \otimes \mathbb{C}h_\alpha$, $n \in \mathbb{Z}\setminus\{0\}$. If $\alpha \in \Delta^{re,l}$ all even or all odd graded components vanish. Consider a subalgebra $\mathfrak{g}(\alpha) \subset \mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Then $\mathfrak{g}(\alpha) \cong \mathfrak{sl}_2$.

Consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g}(\alpha))$. Its center is generated by the Casimir element $z_\alpha = (h_\alpha + 1)^2 + 4e_{-\alpha}e_\alpha$. Remember $\mathfrak{h} = \mathfrak{h}^0 \oplus \mathbb{C}c \oplus \mathbb{C}d$. Define

$$T_\alpha = S(\mathfrak{h}) \otimes \mathbb{C}[z_\alpha].$$

Fix $\lambda \in \mathfrak{h}^*$. Consider the 1-dimensional T_α-module $\mathbb{C}v_{\lambda,\gamma}$ with the action $\left(h \otimes z_\alpha^k\right) v_\lambda = h(\lambda)\gamma^k v_\lambda$ and define the $\mathfrak{h} + \mathfrak{g}(\alpha)$-module

$$V(\lambda, \gamma) = \mathcal{U}(\mathfrak{g}(\alpha) + \mathfrak{h}) \underset{T_\alpha}{\otimes} \mathbb{C}v_\lambda.$$

It has a unique irreducible quotient, say $V_{\lambda,\gamma}$.

Proposition 2.1 ([3]). *If V is an irreducible weight $H + \mathfrak{g}(\alpha)$-module then $V \cong V_{\lambda,\gamma}$ for some $\lambda \in \mathfrak{h}^*$ $\gamma \in \mathbb{C}$.*

Let $\lambda \in \mathfrak{h}^*$, $\gamma \in \mathbb{C}$. Denote

$$\mathcal{N}_\alpha^\pm = \sum_{\varphi \in \Delta_+ \setminus \{\alpha\}} \mathfrak{g}_{\pm\varphi}, \quad E_\alpha^\pm = (\mathfrak{h} + \mathfrak{g}(\alpha)) \oplus \mathcal{N}_\alpha^\pm.$$

Consider $V_{\lambda,\gamma}$ as $E_\alpha^\pm$-module with trivial action of $\mathcal{N}_\alpha^\pm$ and construct the $\mathfrak{g}$-module

$$M_\alpha^\pm(\lambda,\gamma) = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(E_\alpha^\pm)}{\otimes} V_{\lambda,\gamma}.$$

The module $M_\alpha^\pm(\lambda,\gamma)$ is called a *generalized Verma module* following [3]. It has a unique irreducible quotient $L_\alpha^\pm(\lambda,\gamma)$. Notice that $V_{\lambda,\gamma}$ does not have to be finite-dimensional.

Corollary 2.2 ([3])**.** *Let V be an irreducible weight $\mathfrak{g}$-module and $0 \neq v \in V_\lambda$ such that $\mathcal{N}_\alpha^\pm v = 0$, then $V \cong L_\alpha^\pm(\lambda,\gamma)$ for some $\gamma \in \mathbb{C}$.*

3. Loop modules

Consider the Heisenberg subalgebra $G = \sum_{\kappa,n\neq 0} \mathfrak{g}_{n\delta} \oplus \mathbb{C}c \subset \mathfrak{g}$, where $\mathfrak{g}_{n\delta} = 0$ for odd n. Set $G_\pm = \sum_{\kappa,n>0} \mathfrak{g}_{\pm n\delta}$. Let $a \in \mathbb{C}^*$ and $\mathbb{C}v_a$ be the the 1-dimensional $G_\pm \oplus \mathbb{C}c$-module for which $G_\pm v_a = 0$, $cv_a = av_a$. Consider the G-module

$$M^\pm(a) = \mathcal{U}(G) \underset{\mathcal{U}(G_\pm \oplus \mathbb{C}c)}{\otimes} \mathbb{C}v_a.$$

It carries a natural $\mathbb{Z}$-grading with the i-th component $\sigma\left(\mathcal{U}(G_\pm)_{-i}\right) v_a$.

Define another family of modules, so-called loop modules as in [8]. Let $p : \mathcal{U}(G) \to \mathcal{U}(G)/\mathcal{U}(G)c$ be the canonical projection. For $r > 0$, consider the $\mathbb{Z}$-graded ring $L_r = \mathbb{C}[t^{-r}, t^r]$. Denote by P_r the set of graded ring epimorphisms $\Lambda : \mathcal{U}(G)/\mathcal{U}(G)c \to L_r$ with $\Lambda(1) = 1$. Define a G-module structure on L_r by:

$$e_{k\delta} t^{sr} = \Lambda(g(e_{k\delta}))\, t^{sr} = t^{(k+s)r},\ k \in \mathbb{Z}\setminus\{0\},\ ct^{rs} = 0,\ s \in \mathbb{Z}.$$

Denote this G-module by $L_{r,\Lambda}$. Define Λ_0 the trivial homomorphism onto $\mathbb{C}$ with $\Lambda_0(1) = 1$, then L_{0,Λ_0} is the trivial module.

Proposition 3.1. *(i) [8] Every irreducible $\mathbb{Z}$-graded G-module of level zero is isomorphic to $L_{r,\Lambda}$ for some $r \geq 0$, $\Lambda \in P_r$ up to a shifting of gradation,*

(ii) [3] Every irreducible $\mathbb{Z}$-graded G-module of level $a \in \mathbb{C}^*$ with at least one finite-dimensional component is isomorphic to $M^\pm(a)$ up to a shifting of gradation.

If $\alpha \in \Delta^{re,s}$ denote $\mathfrak{n}_\alpha^s = \sum_{n\in\mathbb{Z}} \mathfrak{g}_{\alpha+n\delta}$ and $\mathfrak{n}_\alpha = \mathfrak{n}_\alpha^s \oplus \sum_{i\in\mathbb{Z}} \mathfrak{g}_{2\alpha+(2i+1)\delta}$. If $\alpha \in \Delta^{re,l}$ then there exist $\beta \in \Delta^{re,s}$ and $k \in \mathbb{Z}$ such that $\alpha = 2\beta + k\delta$. Denote $\mathfrak{n}_\alpha = \mathfrak{n}_\beta^s \oplus \sum_{n\in\mathbb{Z}} \mathfrak{g}_{2\beta+(2n+1)\delta}$. The definition of $\mathfrak{n}_\alpha$ depends only on $\alpha \in \Delta_+$ or $\alpha \in \Delta_-$. Write $\mathfrak{n}_+$ or $\mathfrak{n}_-$ in these cases, respectively. In either case $\mathfrak{g} = \mathfrak{n}_{-\alpha} \oplus (\mathfrak{h} + G) \oplus \mathfrak{n}_\alpha$. Set

$$(\mathfrak{h} + G) \oplus \mathfrak{n}_\alpha = \mathfrak{b}$$

Let V be a $\mathbb{Z}$-graded G-module of level $a \in \mathbb{C}$ and $\lambda \in \mathfrak{h}^*$ with $\lambda(c) = a$. Define a $\mathfrak{b}$-module structure on V by the action $hv_i = (\lambda + i\delta)(h)\, v_i$, $\mathfrak{n}_\alpha v_i = 0$ for all $h \in \mathfrak{h}$, $v_i \in V_i$, $i \in \mathbb{Z}$.

Consider the $\mathfrak{g}$-module

$$M_\alpha(\lambda, V) = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V.$$

Proposition 3.2. *(i) $M_\alpha(\lambda, V)$ is $S(\mathfrak{n}_{-\alpha})$-free.*

(ii) $M_\alpha(\lambda, V)$ has a unique irreducible quotient $L_\alpha(\lambda, V)$.

4. The category $\tilde{\mathcal{O}}$ for $A_2^{(2)}$

If $\mathfrak{g}$ is a twisted affine Kac-Moody algebra, π a basis for its root lattice then we define the category $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}(\mathfrak{g})$ of weight $\mathfrak{g}$-modules as follows.

Definition 4.1 ([7])**.** *A $\mathfrak{g}$-module M lies in $\tilde{\mathcal{O}}$ if and only if*

(i) M is a weight module, i.e.

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda, \text{ and}$$

(ii) there exist finitely many elements $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ such that $supp(M) \subset \tilde{D}(\lambda_1) \cup \cdots \cup \tilde{D}(\lambda_k)$, where

$$\tilde{D}(\lambda_i) = \left\{\mu \in \mathfrak{h}^* \mid \lambda_i - \mu \in Q_+ \cup \Delta^{im}\right\},\ Q_+ = \sum_{\alpha \in \pi} \mathbb{Z}_+ \alpha$$

and $supp(M) = \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ as usually.

$\tilde{\mathcal{O}}$ is closed under the operations of taking submodules, quotients and finite direct sums.

Let $\mathfrak{g}$ be again $A_2^{(2)}$ and $\alpha \in \pi$, then $\tilde{D}(\lambda_i) = \{\lambda_i + k\alpha + n\delta \mid k \leq 0, n \in \mathbb{Z}\}$ and $\tilde{D}(\lambda_1) \cup \cdots \cup \tilde{D}(\lambda_k) = \tilde{D}(\lambda_j)$ for j such that $(\lambda_j \mid \alpha)$ is maximal. So $V \in \tilde{\mathcal{O}}$ if and only if there exists an $N \in \mathbb{Z}$ such that $supp(V) \subset \{k\alpha + n\delta \mid k \leq N, n \in \mathbb{Z}\}$. As in [3], Proposition 3.2 leads to the description of the classes of isomorphisms of irreducible modules in $\tilde{\mathcal{O}}$.

Proposition 4.2. *[[3]] Let $\tilde{V}$ be an irreducible object in $\tilde{\mathcal{O}}$. Then there exist $\lambda \in \mathfrak{h}^*$ and an irreducible G-module V such that $\tilde{V} \cong L_\alpha(\lambda, V)$.*

Theorem 4.3 ([7]). *Let $\tilde{V}$ be an irreducible object in $\tilde{\mathcal{O}}$.*

(i) If $\tilde{V}$ is of level zero then $\tilde{V} \cong L_\alpha(\lambda, L_{r,\Lambda})$ for some $\lambda \in \mathfrak{h}^*$, $\lambda(c) = 0$, $\Lambda \in P_r$.

(ii) If $\tilde{V}$ is of level $a \in \mathbb{C}^*$ and dim $\tilde{V}_\mu < \infty$ for at least one $\mu \in$ supp$(\tilde{V})$ then then $\tilde{V} \cong L_\alpha(\lambda, M^\pm(a))$ for some $\lambda \in \mathfrak{h}^*$, $\lambda(c) = a$.

Remark 4.4. By [7] the level zero modules are the only irreducible integrable ones in $\tilde{\mathcal{O}}$.

5. Classification of non-dense $\mathfrak{g}$-modules

In this section we prove the main result. The major part is the content of a Lemma which proves the result assuming the whole in the root lattice at $\lambda + k\delta$, $k \in \mathbb{Z}_+$. The proof is structured in form of a binary tree where in each leaf we construct a vector that generates an irreducible quotient. The result is an analog to the $A_1^{(1)}$-case treated in [3].

Definition 5.1. *An irreducible weight $\mathfrak{g}$-module V is called* dense *if supp$(V) = \lambda + Q$ for some $\lambda \in \mathfrak{h}^*$ and* non-dense *otherwise.*

Now we can state the main theorem.

Theorem 5.2. *If $\tilde{V}$ is an irreducible non-dense $\mathfrak{g}$-module then either $\tilde{V} \cong L_\alpha^+(\lambda, \gamma)$ or $\tilde{V} \cong L_\alpha^-(\lambda, \gamma)$ or $\tilde{V} \cong L_\alpha(\lambda, V)$ for some $\alpha \in \Delta^{re}$, $\lambda \in \mathfrak{h}^*$, $\lambda(c) = a$, $\gamma \in \mathbb{C}$ and some irreducible G-module V.*

The rest of the section is devoted to the proof the Theorem.

Definition 5.3. *A subset $P \subset \Delta$ is called* closed *if $\beta_1, \beta_2 \in P$, $\beta_1 + \beta_2 \in \Delta$ imply $\beta_1 + \beta_2 \in P$. It is called* partition *if in addition $P \cap -P = \varnothing$ and $P \cup -P = \Delta$. Two partitions are called* equivalent *if they lie on the same $W \times \{\pm 1\}$ orbit.*

Denote by $\mathbb{Z}_{\geq s}$ the set $\{s, s+1, \dots\}$ by $\mathbb{Z}_+$ the set of positive integers. From ([5] Chapt. 2) we derive that there exist to non-equivalent partitions of the rootsystem of $\mathfrak{g}$, in particular $P_1 = \Delta_+$ and $P_0 = \left\{\alpha + \mathbb{Z}\delta \mid \alpha \in \Delta_+^0\right\} \cup \mathbb{Z}_+\delta$. They are called real (or classical) and imaginary, respectively.

Lemma 5.4. *Let P be a partition, $P \ni \delta$, $P^{re} = P \cap \Delta^{re}$, $P_\pm = P \cap \Delta_\pm$,$\beta \in \Delta^{re}$.*

If $|P^{re} \cap \{\beta + \mathbb{Z}_{\geq 0}\delta\}| < \infty$ or $|P^{re} \cap \{-\beta + \mathbb{Z}_{\geq 0}\delta\}| < \infty$ then

$$P^{re} = \{\varphi + \mathbb{Z}\delta\} \cup \{2\varphi + (2\mathbb{Z}+1)\delta\}$$

for some $\varphi \in \Delta^{re,s}$ else $P^{re} = \Delta_+(\tilde{\pi})$ for some basis $\tilde{\pi}$ of Δ.

Proof. Recall that there exist exactly two non-equivalent classes of partitions, those equivalent to $\Delta_+^{re}(\pi)$ and to $\left\{\alpha+\mathbb{Z}\delta \mid \alpha\in\Delta_+^0\right\}\cup\Delta_+^{im}$ respectively. Now with [5] Propostion 2.3 (ii) the statement follows. □

Corollary 5.5. *Let $\Gamma\subset\Delta$ be a partition containg δ. If $\left|\Delta_+^{re}\cap\Gamma\right| = \left|\Delta_-^{re}\cap\Gamma\right| = \infty$, then there exists an $n\in\mathbb{Z}$ such that $\Gamma=\Delta_+(\tilde{\pi})$ for $\tilde{\pi}=\{\varphi',\delta-\varphi'\}$, $\varphi'=\varphi+n\delta$, explicitely*

$$\begin{aligned}\Delta_+(\tilde{\pi}) = &\{\varphi+\mathbb{Z}_{\geq n}\delta\}\cup\{-\varphi+\mathbb{Z}_{\geq -n+1}\delta\}\cup\{2\varphi+(2\mathbb{Z}_{\geq n}+1)\delta\}\cup\\ &\cup\{-2\varphi+(2\mathbb{Z}_{\geq -n+1}-1)\delta\}\cup\mathbb{Z}_+\delta.\end{aligned}$$

Proof. Recall the action of the affine Weyl group and apply it to the Lemma. □

Definition 5.6. *Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$. A non-zero element v of a $\mathfrak{g}$-module V is called* $\mathfrak{a}$-primitive *if $\mathfrak{a}v=0$. A non-zero element v of a $\mathfrak{g}$-module V is called* primitive *iff $\mathcal{N}_\varphi^+v=0$, $\mathcal{N}_\varphi^-v=0$ or $\mathfrak{n}_\varphi v=0$ for some $\varphi\in\Delta^{re}$, i.e. iff it is $\mathcal{N}_\varphi^+$-primitive or $\mathcal{N}_\varphi^-$-primitive or $\mathfrak{n}_\varphi$-primitive. Denote $N(v)\subset\Delta$ the set of roots ψ such that $e_\psi v=0$.*

Remark 5.7. (i) Primitive vectors were originally called admissible. For $\varphi\in\Delta^{re}$, a $\mathfrak{n}_\varphi$-primitive element $v\in V$ is also called *singular.*

(ii) If some $v\in V$ is $\mathcal{N}_+$-primitive then it is obviously already $\mathcal{N}_\varphi^+$-primitive.

(iii) On order to classify $\mathfrak{g}$-modules we have to look for primitive elements. Each of those generate irreducible quotient in terms of $\tilde{V}\cong L_\alpha^\pm(\lambda,\gamma)$, or $\tilde{V}\cong L_\alpha(\lambda,V)$ as in Corollary 2.2 and the proof of Proposition 4.2, respectively.

Lemma 5.8. *If the $\mathfrak{g}$-module V contains a non-zero vector $v\in V_\lambda$ such that $e_\varphi v=0$ for some $\varphi\in\Delta^{re}$ and $\lambda+k\delta\notin supp(V)$ for some $k\in\mathbb{Z}\setminus\{0\}$ then V contains a primitive vector.*

Proof. We will assume that $k>0$. The case $k<0$ can be considered analogously. We prove the Lemma by induction on k. Let $k=1$.

1. In the first step assume that $\varphi\in\Delta^{re,s}$, so $e_\varphi v=0$.

As $\lambda+\delta\notin supp(V)$ we have $e_\delta v=0$ and $e_{\varphi+m\delta}v=0$ for all $m\geq 0$ (by induction on m: $e_{\varphi+(m+1)\delta}v=[e_\delta,e_{\varphi+m\delta}]v=0$ by induction assumption). If $e_{\varphi-n\delta}v=0$ for all $n>0$ then $\mathfrak{n}_\varphi^s v=0$. Because of $[e_{\varphi+k\delta},e_{\varphi+m\delta}]=e_{2\varphi+(k+m)\delta}$, also $\mathfrak{n}_\varphi^l v=\sum_{i\in\mathbb{Z}}\mathfrak{g}_{2\varphi+i\delta}v=0$ and v is primitive.

1.1. If $e_{-\varphi+n\delta}v=0$ for all $n<0$ then $\mathfrak{n}_{-\varphi}v=0$.

1.2. Thus we can assume $e_{-\varphi+n\delta}v \neq 0$ for some $n \in \mathbb{Z}$. If $n < 0$ then v is already $\mathcal{N}_+$-primitive. If $n = 0$ we have immediately $\mathcal{N}^+_{-2\varphi+\delta}v = 0$ as in Corollary 5.5.

1.2.1. If $e_{l\delta}v \neq 0$ for some $l \in \mathbb{Z}_+$ then set $v_{l\delta} = e_{l\delta}v$ for the least of such l. By hypothesis $e_{-(l-1)\delta}v_{l\delta} \in V_{\lambda+\delta} = 0$ and also $e_{\varphi-k\delta}v_{l\delta} = \left[e_{-(l-1)\delta}, e_{\varphi-k+(l-1)\delta}\right] v_{l\delta} = 0$ for all $k \leq l-1$ and thus for all $k \in \mathbb{Z}$. We thus derived $\mathfrak{n}_\varphi v = 0$.

1.2.2. Thus we can assume $e_{l\delta}v = 0$ for all $l \in \mathbb{Z}_+$.

1.2.2.1. If possible choose $n > 0$ the greatest number such that $e_{-\varphi+n\delta}v \neq 0$ and set $v_{-\varphi+n\delta} = e_{-\varphi+n\delta}v$. By assumption $e_{\varphi-(n-1)\delta}v_{-\varphi+n\delta} \in V_{\lambda+\delta} = 0$. Therefore $\{\varphi + \mathbb{Z}_{\geq -n+1}\delta\} \cup \{-\varphi + \mathbb{Z}_{\geq n+1}\delta\} \subset N\left(v_{-\varphi+n\delta}\right)$. Thus,
$\{\varphi' + \mathbb{Z}_{\geq 2}\delta\} \cup \{-\varphi + \mathbb{Z}_{\geq 0}\delta\} \subset N\left(v_{-\varphi+n\delta}\right)$ for $\varphi' = \varphi - (n+1)\delta$. If not already zero set $v_{n\delta} = e_{\varphi'-(n+1)\delta}v_{-\varphi'+(2n+1)\delta}$ (otherwise $v_{-\varphi'+(2n+1)\delta}$ is immediately $\mathcal{N}_+$-primitive). Again, if possible set $v_{\varphi'} = e_{\varphi'-n\delta}v_{n\delta} \neq 0$ (otherwise $v_{n\delta}$ is immediately $\mathcal{N}_+$-primitive). But now, $e_{\varphi'+\delta}v_{\varphi'} \in V_{\lambda+\delta} = 0$ by assumption and $v_{\varphi'}$ is $\mathcal{N}^+_{-2\varphi'+\delta}$-primitive for some $\varphi' \in \Delta^{re}$.

1.2.2.2. Thus we can assume that $e_{-\varphi+n\delta}v \neq 0$ for all $n \in \mathbb{Z}_+$. Choose an arbitrary n out of such and set $v_{-\varphi+n\delta} = e_{-\varphi+n\delta}v$. Then $e_{\varphi-(n-1)\delta}v_{-\varphi+n\delta} \in V_{\lambda+\delta} = 0$. Assume $e_{\varphi-l\delta}v_{-\varphi+n\delta} \neq 0$ for some $l \geq n$ and set
$v_{(n-l)\delta} = e_{\varphi-l\delta}v_{-\varphi+n\delta}$ (otherwise $v_{-\varphi+n\delta}$ is $\mathfrak{n}_\varphi$-primitive) and we are in a situation analougously to case 1.2.2.1.

2. In the second step choose $\varphi = 2\alpha + \delta \in \Delta^{re,l}$ i.e. $e_{2\alpha+\delta}v = 0$ by assumption and $e_\delta v \in V_{\lambda+\delta} = 0$.

2.1. If $e_{-2\alpha+\delta}v = 0$ then $[e_{2\alpha+\delta}, e_{-2\alpha+\delta}]\, v = e_{2\delta}v = 0$ and $e_{\pm 2\alpha+m\delta}v = 0$ for all $m \in \mathbb{Z}_+$ thus $e_\psi v = 0$ for all $\psi \in \Delta^{re,l}_+$. We can assume that $e_\alpha v = 0$ (if $\tilde{v} = e_\alpha v \neq 0$, by assumption $e_{-\alpha+\delta}\tilde{v} = 0$, hence $[e_{2\alpha-\delta}e_{-\alpha+\delta}]\,\tilde{v} = e_\alpha\tilde{v} = 0$, contradiction) then $[e_\alpha, e_{-2\alpha+\delta}]\, v = e_{-\alpha+\delta}v = 0$ and $[e_{k\delta}, e_\alpha]\, v = e_{\alpha+k\delta}v = 0$ for all $k \in \mathbb{Z}_{\geq 0}$ thus $\mathcal{N}_+v = 0$ and v is primitive,

2.2. Otherwise, if $e_{-2\alpha+\delta}v \neq 0$ assume again that $e_{\alpha-k\delta}v \neq 0$ for some $k \in \mathbb{Z}_+$ and set $v_{\alpha-k\delta} = e_{\alpha-k\delta}v$. By assumption $e_{\alpha+(k+1)\delta}v_{-\alpha-k\delta} = 0$.

2.2.1. If $e_{-\alpha-k\delta}v_{-\alpha-k\delta} = 0$ then $N\left(v_{-\alpha-k\delta}\right) \cup \{-2\varphi' + \delta, 2\varphi' + \delta\}$ contains the partition $\Delta_+\left(\tilde{\pi}\right)$, $\tilde{\pi} = \{\varphi', \delta - \varphi'\}$, $\varphi' = \alpha + k\delta$. Note that $e_{2\delta}v_{-\varphi'} = \left[e_{\varphi'+\delta}, e_{-\varphi'+\delta}\right] v_{-\varphi'} = 0$. Assume both of the $e_{\pm 2\varphi'+\delta}v_{-\varphi'}$ not to be zero and $e_{-\varphi'-l\delta}v_{-\varphi'} \neq 0$ for some $l \in \mathbb{Z}_+$ (otherwise we are done). Choose l to be minimal in that sense and set $v_{-2\varphi'-l\delta} = e_{-\varphi'-l\delta}v_{-\varphi'} \neq 0$, then $e_{2\varphi'+(l+1)\delta}v_{-\varphi'} \in V_{\lambda+\delta} = 0$ wich gives $\mathcal{N}^+_{-2\varphi'+\delta}v_{-2\varphi'-l\delta} = 0$ with respect to $\Delta_+\left(\pi''\right)$, $\varphi'' = -\varphi' - (l-1)\,\delta$.

2.2.2. Else $v_{-2\varphi'} = e_{-\varphi'}v_{-\varphi'} \neq 0$. By assumption $e_{2\varphi'+\delta}v_{-2\varphi'} = 0$.

Now $N\left(v_{-2\varphi'}\right) \cup \{\varphi', \delta, -2\varphi' + \delta\} \cup \{-\varphi' + \mathbb{Z}_+\delta\}$ contains the partition $\Delta_+(\tilde{\pi})$, $\tilde{\pi} = \{\varphi', \delta - \varphi'\}$, $\varphi' = \alpha + k\delta$. Assuming successively $v_{-\varphi'} = e_{\varphi'}v_{-2\varphi'} \neq 0$ (otherwise there is an l, minimal by choice, as in 2.2.1. etc.), $v_0 = e_{\varphi'}v_{-\varphi'} \neq 0$, $v_{\varphi'} = e_{\varphi'}v_0 \neq 0$ (now $e_{-\varphi'+\delta}v_{\varphi'} = e_\delta v_{\varphi'} = 0$), $v_{-\varphi'+\delta} = e_{-2\varphi'+\delta}v_{\varphi'} \neq 0$ we argued $e_{\varphi'}v_{-\varphi'+\delta} \in V_{\lambda+\delta} = 0$ down to zero and thus proved the basis of induction.

Assume now that the Lemma is proved for all $k' = 1, \dots, k-1$ and consider another tree of cases:

1. If there exists an $n \in \{1, \dots, k-1\}$ such that $e_{i\delta}v = 0$ for all $i = 0, \dots, n-1$ but $e_{n\delta}v \neq 0$. Set $v_{n\delta} = e_{n\delta}v$ and we can apply induction hypothesis.

2. Thus assume $e_{i\delta}v = 0$ for all $i = 1, \dots, k$. Let $\varphi \in \Delta^{re}$ such that $e_\varphi v = 0$. We can also assume that $e_{-\varphi+l\delta}v \neq 0$ for some $l \in \mathbb{Z}_+$ (otherwise $\mathfrak{n}_{-\varphi}v = 0$ and we are done). Choosing the highest of such l, we have thus established $N(v) \supset \{\varphi + \mathbb{Z}_{\geq 0}\delta\} \cup \{\varphi + (2\mathbb{Z}_{\geq 0} + 1)\delta\} \cup \{-\varphi + \mathbb{Z}_{\geq l+1}\delta\} \cup \{-2\varphi + (2\mathbb{Z}_{\geq l+1} + 1)\delta\} \cup \mathbb{Z}_+\delta$. Assume also $\varphi - \delta \notin N(v)$ as otherwise, we reduce immediately to the case $l' = l - 1$.

2.1. If $l = 0$ like in Corollary 5.5 we obtain a partition for which $\mathcal{N}^+_{-2\varphi+\delta}v = 0$.

2.2. For $l > 0$ we may define $v_{-\varphi+l\delta} = e_{-\varphi+l\delta}v \neq 0$. Still $e_{i\delta}v_{-\varphi+l\delta} = e_{-\varphi+(l+i)\delta}v + e_{-\varphi+l\delta}e_{i\delta}v = 0$ for all $i = 1, \dots, k$ and $e_{-\varphi+i\delta}v_{-\varphi+l\delta} = e_{-2\varphi+(l+i)\delta}v + e_{-\varphi+l\delta}e_{-\varphi+i\delta}v = 0$ for $i = l+2$ (because $i + l$ is even in this case) and thus for all $i \geq l+2$.

By assumption $e_{\varphi+(k-l)\delta}v_{-\varphi+l\delta} \in V_{\lambda+k\delta} = 0$. Thus if $l > k$ choose the largest $m < k - l$ such that $e_{\varphi+m\delta}v_{-\varphi+l\delta} \neq 0$ and denote this vector $v_{(m+l)\delta}$. If $0 < m + l < k$ then we are in the case of the induction hypothesis, else $m + l \leq 0$. So we can assume that $m \leq -l$. But this means $e_{\varphi-(l-1)\delta}v_{-\varphi+l\delta} = 0$ by choice of m. Set $\varphi' = \varphi - (l-1)\delta$ and we have $N\left(v_{-\varphi'+\delta}\right) \supset \{\varphi' + \mathbb{Z}_{\geq 0}\delta\} \cup \{\varphi' + (2\mathbb{Z}_{\geq 0} + 1)\delta\} \cup \{-\varphi' + \mathbb{Z}_{\geq 3}\delta\} \cup \{-2\varphi' + (2\mathbb{Z}_{\geq 3} + 1)\delta\} \cup \mathbb{Z}_+\delta$.

2.2.1. Assume $e_{\varphi'-(k-1)\delta}v_{-\varphi'+\delta} \neq 0$ and set $v_{-k\delta} = e_{\varphi'-k\delta}v_{-\varphi'+\delta}$ (otherwise clear). Note that it may only happen that $e_{i\delta}v_{-k'\delta} \neq 0$ for $i \leq 2$, because
$\left[e_{\varphi'}, e_{-\varphi'+i\delta}\right] v_{-\varphi'+\delta} = e_{i\delta}v_{-\varphi'+\delta} = 0$ for all $i \geq 3$.

We proceed with a little iteration:

010 $k' = k$
020 IF $e_{i\delta} v_{-k'\delta} \neq 0$ for some $i \in \{1, 2\}$
THEN set $v_{(i-k')\delta} = e_{i\delta} v_{-k'\delta}$ for the highest of such i
ELSE {PRINT''$v_{-k'\delta}$'' :
STOP}
030 IF $(i - k') \geq 1$ &&(this can actually at most be equal 1 because the previous note)
THEN {PRINT''$v_{(i-k')\delta}$ fulfills the condition of induction hypothesis" :
STOP}
ELSE {set $k' = -(i - k')$: GOTO 020}
040 END

It is easy to see, that the iteration always terminates. Assume the program returns $v_{-k'\delta}$. Note that $k' \in \{0, \ldots, k\}$. Set $j = k - k' \in \{0, \ldots k\}$. In order to annihilate the missing vector, we have to climb up. We do this by means of the following loop:

110 WHILE $-k' \neq -1$
IF $e_{-\varphi'+2\delta} v_{-k'\delta} \neq 0$
THEN set $v_{-\varphi'-(k'-2)\delta} = e_{-\varphi'+2\delta} v_{-k'\delta}$
ELSE {PRINT''$v_{-k'\delta}$'' :
STOP} &&(call this „singular case I")
IF $e_{\varphi'-\delta} v_{-\varphi'-(k'-2)\delta} \neq 0$
THEN set $v_{-(k'-1)\delta} = e_{\varphi'-\delta} v_{-\varphi'-(k'-2)\delta}$: $k' = k' - 1$
ELSE {PRINT''$v_{-\varphi'-(k'-2)\delta}$'' :
STOP} && (call this „singular case II")
WHILEEND
120 PRINT''v_δ fulfills the condition of induction hypothesis"
130 END

In both of the singular cases we end up in the following situation $N(w_{k'}) \supset \{\psi + \mathbb{Z}_{\geq 0}\delta\} \cup \{\psi + (2\mathbb{Z}_{\geq 0} + 1)\delta\} \cup \{-\psi + \mathbb{Z}_{\geq 2}\delta\} \cup \{-2\psi + (2\mathbb{Z}_{\geq 2} + 1)\delta\} \cup \mathbb{Z}_+\delta$ for some $\psi \in \Delta^{re}$ and one of the vectors $v_{-k'\delta}$ and $v_{-\varphi'-(k'-2)\delta}$. Note that $-k' \leq 0$. We proceed with another loop for $v_{-k'\delta}$ (singular case I). Singular case II $\left(v_{-\varphi'-(k'-2)\delta}\right)$ goes analogously.

```
210 WHILE −k' ≠ 1 or 2
        IF e_{−φ'+δ} v_{−k'δ} ≠ 0
          THEN set v_{−φ'−(k'−1)δ} = e_{−φ'+δ} v_{−k'δ}
          ELSE {PRINT"v_{−k'δ}" :
            STOP}          &&(call this „singular case A")
        IF e_{−φ'+δ} v_{−φ'−(k'−1)δ} ≠ 0
          THEN set v_{−2φ'−(k'−2)δ} = e_{−φ'+δ} v_{−φ'−(k'−1)δ}
          ELSE {PRINT"v_{−(k'−1)δ}" :
            STOP}          &&(call this „singular case B")
        IF e_{2φ'−δ} v_{−2φ'−(k'−2)δ} ≠ 0
          THEN set v_{−(k'−1)δ} = e_{2φ'−δ} v_{−2φ'−(k'−2)δ} and k' = k' − 1
          ELSE {PRINT"v_{−2φ'−(k'−2)δ}" :
            STOP}          &&(call this „singular case C")
    WHILEEND
220 PRINT "v_{−k'δ}"
230 END
```

As in the previous loop, the program returns always a vector, say w.

In the singular case A and B we have $-\varphi'+\delta \in N(w)$, thus $\mathcal{N}^+_{-2\varphi'+\delta} w = 0$.

In the singular case C we have $2\varphi' - \delta \in N(w)$, thus $\mathcal{N}^+_{-2\varphi''+\delta} w = 0$ with respect to $\Delta_+ (\{\varphi'', \delta - \varphi''\})$ for $\varphi'' = -\varphi' + \delta$ and thus a primitive vector, which proves the Lemma. □

Proposition 5.9. *Let V be an irreducible non-dense $\mathfrak{g}$-module. Then V contains a primitive element.*

Proof. Let $\lambda \in supp(V)$ and $\lambda + \varphi \notin supp(V)$ for some $\varphi \in \Delta$. Choose a non-zero vector $v \in V_\lambda$. Consider another tree of cases in order to construct a primitive element or provide the assumption of the Lemma above.

1. Assume $\varphi \in \Delta^{im}$, i.e. $\varphi = k\delta$, $k \in \mathbb{Z}\setminus\{0\}$.

1.1. If $e_\alpha v = 0$ for some $\alpha \in \Delta^{re,s}$ then the statement follows from the Lemma above,

1.2. else $e_\alpha v \neq 0$.

1.2.1. If $e_{-\alpha} v = 0$ then the statement follows from the Lemma.

1.2.2. else $v' = e_{-\alpha} v \neq 0$. As $\lambda + k\delta \notin supp(V)$ we have $\lambda' + \alpha + k\delta \notin supp(V)$ for $\lambda' = \lambda - \alpha$. Thus $e_{\alpha+k\delta} v' = 0$. Also $e_{\alpha+n\delta} v' = 0$ for all $n = k, 2k, 3k, \ldots$.

1.2.2.1. If $e_{\alpha+l\delta} v' = 0$ for all $l' \in \mathbb{Z}$ then v' is $\mathfrak{n}_\alpha$-primitive,

1.2.2.2. else we may define $v'' = e_{\alpha+l'\delta} v' \neq 0$ for some $l' \in \mathbb{Z}$, $l' \neq k, 2k, 3k, \ldots$. Then $\lambda'' + (k - l')\,\delta \notin supp(V)$ for $\lambda'' = \lambda' + \alpha + l'\delta$

but still $e_{-\alpha+n\delta}v'' = 0$ for any $n = k, 2k, 3k, \ldots$. and $-\alpha + k\delta \in \Delta^{re}$ what brings us in the situation of the Lemma.

2. Assume $\varphi \in \Delta^{re}$. Then we have $e_\varphi v \in V_{\lambda+\varphi} = 0$ by assumption.

2.1. If there exists $v' = e_{\varphi-n\delta}v \neq 0$ for some $n \in \mathbb{Z}\setminus\{0\}$ then $v' \in V_{\lambda'}$ for $\lambda' = \lambda + \varphi - n\delta$ and $V_{\lambda'+n\delta} = 0$. But these are the assumptions of case 1 in this proof.

2.2. If $e_{\varphi-n\delta}v = 0$ for all $n \in \mathbb{Z}$ then v is $\mathfrak{n}_\varphi$-primitive. □

Now Theorem 5.2 follows from the Proposition, Corollary 2.2 and Proposition 4.2.

6. Classification of supports

Now we are able to classify all possible supports of irreducible $\mathfrak{g}$-modules. Denote $\mathbb{Z}_+\pi = \left\{\sum_{x_i\in\pi} a_i x_i \neq 0 \mid a_i \in \mathbb{Z}_{\geq 0}\right\}$ for a set π.

Theorem 6.1. *Let $\pi = \{\varphi, \delta - \varphi\}$ be a basis of the root lattice. The support of an irreducible $\mathfrak{g}$-module is of one (and only one) of the following equivalence classes (w.r.t. the affine Weyl group) for some $\lambda \in \mathfrak{h}^*$,*

(i) $S_{dense} = \lambda + Q$,

(ii) $S_{Verma} \subset \lambda \pm \mathbb{Z}_+\pi$, *for a highest or lowest weight module*

(iii) $S^{\pm}_{real} = \lambda \pm \mathbb{Z}_+\pi$ *(2 classes),*

(iv) $S^{\pm}_{real,\varphi} = \lambda \pm \mathbb{Z}_+\pi + \mathbb{Z}\varphi$ *(2 classes),*

(v) $S^{(\pm,\pm)}_{real,\alpha} = \lambda \pm \mathbb{Z}_+\pi + \mathbb{Z}\alpha$ *where* $\alpha = 2\varphi \pm \delta$ *(4 classes),*

(vi) $S^{(\pm,\pm)}_{im} = \lambda + \mathbb{Z}_\pm\delta \cup \{\mathbb{Z}_\pm\varphi + \mathbb{Z}\delta\}$ *for* $\lambda(c) \neq 0$ *(4 classes),*

(vii) $S_{\lambda(c)=0} = \{\lambda \pm \mathbb{Z}_+\varphi + \mathbb{Z}\delta\} \cup \{\lambda\}$,
for $\lambda(c) = 0$ *and* $L_{r,\Lambda} = L_{0,\Lambda_0}$,

(viii) $S_{trivial} = \lambda$, *if* $\lambda(c) = \lambda(h) = 0$.

Proof. Follows immediately from Proposition 5.9. □

References

[1] Kac, V., *Infinite dimensional Lie algebras, 3rd Ed.*, Cambridge, 1990

[2] Carter, R., *Lie algebras of finite and affine type*, Cambridge, 2005

[3] Futorny, V., *Irreducible non-dense $A_1^{(1)}$-modules*, Pacific J. of Math. Vol. 172, No. 1, 83-97, 1996

[4] Futorny, V., Tsylke, .A., *Irreducible non-zero level modules with finite- dimensional weight spaces for Affine Lie algebras*, J. of Algebra 238, 426-441, 2001

[5] Futorny, V., *Representations of Affine Lie algebras*, Queen's Papers in Pure and Applied Mathematics 106, v. 1, Kingston, 1997.

[6] Futorny, V., Imaginary Verma modules for affine Lie algebras, Canad. Math. Bull., 37, 213-218, 1994

[7] Chari, V., Pressley, A., *Integrable Representations of Twisted Affine Lie Algebras*, J. of Algebra 113, 438-46, 1986

[8] Chari, V., *Integrable Representations of Affine Lie Algebras*, Invent. math. 85, 317-335, 1986

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