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Biserial minor degenerations of matrix algebras over a field

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Abstract. Let $n \geq 2$ be a positive integer, K an arbitrary field, and $q = [q^{(1)}| \dots |q^{(n)}]$ an n-block matrix of $n \times n$ square matrices $q^{(1)}, \dots, q^{(n)}$ with coefficients in K satisfying the conditions (C1) and (C2) listed in the introduction. We study minor degenerations $\mathbb{M}_n^q(K)$ of the full matrix algebra $\mathbb{M}_n(K)$ in the sense of Fujita-Sakai-Simson [7]. A characterisation of all block matrices $q = [q^{(1)}| \dots |q^{(n)}]$ such that the algebra $\mathbb{M}_n^q(K)$ is basic and right biserial is given in the paper. We also prove that a basic algebra $\mathbb{M}_n^q(K)$ is right biserial if and only if $\mathbb{M}_n^q(K)$ is right special biserial. It is also shown that the K-dimensions of the left socle of $\mathbb{M}_n^q(K)$ and of the right socle of $\mathbb{M}_n^q(K)$ coincide, in case $\mathbb{M}_n^q(K)$ is basic and biserial.

Introduction

Throughout this paper, $n \geq 2$ is an integer and K an arbitrary field. We denote by $\mathbb{M}_n(K)$ the K-algebra of all square $n \times n$ matrices with coefficients in K. Following [7], by a **minor constant structure matrix** of size $n \times n^2$ with coefficients in K we mean any n-block matrix $q = [q^{(1)}|q^{(2)}| \dots |q^{(n)}]$, where $q^{(1)} = [q_{ij}^{(1)}], \dots, q^{(n)} = [q_{ij}^{(n)}] \in \mathbb{M}_n(K)$ are $n \times n$ matrices satisfying the following two conditions:

(C1) $q_{rj}^{(r)} = 1$ and $q_{jr}^{(r)} = 1$, for all $j, r \in \{1, \dots, n\}$.

(C2) $q_{ij}^{(r)} q_{is}^{(j)} = q_{is}^{(r)} q_{rs}^{(j)}$, for all $i, j, r, s \in \{1, \dots, n\}$.

We call q **basic** if, in addition, the following condition is satisfied:

(C3) $q_{jj}^{(r)} = 0$, for $r = 1, \dots, n$ and all $j \in \{1, \dots, n\}$ such that $j \neq r$.

The set of all minor constant structure matrices q of size $n \times n^2$, with coefficients in K is denoted by $\mathbb{ST}_n(K) \subseteq \mathbb{M}_{n \times n^2}(K)$. A matrix q in $\mathbb{ST}_n(K)$ is

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called $(0,1)$**-matrix**, if each entry $q_{ij}^{(r)}$ is either 0 or 1. Throughout this paper, any matrix q in $\mathbb{ST}_n(K)$ will be simply called a **structure matrix**.

Given $q \in \mathbb{ST}_n(K)$, a minor q-degeneration $\mathbb{M}_n^q(K)$ of the full matrix K-algebra $\mathbb{M}_n(K)$ is defined in [7] to be the K-vector space $\mathbb{M}_n(K)$ equipped with the multiplication

$$\cdot_q : \mathbb{M}_n(K) \otimes_K \mathbb{M}_n(K) \longrightarrow \mathbb{M}_n(K) \tag{1}$$

given by the formula $\lambda' \cdot_q \lambda'' = [\lambda_{ij}]$, where $\lambda_{ij} = \sum\limits_{s=1}^{n} \lambda'_{is} q_{ij}^{(s)} \lambda''_{sj}$, for $i,j \in \{1,\dots,n\}$ and $\lambda' = [\lambda'_{ij}], \lambda'' = [\lambda''_{ij}] \in \mathbb{M}_n(K)$. It is easy to see that $\cdot_q$ defines a K-algebra structure on $\mathbb{M}_n(K)$ and the unity matrix E is the identity element of the algebra $\mathbb{M}_n^q(K)$. If $n \geq 2$ and q is basic then the global homological dimension of the algebra $\mathbb{M}_n^q(K)$ is infinite.

We recall that a class of algebras of type $\mathbb{M}_n^q(K)$ were studied by Fujita in [5] (called full matrix algebras with structure systems) as a framework for a study of factor algebras of tiled R-orders Λ, in relation with the results of the papers [4], [11], [14] (see also [6] and [8]), where R is a discrete valuation domain. The results in [7] show that one can treat the algebras $\mathbb{M}_n^q(K)$ by an elementary algebraic geometry technique and study them in a deformation theory context. Note also that the authors in [7] follow an old idea of the skew matrix ring construction by Kupisch in [12], see also Oshiro and Rim [13].

The minor degenerations $\mathbb{M}_n^q(K)$ of the algebra $\mathbb{M}_n(K)$ and their modules are investigated in [7] by means of the properties of the coefficients of the matrix q and by applying quivers with relations. In particular, the Gabriel quiver of $\mathbb{M}_n^q(K)$ is described and conditions for q to be $\mathbb{M}_n^q(K)$ a Frobenius algebra are given.

In the present paper we give necessary and sufficient conditions for coefficients of $q \in \mathbb{ST}_n(K)$ to be $\mathbb{M}_n^q(K)$ a right biserial algebra or a right special biserial algebra, see [9], [18] and Sections 2 and 3 for definitions. One of the main results of the paper is the following theorem.

Theorem 1. *Assume that K is a field, $n \geq 2$, $q = [q^{(1)}|\dots|q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix and, given $j, l \in \{1,\dots,n\}$, we set*

$$\mathcal{M}_{(j,l)} = \{p \in \{1,\dots,n\}; q_{jp}^{(l)} \neq 0\} \text{ and } m_{(j,l)} = |\mathcal{M}_{(j,l)}|. \tag{2}$$

The following four conditions are equivalent.

(a) *The algebra $\mathbb{M}_n^q(K)$ is right biserial* (*see Section* 2).

(b) *The algebra $\mathbb{M}_n^q(K)$ is right special biserial* (*see Section* 3).

(c) *For each $i \in \{1,\dots,n\}$, at least one of the following two conditions is satisfied.*

(c_1) *There exists a permutation $\tau_i : \{1,\dots,n\} \to \{1,\dots,n\}$ such that the equality $q_{ip}^{(\tau_i(l))} = 0$ implies the equality $q_{ip}^{(\tau_i(j))} = 0$, for $l < j$ and each $p \in \{1,\dots,n\}$.*

(c_2) *There are two indices $s_i < r_i$ such that the sets $\mathcal{M}_{(i,s_i)}$ and $\mathcal{M}_{(i,r_i)}$ have the following properties:*

(c_{21}) $|\mathcal{M}_{(i,s_i)} \cup \mathcal{M}_{(i,r_i)}| = n-1$,

(c_{22}) *the set $\mathcal{M}_{(i,s_i)} \cap \mathcal{M}_{(i,r_i)}$ is empty or has precisely one element,*

(c$_{23}$) *there exist two bijections*
$\tau_{(i,s_i)} : \{1,\ldots,m_{(i,s_i)}\} \to \mathcal{M}_{(i,s_i)}$ *and* $\tau_{(i,r_i)} : \{1,\ldots,m_{(i,r_i)}\} \to \mathcal{M}_{(i,r_i)}$ *such that, given* $\tau \in \{\tau_{(i,s_i)}, \tau_{(i,r_i)}\}$, *the equality* $q_{ip}^{(\tau(l))} = 0$ *implies the equality* $q_{ip}^{(\tau(j))} = 0$, *for* $l < j$ *and all* $p \in \{1,\ldots,n\}$.

(d) *For any* $i \in \{1,\ldots,n\}$, *each of the following two conditions is satisfied.*

(d$_1$) *There is one or two indices* $r_i \in \{1,\ldots,n\}$ *such that* $r_i \neq i$ *and* $q_{ir_i}^{(t)} = 0$, *for all* $t \notin \{i, r_i\}$.

(d$_2$) *For any* $s \neq i$ *such that* $q_{is}^{(t')} = 0$, *for all* $t' \notin \{i,s\}$, *there is at most one index* $l_{(i,s)} \in \{1,\ldots,n\}$ *such that* $l_{(i,s)} \neq s$, $q_{il_{(i,s)}}^{(s)} \neq 0$ *and* $q_{sl_{(i,s)}}^{(p')} = 0$, *for all* $p' \notin \{s, l_{(i,s)}\}$.

The equivalence of (a) and (c) is proved in Section 2, and the equivalence of the statements (a), (b), and (d) is proved in Section 3, where we also collect basic facts on the algebras $\mathbb{M}_n^q(K)$ that are special biserial. In Corollary 3 we show that $\dim_K \mathrm{soc}(A_{q\,A_q}) = \dim_K \mathrm{soc}({}_{A_q}A_q)$, for any biserial and basic algebra $A_q = \mathbb{M}_n^q(K)$. Moreover, we give an example of a non-biserial algebra A_q such that $\dim_K \mathrm{soc}(A_{q\,A_q}) \neq \dim_K \mathrm{soc}({}_{A_q}A_q)$.

Throughout this paper we use the standard terminology and notation introduced in [1], [2], [3], [15], [17]. Given a ring R with an identity element, we denote by $J(R)$ the Jacobson radical of R, and by $\mathrm{mod}(R)$ the category of finitely generated right R-modules, and by $\mathrm{pr}(R)$ the full subcategory of $\mathrm{mod}(R)$ of right projective R-modules. For any homomorphism $h : M \longrightarrow N$ in $\mathrm{mod}(R)$, we denote by $\mathrm{Im}\, h$ the image of h. Given $n \geq 1$, we denote by e_{ij} the matrix unit in $\mathbb{M}_n(K)$ with 1 on the (i,j) entry, end zeros elsewhere. We fix $n \geq 2$ and we set

$$A_q = \mathbb{M}_n^q(K) = e_1 A_q \oplus \ldots \oplus e_n A_q, \tag{3}$$

for $q \in \mathbb{ST}_n(K)$. Obviously, $e_1 = e_{11}, \ldots, e_n = e_{nn}$ is a complete set of pairwise orthogonal primitive idempotents of A_q. We recall that A_q is said to be basic, if $e_i A_q \ncong e_j A_q$, for $i \neq j$. The paper contains part of author's doctoral dissertation written in Department of Algebra and Geometry of Nicolaus Copernicus University.

1. Preliminaries

Throughout, we use the notation $\mathcal{M}_{(j,l)}$ and $m_{(j,l)}$ as defined in (2) and, given $\lambda, \lambda' \in \mathbb{M}_n^q(K)$, we often write simply $\lambda\lambda'$ instead of $\lambda \cdot_q \lambda'$.

Note that, in view of the definition (1) of $\cdot_q$, we have

$$e_{rs} \cdot_q e_{jl} = \begin{cases} q_{rl}^{(s)} e_{rl}, & \text{for } \ s = j, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

Recall that a right module M over a ring R is called **serial** (or **uniserial**), if M has a unique composition series, see [1].

The following lemma collects elementary properties of the algebra $A_q = \mathbb{M}_n^q(K)$ which we frequently use in the paper.

Lemma 1. *Assume that $n \geq 2$, $q = [q^{(1)}|\dots|q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix, $A_q = \mathbb{M}_n^q(K)$ and $i, r, s \in \{1, \dots, n\}$.*

(a) $e_{rs}A_q = \sum\limits_{l \in \mathcal{M}_{(r,s)}} e_{rl}K$.

(b) $e_{ir}A_q \subseteq e_{is}A_q$ *if and only if* $q_{ir}^{(s)} \neq 0$. *Moreover,* $e_{ir}A_q \neq e_{is}A_q$, *for* $r \neq s$.

(c) *If L is a right submodule of e_iA_q, then $L = e_{ii_1}A_q + \ldots + e_{ii_s}A_q$, for some $i_1, \dots, i_s \in \{1, \dots, n\}$. If, in addition, L is serial, then $L = e_{it}A_q$, for some $t \in \{1, \dots, n\}$.*

(d) *A right ideal S of A_q is simple if and only if S has the form $S = e_{rs}K$, where e_{rs} is a matrix unit such that $r \neq s$ and $q_{rl}^{(s)} = 0$, for all $l \neq s$.*

(e) *The Jacobson radical $J(A_q)$ of A_q consists of all matrices $\lambda = [\lambda_{ij}] \in \mathbb{M}_n(K)$ such that $\lambda_{11} = \ldots = \lambda_{nn} = 0$.*

(f) *Assume that $q \in \mathbb{ST}_n(K)$ is an arbitrary structure matrix. The algebra A_q is basic if and only if the matrix q satisfies the condition* (S3).

Proof. (a) If $\lambda = \sum\limits_{j,l} \lambda_{jl}e_{jl} \in A_q$, where $\lambda_{jl} \in K$, then (4) implies

$$e_{rs} \cdot_q \lambda = \sum_{j,l} \lambda_{jl} e_{rs} \cdot_q e_{jl} = \sum_{l=1}^{n} \lambda_{sl} q_{rl}^{(s)} e_{rl} = \sum_{l \in \mathcal{M}_{(r,s)}} \lambda_{sl} q_{rl}^{(s)} e_{rl}$$

and we get $e_{rs}A_q \subseteq \sum\limits_{l \in \mathcal{M}_{(r,s)}} e_{rl}K$. The inverse inclusion holds, because the equality (4) yields $e_{rl} = \frac{1}{q_{rl}^{(s)}} e_{rs} \cdot_q e_{sl}$, for all $l \in \mathcal{M}_{(r,s)}$.

(b) By (a) and (4), we get $q_{ir}^{(s)} \neq 0$ if and only if $e_{ir}A_q \subseteq e_{is}A_q$. This proves the first part of (b). To prove the second part assume, to the contrary, that $r \neq s$ and $e_{ir}A_q = e_{is}A_q$. Then, in view of (a), we have $\mathcal{M}_{(i,r)} = \mathcal{M}_{(i,s)}$, and consequently, we get the contradiction $0 \neq q_{ir}^{(s)} q_{is}^{(r)} = q_{is}^{(s)} q_{ss}^{(r)} = 0$, because of (C2) and (C3). This finishes the proof of (b).

(c) Assume that L is a right submodule of e_iA_q. If $\lambda \in L$, then $\lambda = \sum\limits_{p=1}^{m} (\sum\limits_{j=1}^{n} \mu_{ij}^{(p)} e_{ij}) \cdot_q \lambda_p$, for some $m \geq 1$, $\mu_{ij}^{(p)} \in K$ and $\lambda_p = [\lambda_{r'l'}^{(p)}] \in A_q$. Then, according to (4), we get

$$\lambda \cdot_q e_l = \sum_{p=1}^{m} (\sum_{j=1}^{n} \mu_{ij}^{(p)} e_{ij}) \cdot_q \lambda_p \cdot_q e_l = \sum_{p=1}^{m} \sum_{j=1}^{n} \mu_{ij}^{(p)} \lambda_{jl}^{(p)} q_{il}^{(j)} e_{il},$$

for any $l \in \{1, \dots, n\}$. Hence, given l such that $\lambda \cdot_q e_l \neq 0$, the element

$$e_{il} = (\sum_{p=1}^{m} \sum_{j=1}^{n} \mu_{ij}^{(p)} \lambda_{jl}^{(p)} q_{il}^{(j)})^{-1} \lambda \cdot_q e_l$$

belongs to L, and consequently $L = e_{ii_1}A_q + \ldots + e_{ii_s}A_q$, for some $i_1, \dots, i_s \in \{1, \dots, n\}$. Hence, if L is serial, then the right modules $e_{ii_1}A_q, \dots, e_{ii_s}A_q$ form a chain and there is an index $t \in \{i_1, \dots, i_s\}$ such that $L = e_{it}A_q$.

For the proof of (d), (e), and (f) we refer to [7]. $\square$

Recall from [1] that to any basic and connected finite dimensional K-algebra A, with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$, we associate the **Gabriel quiver** $Q_A = (Q_0^A, Q_1^A)$ as follows, see [10]. The set $Q_0^A = \{1, \ldots, n\}$ is the set of points of Q_A, which elements are in bijective correspondence with the idempotents $e_1, e_2, \ldots, e_n$. Given two points $i, j \in Q_0^A$, the arrows $\beta : i \to j$ in Q_1^A are in bijective correspondence with the vectors in a fixed basis of the K-vector space $e_i[J(A)/J(A)^2]e_j$. The following simple observation was made in [7, Corolary 2.20].

Lemma 2. *Assume that $n \geq 2$, $q = [q^{(1)}|\ldots|q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix and let $A_q = \mathbb{M}_n^q(K)$.*

(a) $Q_0^{A_q} = \{1, \ldots, n\}$.

(b) *Given $i, j \in Q_0^{A_q}$, there exists an arrow $i \to j$ in $Q_1^{A_q}$ if and only if $i \neq j$ and $q_{ij}^{(r)} = 0$, for all $r \notin \{i, j\}$. In this case, there is a unique arrow $\beta_{ij} : i \to j$ that corresponds to the coset $\overline{q}e_{ij} \in e_i[J(A_q)/J(A_q)^2]e_j$ of the matrix unit e_{ij}.*

(c) *The quiver Q_{A_q} is connected and has no loops.*

2. When $A_q = \mathbb{M}_n^q(K)$ is a biserial algebra?

One of the aims of this section is to give a characterisation of the right biserial algebras $\mathbb{M}_n^q(K)$ in terms of the coefficients of the structure matrix q.

Now, we describe serial submodules of the projective A_q-modules e_iA_q in terms of the coefficients of q.

Lemma 3. *Assume that K is a field, $n \geq 2$, $q = [q^{(1)}|\ldots|q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix, given $i, r \in \{1, \ldots, n\}$. Let $\mathcal{M}_{(i,r)}$ be the set* (2).

(a) *A right A_q-module $e_{ir}A_q$ is serial if and only if there exists a bijection $\tau : \{1, \ldots, m_{(i,r)}\} \to \mathcal{M}_{(i,r)}$ such that the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \ldots, n\}$.*

(b) *A right A_q-module e_iA_q is serial if and only if there exists a permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \ldots, n\}$.*

Proof. (a) Fix $i, r \in \{1, \ldots, n\}$. Note that, by Lemma 1(a),(b), the module $e_{ir}A_q$ is serial if and only if the submodules $e_{it}A_q$ of $e_{ir}A_q$, with $t \in \mathcal{M}_{(i,r)}$, form a chain, or equivalently (by Lemma 1(a)) if and only if there exists a bijection $\tau : \{1, \ldots, m_{(i,r)}\} \to \mathcal{M}_{(i,r)}$ such that the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \ldots, n\}$. Consequently, (a) follows.

(b) By applying (a) to $e_i = e_{ii}$, we get $e_iA_q = e_{ii}A_q$, $m_{(i,i)} = n$ and $\mathcal{M}_{(i,i)} = \{1, \ldots, n\}$. Thus, by the arguments given above, e_iA_q is serial if and only if there exists a permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \ldots, n\}$. □

In the following two lemmata we study the structure of the Jacobson radical $J(e_iA_q)$ of e_iA_q in terms of the coefficients of q.

Lemma 4. *Assume that K is a field, $n \geq 2$, $q = [q^{(1)}| \dots |q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix and $i \in \{1, \dots, n\}$. Then the Jacobson radical $J(e_iA_q)$ of e_iA_q is a direct sum of two serial proper submodules if and only if there are two indices $s < r$ such that the sets $\mathcal{M}_{(i,s)}$, $\mathcal{M}_{(i,r)}$ (2) have the following properties:*

- $|\mathcal{M}_{(i,s)} \cup \mathcal{M}_{(i,r)}| = n - 1$,
- *the set* $\mathcal{M}_{(i,s)} \cap \mathcal{M}_{(i,r)}$ *is empty,*
- *there exist two bijections*

$$\tau_{(i,s)} : \{1, \dots, m_{(i,s)}\} \to \mathcal{M}_{(i,s)} \ \text{and}\ \ \tau_{(i,r)} : \{1, \dots, m_{(i,r)}\} \to \mathcal{M}_{(i,r)}$$

such that, given $\tau \in \{\tau_{(i,s)}, \tau_{(i,r)}\}$, the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \dots, n\}$.

Proof. Fix $i \in \{1, \dots, n\}$. By Lemma 1(c), $J(e_iA_q)$ is a direct sum of two serial proper submodules if and only if there are two indices $s < r$ such that $J(e_iA_q) = e_{is}A_q \oplus e_{ir}A_q$ and $e_{is}A_q, e_{ir}A_q$ are serial. According to Lemma 1(a), $e_{is}A_q \cap e_{ir}A_q = 0$ if and only if the set $\mathcal{M}_{(i,s)} \cap \mathcal{M}_{(i,r)}$ is empty. Moreover, by [1, Proposition 4.5(c)] and Lemma 1(a),(e), we have $J(e_iA_q) = e_{is}A_q + e_{ir}A_q$ if and only if $|\mathcal{M}_{(i,s)} \cup \mathcal{M}_{(i,r)}| = n - 1$. By Lemma 3(a), the right modules $e_{is}A_q, e_{ir}A_q$ are serial if and only if there exist two bijections $\tau_{(i,s)} : \{1, \dots, m_{(i,s)}\} \to \mathcal{M}_{(i,s)}$ and $\tau_{(i,r)} : \{1, \dots, m_{(i,r)}\} \to \mathcal{M}_{(i,r)}$ such that, given $\tau \in \{\tau_{(i,s)}, \tau_{(i,r)}\}$ the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \dots, n\}$. Hence, the required equivalence follows. □

Lemma 5. *Assume that K is a field, $n \geq 2$, $q = [q^{(1)}| \dots |q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix, given $i \in \{1, \dots, n\}$. The Jacobson radical $J(e_iA_q)$ of e_iA_q is a sum of two serial submodules L' and L'' such that $L' \cap L''$ is a simple module if and only if there are two indices $s < r$ such that the sets $\mathcal{M}_{(i,s)}, \mathcal{M}_{(i,r)}$ (2) have the following properties:*

- $|\mathcal{M}_{(i,s)} \cup \mathcal{M}_{(i,r)}| = n - 1$,
- *the set* $\mathcal{M}_{(i,s)} \cap \mathcal{M}_{(i,r)}$ *has precisely one element,*
- *there exist two bijections*

$$\tau_{(i,s)} : \{1, \dots, m_{(i,s)}\} \to \mathcal{M}_{(i,s)} \ \text{and}\ \ \tau_{(i,r)} : \{1, \dots, m_{(i,r)}\} \to \mathcal{M}_{(i,r)}$$

such that, given $\tau \in \{\tau_{(i,s)}, \tau_{(i,r)}\}$ the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \dots, n\}$.

Proof. Fix $i \in \{1, \dots, n\}$. By Lemma 1(c), there exist serial submodules L' and L'' of $J(e_iA_q)$ such that $J(e_iA_q) = L' + L''$ and the module $L' \cap L''$ is simple if and only if there exit two indices $s < r$ such that $J(e_iA_q) = e_{is}A_q + e_{ir}A_q$, the module $e_{is}A_q \cap e_{ir}A_q$ is simple and $e_{is}A_q, e_{ir}A_q$ are serial. According to Lemma 1(a) and (d), the module $e_{is}A_q \cap e_{ir}A_q$ is simple if and only the set $\mathcal{M}_{(i,s)} \cap \mathcal{M}_{(i,r)}$ has precisely one element. Hence the equivalence follows as in the proof of Lemma 4. □

We recall from [9] that a finite dimensional K-algebra A is **right** (resp. **left**) **biserial** if every indecomposable projective right (resp. left) A-module P is serial, or the Jacobson radical $J(P)$ of P is a sum of two serial submodules P_1 and P_2 such that the module $P_1 \cap P_2$ is zero or simple. An algebra A is said to be **biserial**, if it is both left and right biserial.

The following corollary proves the equivalence of (a) and (c) in Theorem 1.

Corollary 1. *Assume that K is a field, $n \geq 2$ and $q = [q^{(1)}| \dots |q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix. Then the following conditions are equivalent.*

(a) *The algebra $\mathbb{M}_n^q(K)$ is right biserial.*

(b) *For each $i \in \{1, \dots, n\}$, at least one of the following two conditions is satisfied:*

(b_1) *there exists a permutation $\tau_i : \{1, \dots, n\} \to \{1, \dots, n\}$ such that the equality $q_{ip}^{(\tau_i(l))} = 0$ implies the equality $q_{ip}^{(\tau_i(j))} = 0$, for $l < j$ and each $p \in \{1, \dots, n\}$,*

(b_2) *there are two indices $s_i < r_i$ such that the sets $\mathcal{M}_{(i,s_i)}$, $\mathcal{M}_{(i,r_i)}$ have the following properties:*

(b_{21}) $|\mathcal{M}_{(i,s_i)} \cup \mathcal{M}_{(i,r_i)}| = n - 1$,

(b_{22}) *the set $\mathcal{M}_{(i,s_i)} \cap \mathcal{M}_{(i,r_i)}$ is empty or has precisely one element,*

(b_{23}) *there exist two bijections*
$\tau_{(i,s_i)} : \{1, \dots, m_{(i,s_i)}\} \to \mathcal{M}_{(i,s_i)}$ and $\tau_{(i,r_i)} : \{1, \dots, m_{(i,r_i)}\} \to \mathcal{M}_{(i,r_i)}$ such that, given $\tau \in \{\tau_{(i,s_i)}, \tau_{(i,r_i)}\}$, the equality $q_{ip}^{(\tau(l))} = 0$ implies the equality $q_{ip}^{(\tau(j))} = 0$, for $l < j$ and each $p \in \{1, \dots, n\}$.

Proof. Apply [1, Corollary 5.17] and Lemmata 3, 4, and 5. □

As an immediate consequence of Corollary 1 we get the following corollary.

Corollary 2. *Assume that $q \in \mathbb{ST}_n(K)$ and $\overline{q} \in \mathbb{ST}_n(K)$ is its $(0,1)$-limit in the sense of [7]. The algebra A_q is right biserial if and only if the algebra $A_{\overline{q}}$ is right biserial.*

3. Special biserial algebras $\mathbb{M}_n^q(K)$

In this section we study basic special biserial minor degenerations $\mathbb{M}_n^q(K)$ of $\mathbb{M}_n(K)$ and we prove that the algebra $\mathbb{M}_n^q(K)$ is right special biserial if and only if the algebra $\mathbb{M}_n^q(K)$ is right biserial.

We recall from [18] (see also [16]) that a K-algebra of the form KQ/Ω, where Q is an quiver and Ω is an admissible ideal of the path K-algebra KQ of Q is called a **right special biserial**, if the following two conditions are satisfied:

(a) any vertex of Q is a starting point of at most two arrows, and

(b) given an arrow $\beta : i \to j$ in Q, there is at most one arrow $\gamma : j \to r$ in Q such that $\beta\gamma \notin \Omega$.

Lemma 6. *Assume that K is a field, $n \geq 2$, $q = [q^{(1)}| \dots |q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix and $A_q = \mathbb{M}_n^q(K)$. Let $Q_{A_q} = (Q_0^{A_q}, Q_1^{A_q})$ be the Gabriel quiver of A_q and $i \in \{1, \dots, n\}$ is viewed as a vertex of Q_{A_q}.*

(a) *If $e_i A_q$ is serial, then i is a starting point of precisely one arrow in Q_{A_q}.*

(b) *If $J(e_i A_q) = L' + L''$, where $L' \neq L''$ are serial proper submodules of $J(e_i A_q)$ and the module $L' \cap L''$ is simple or zero, then i is a starting point of precisely two arrows in Q_{A_q}.*

(c) *If A_q is right biserial, then each vertex of Q_{A_q} is a starting point of at most two arrows.*

Proof. Let $A_q = \mathbb{M}_n^q(K)$ and let $Q_{A_q} = (Q_0^{A_q}, Q_1^{A_q})$. Fix i, l in $Q_0^{A_q}$ such that $i \neq l$. Note that, by Lemma 1(e),(f), [1, Lemma I.4.2(a)] and [1, Appendix 3.5(b)], we have

$$e_iA_qe_l/e_iJ(A_q)^2e_l \cong \mathrm{Hom}_{A_q}(e_lA_q, e_iA_q)/\mathrm{rad}^2_{\mathrm{pr}(A_q)}(e_lA_q, e_iA_q). \tag{5}$$

where $\mathrm{rad}^2_{\mathrm{pr}(A_q)}$ is the square of the Jacobson radical $\mathrm{rad}_{\mathrm{pr}(A_q)}$ of the category $\mathrm{pr}(A_q)$. A homomorphism $f : e_lA_q \to e_iA_q$ is irreducible in the category $\mathrm{pr}(A_q)$ if and only if f is a non-isomorphism and $f \notin \mathrm{rad}^2_{\mathrm{pr}(A_q)}(e_lA_q, e_iA_q)$, or equivalently, there is an arrow $\beta_{il} : i \to l$ in Q_{A_q}.

(a) Assume that the module e_iA_q is serial. Then $J(e_iA_q)$ contains a unique maximal submodule $J(e_iA_q)'$, the module $J(e_iA_q)/J(e_iA_q)'$ is simple and, hence, the projective cover of $J(e_iA_q)$ has the form $h' : e_jA_q \to J(e_iA_q)$, for some $j \neq i$. Moreover, the composite homomorphism $h = (e_jA_q \xrightarrow{h'} J(e_iA_q) \subset e_iA_q)$ is irreducible in $\mathrm{pr}(A_q)$. To show it, assume to the contrary that h is not irreducible. It follows that $h \in \mathrm{rad}^2_{\mathrm{pr}(A_q)}(e_jA_q, e_iA_q)$. Hence, there are two non-zero non-isomorphisms

$$e_jA_q \xrightarrow{f'} e_sA_q \xrightarrow{f''} e_iA_q,$$

for some $s \notin \{i, j\}$, such that $f'' \circ f' \neq 0$. It follows that $\mathrm{Im}\, f'' \subseteq J(e_iA_q)$ and there is $g : e_sA_q \to e_jA_q$ such that $f'' = h \circ g$, that is, the diagram

$$\begin{tikzcd} e_jA_q \arrow[r, "h"] & e_iA_q \\ & e_sA_q, \arrow[ul, "g"] \arrow[u, "f''"'] \end{tikzcd}$$

is commutative. Hence, we get $g \circ f' \neq 0$, because $h \circ g \circ f' = f'' \circ f' \neq 0$. Since $0 \neq g \circ f' \in \mathrm{End}(e_jA_q) \cong K$, then $g \circ f' = \mu \cdot \mathrm{id}$, for some non-zero $\mu \in K$. It follows that f' is an isomorphism and we get a contradiction. Consequently, h is an irreducible homomorphism and there is an arrow $\beta_{ij} : i \to j$ in Q_{A_q}.

Assume that there is an arrow $\beta_{ip} : i \to p$ in Q_{A_q} starting from i. Then there is an irreducible homomorphism $g' : e_pA_q \to e_iA_q$ and, by the arguments used earlier, there is a commutative diagram

$$\begin{tikzcd} e_jA_q \arrow[r, "h"] & e_iA_q \\ & e_pA_q, \arrow[ul, "u"] \arrow[u, "g'"'] \end{tikzcd}$$

It follows that u is an isomorphism and, hence, $p = j$ and $\beta_{ip} = \beta_{ij}$. This finishes the proof of (a).

(b) Assume that $J(e_iA_q) = L' + L''$, where $L' \neq L''$ are serial proper submodules of $J(e_iA_q)$ and the module $L' \cap L''$ is simple or zero. One can show, as in the proof of (a), that there are homomorphisms $h' : e_jA_q \to J(e_iA_q)$ and $h'' : e_rA_q \to J(e_iA_q)$, for some $j \neq r$ (because $\dim_K \mathrm{Hom}_K(e_jA_q, e_iA_q) = 1$), such that the homomorphism $(h', h'') : e_jA_q \oplus e_rA_q \longrightarrow L' + L'' = J(e_iA_q)$ is a projective cover of $J(e_iA_q)$ (because $L' \neq L''$ are serial proper submodules of $J(e_iA_q)$), and that the composite homomorphism

$$h = (e_jA_q \oplus e_rA_q \overset{(h',h'')}{\longrightarrow} L' + L'' = J(e_iA_q) \subset e_iA_q)$$

is irreducible in $\mathrm{pr}(A_q)$. It follows that the composite homomorphisms $\widetilde{h}' = (e_jA_q \to e_jA_q \oplus e_rA_q \xrightarrow{h} e_iA_q)$, $\widetilde{h}'' = (e_rA_q \to e_jA_q \oplus e_rA_q \xrightarrow{h} e_iA_q)$ are irreducible homomorphisms in $\mathrm{pr}(A_q)$. Since $j \neq r$, then in view of the isomorphism (5), the irreducible homomorphisms $\widetilde{h}'$ and $\widetilde{h}''$ correspond to two different arrows $\beta_{ij} : i \to j$ and $\beta_{ir} : i \to r$ in Q_{A_q} starting from i.

To finish the proof of (b), assume that there is an arrow $\beta_{it} : i \to t$ in Q_{A_q} starting from i. Then there is an irreducible homomorphism $g : e_tA_q \to e_iA_q$ and, by the arguments used earlier, there is a commutative diagram

$$\begin{tikzcd}
e_jA_q \oplus e_rA_q \arrow[r, "h"] & e_iA_q \\
& e_tA_q, \arrow[ul, "u"] \arrow[u, "g"']
\end{tikzcd}$$

where $u = (u_j, u_r)$ and $u_j : e_tA_q \to e_jA_q$, $u_r : e_tA_q \to e_rA_q$. Since g is irreducible and $\widetilde{h}'$, $\widetilde{h}''$ belong to the Jacobson radical of the category $\mathrm{pr}(A_q)$, then one of the maps u_j, u_r is an isomorphism, see [1, Appendix 3.5(b)]. If u_j is an isomorphism, then $t = j$ and $\beta_{it} = \beta_{ij}$. If u_r is an isomorphism, then $t = r$ and $\beta_{it} = \beta_{ir}$. This finishes the proof of (b).

Since (c) is a consequence of (a) and (b), the proof is complete. □

Now we describe the matrices $q \in \mathbb{ST}_n(K)$ such that the algebra $\mathbb{M}_n^q(K)$ is right special biserial.

Theorem 2. *Assume that K is a field, $n \geq 2$ and $q = [q^{(1)}| \dots |q^{(n)}] \in \mathbb{ST}_n(K)$ is a basic structure matrix. The following conditions are equivalent.*

(a) *The algebra $\mathbb{M}_n^q(K)$ is right biserial.*

(b) *The algebra $\mathbb{M}_n^q(K)$ is right special biserial.*

(c) *For any $i \in \{1, \dots, n\}$, each of the following two conditions is satisfied:*

(c_1) *there is one or two indices $r_i \in \{1, \dots, n\}$ such that $r_i \neq i$ and $q_{ir_i}^{(t)} = 0$, for all $t \notin \{i, r_i\}$,*

(c_2) *for any $s \neq i$ such that $q_{is}^{(t')} = 0$, for all $t' \notin \{i, s\}$, there is at most one index $l_{(i,s)} \in \{1, \dots, n\}$ such that $l_{(i,s)} \neq s$, $q_{il_{(i,s)}}^{(s)} \neq 0$ and $q_{sl_{(i,s)}}^{(p')} = 0$, for all $p' \notin \{s, l_{(i,s)}\}$.*

Proof. Let $A = \mathbb{M}_n^q(K)$ and let $Q_A = (Q_0^A, Q_1^A)$. By the proof of Gabriel's Theorem given in [1, Chapter III], the map $h : KQ_A \longrightarrow A$ defined on arrows $\beta_{ij} : i \to j$ by the formula $h(\beta_{ij}) = e_{ij}$ uniquely extends to a K-algebra surjective homomorphism $h : KQ_A \longrightarrow A$, such that $h(\omega) = q_{i_1i_3}^{(i_2)} q_{i_1i_4}^{(i_3)} \cdot \ldots \cdot q_{i_1i_l}^{(i_{l-1})} e_{i_1i_l}$, for any path $\omega = \beta_{i_1i_2}\beta_{i_2i_3}\dots\beta_{i_{l-1}i_l}$. Moreover, the ideal $\Omega = \mathrm{Ker}\, h$ is an admissible and h induces a K-algebra isomorphism $KQ_A/\Omega \cong A$. Hence, in view of the assumption that q is the basic structure matrix, the ideal Ω contains the elements $\beta_{ii_1}\beta_{i_1i_2}\dots\beta_{i_mi}$, for any cycle and the path $\beta_{ij}\beta_{jl'}$, if $q^{(j)}_{il'} = 0$, for $i, j, l' \in \{1, \dots, n\}$. Throughout the proof, we view A as the bound quiver algebra $A \cong KQ_A/\Omega$.

(a)$\Rightarrow$(b) Assume that $A = \mathbb{M}^q_n(K) \cong KQ_A/\Omega$ is right biserial. By Lemma 6(c), every vertex of Q_A is a starting point of at most two arrows. It remains to show that, for any arrow $\beta_{ij} : i \to j$ in Q_A, there exists at most one arrow $\beta_{jr} : j \to r$ in Q_A such that $\beta_{ij}\beta_{jr} \notin \Omega$, or equivalently, $q^{(j)}_{ir} \neq 0$.

If $n = 2$, then according to [7, Example 2.8] and Lemma 1(f), up to isomorphism, there exists precisely one basic algebra, namely the algebra $A \cong KQ_A/\Omega$, given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta_{12}"] & 2 \arrow[l, shift left, "\beta_{21}"] \end{tikzcd}$$

and the relations $\beta_{12}\beta_{21}$ and $\beta_{21}\beta_{12}$. Thus, in case $n = 2$, our claim follows.

If $n = 3$, then there are precisely five such algebras listed in [7, Theorem 4.1], up to isomorphism, and described by means of quivers with relations. A case by case inspection shows that the implication (a)$\Rightarrow$(b) holds, for each of the five algebras listed in [7, Theorem 4.1].

Assume that $n \geq 4$ and there exists an arrow $\beta_{ij} : i \to j$ in Q_A. Suppose, to the contrary, that there exist two different arrows $\beta_{jr} : j \to r$ and $\beta_{jp} : j \to p$ in Q_A such that $\beta_{ij}\beta_{jr} \notin \Omega$ and $\beta_{ij}\beta_{jp} \notin \Omega$. The arguments given above yield $q^{(j)}_{ir} \neq 0$ and $q^{(j)}_{ip} \neq 0$. By our assumption and Lemma 2, $q^{(p)}_{jr} = q^{(r)}_{jp} = 0$. Hence we conclude that

$$q^{(j)}_{ir} q^{(r)}_{ip} = q^{(j)}_{ip} q^{(r)}_{jp} = 0 \text{ and } q^{(j)}_{ip} q^{(p)}_{ir} = q^{(j)}_{ir} q^{(p)}_{jr} = 0,$$

because of (C2). Hence, in view of (C1), we get

$$q^{(r)}_{ip} = q^{(p)}_{ir} = 0 \text{ and } q^{(p)}_{ip} = q^{(r)}_{ir} = 1. \tag{6}$$

It follows that there is no permutation τ_i satisfying the condition (b_1) of Corollary 1. Because A is a right biserial, for the algebra A the condition (b_2) of Corollary 1 is satisfied and we have two sets $\mathcal{M}_{(i,s_i)}, \mathcal{M}_{(i,r_i)}$ such that $|\mathcal{M}_{(i,s_i)} \cup \mathcal{M}_{(i,r_i)}| = n - 1$, for $s_i < r_i$. Hence we get $j \in \mathcal{M}_{(i,s_i)}$ or $j \in \mathcal{M}_{(i,r_i)}$, because q is basic. Without loss of generality, we can assume that $j \in \mathcal{M}_{(i,s_i)}$. Thus by (2) and Lemma 1(b), we have $e_{is_i}A \supseteq e_{ij}A \supset e_{ir}A$ and $e_{ij}A \supset e_{ip}A$. According to Lemma 1(b), this implies $q^{(s_i)}_{ir} \neq 0$ and $q^{(s_i)}_{ip} \neq 0$. Hence, in view of (6), the there is no bijection $\tau_{(i,s_i)} : \{1, \dots, m_{(i,s_i)}\} \to \mathcal{M}_{(i,s_i)}$ such that the condition (b_{23}) of Corollary 1 is satisfied and we get a contradiction. Consequently, the algebra A, with $q \in \mathbb{ST}_n(K)$ and $n \geq 4$ is right special biserial. This finishes the proof of the implication (a)$\Rightarrow$(b).

The implication (b)$\Rightarrow$(a) holds, for any basic algebra A, see [18, Lemma 1].

(b)$\Leftrightarrow$(c) Recall that $A \cong KQ_A/\Omega$ and fix $i \in \{1, \dots, n\}$. By Lemma 2(b), the condition (c_1) is satisfied if and only if the vertex i in Q_A is a starting point of at most two arrows in Q_A. Moreover, according to Lemma 2(b) and the property $\beta_{ii_1}\beta_{i_1i_2} \in \Omega$, if $q^{(i_1)}_{ii_2} = 0$, the condition (c_2) is satisfied if and only if for any arrow $\beta_{is} : i \to s$ in Q_A there is at most one arrow $\beta_{sl_{(i,s)}} : s \to l_{(i,s)}$ in Q_A such that $\beta_{is}\beta_{sl_{(i,s)}} \notin \Omega$. Consequently, the equivalence of (b) and (c) is proved and the proof is complete. $\square$

Note that, together with Corollary 1, Theorem 2 completes the proof of Theorem 1. We recall from [18] that the implication (a)$\Rightarrow$(b) does not hold, for arbitrary basic algebra.

Now we prove an interesting property of the socle of the algebras $A_q = \mathbb{M}_n^q(K)$. For this purpose, we recall from [7] that the **transpose** of $q \in \mathbb{ST}_n(K)$ is defined to be the n-block matrix $q^{tr} = \widetilde{q} = [\widetilde{q}^{(1)}|\ldots|\widetilde{q}^{(n)}]$, where $\widetilde{q}^{(j)} = [q^{(j)}]^{tr}$ is the transpose of $q^{(j)}$, for $j = 1, \ldots, n$.

Corollary 3. *Assume that K is a field, $n \geq 2$ and the structure matrix $q = [q^{(1)}|\ldots|q^{(n)}] \in \mathbb{ST}_n(K)$ is basic. If the algebra $A_q = \mathbb{M}_n^q(K)$ is biserial, then $\dim_K \mathrm{soc}(A_{q A_q}) = \dim_K \mathrm{soc}({}_{A_q}A_q)$.*

Proof. Assume that $n \geq 2$, $q = [q^{(1)}|\ldots|q^{(n)}] \in \mathbb{ST}_n(K)$ is basic and the algebra $A_q = \mathbb{M}_n(K)$ is biserial. It follows from Lemma 1(d) that

$$\dim_K \mathrm{soc}(e_j A_q) \in \{1, 2\}, \tag{7}$$

for any $j \in \{1, \ldots, n\}$ Note that, according to [7, Lemma 2.15(a)], there is an algebra isomorphism $(A_q)^{op} \cong A_{q^{tr}}$. It follows that the algebra $A_{q^{tr}}$ is biserial and, by (7), we have

$$\dim_K \mathrm{soc}(A_q e_j) \in \{1, 2\}, \tag{8}$$

for any $j \in \{1, \ldots, n\}$.

Fix $j \in \{1, \ldots, n\}$. First, we prove that, $\dim_K \mathrm{soc}(A_q e_i) = 1$, if $\mathrm{soc}(e_j A_q) = e_{ji}K$, for some $i \in \{1, \ldots, n\}$. Assume that $\mathrm{soc}(e_j A_q) = e_{ji}K$, for some $i \in \{1, \ldots, n\}$. Then, by Lemma 1(b), we get $q_{ji}^{(s)} \neq 0$, for all $s \in \{1, \ldots, n\}$. Moreover, the definition of q^{tr} yields $(q^{tr})_{ij}^{(s)} \neq 0$, for each $s \in \{1, \ldots, n\}$.

The condition (C1) yields the equality $(q^{tr})_{is}^{(s)} = 1$, for each $s \in \{1, \ldots, n\}$. Hence and from Lemma 1(d), we conclude that the equality $(q^{tr})_{ip}^{(j)} = 0$, for $p \neq j$ holds only for $j \in \{1, \ldots, n\}$. Equivalently, by Lemma 1(d) and the isomorphism $(A_q)^{op} \cong A_{q^{tr}}$, we have $\dim_K \mathrm{soc}(A_q e_i) = \dim_K \mathrm{soc}(e_i A_{q^{tr}}) = 1$.

In the sequel, we denote by l_r^q (resp. l_l^q) the number of indecomposable projective right (resp. left) A_q-modules of the form $e_t A_q$ (resp. $A_q e_t$) with simple socle. Note that, if $e_t A_q \ncong e_{p'} A_q$ and the modules $\mathrm{soc}(e_t A_q)$, $\mathrm{soc}(e_{p'} A_q)$ are simple, then $\mathrm{soc}(e_t A_q) \ncong \mathrm{soc}(e_{p'} A_q)$, because the modules $e_t A_q$, $e_{p'} A_q$ are injective.

By the argument applied above and Lemma 1(f), we obtain $l_r^q \leq l_l^q$ and $l_r^{q^{tr}} \leq l_l^{q^{tr}}$. Since, in view of the isomorphism $(A_q)^{op} \cong A_{q^{tr}}$, we have $l_l^q = l_r^{q^{tr}}$ and $l_r^q = l_l^{q^{tr}}$, then $l_l^q \leq l_r^q$, that is, $l_r^q = l_l^q$. Because $A_q e_1 \oplus \ldots \oplus A_q e_n = A_q = e_1 A_q \oplus \ldots \oplus e_n A_q$ and $l_r^q = l_l^q$, then the formulae (7) and (8) yield the required equality $\dim_K \mathrm{soc}({}_{A_q}A_q) = \dim_K \mathrm{soc}(A_{q A_q})$. $\square$

We end the paper by an example of a non-biserial basic minor degeneration A_q of $\mathbb{M}_n(K)$ such that $\dim_K \mathrm{soc}({}_{A_q}A_q) \neq \dim_K \mathrm{soc}(A_{q A_q})$.

Example 1. Assume that $n = 4$ and let $A_q = \mathbb{M}_4^q(K)$ is given by the basic structure matrix

$$q = \begin{bmatrix} 1111 & 0100 & 0010 & 0001 \\ 1011 & 1111 & 0010 & 0001 \\ 1000 & 0100 & 1111 & 0001 \\ 1000 & 0100 & 0010 & 1111 \end{bmatrix}$$

It follows from Lemma 1(d) that we have
$\mathrm{soc}(e_1 A_q) = e_{12}K \oplus e_{13}K \oplus e_{14}K$,
$\mathrm{soc}(e_2 A_q) = e_{23}K \oplus e_{24}K$,
$\mathrm{soc}(e_3 A_q) = e_{31}K \oplus e_{32}K \oplus e_{34}K$, and
$\mathrm{soc}(e_4 A_q) = e_{41}K \oplus e_{42}K \oplus e_{43}K$
and hence $\dim_K \mathrm{soc}(A_{q\,A_q}) = 11$. Moreover, in view of (7), the algebra A_q is not biserial, because $\dim_K \mathrm{soc}(e_1 A_q) = 3$. On the other hand, since the algebra $A_{q^{tr}} \cong A^{op}$ is given by the structure matrix

$$q^{tr} = \begin{bmatrix} 1111 & 0100 & 0010 & 0001 \\ 1000 & 1111 & 0010 & 0001 \\ 1100 & 0100 & 1111 & 0001 \\ 1100 & 0100 & 0010 & 1111 \end{bmatrix}$$

we get
$\mathrm{soc}(e_1 A_{q^{tr}}) = e_{12}K \oplus e_{13}K \oplus e_{14}K$,
$\mathrm{soc}(e_2 A_{q^{tr}}) = e_{21}K \oplus e_{23}K \oplus e_{24}K$,
$\mathrm{soc}(e_3 A_{q^{tr}}) = e_{32}K \oplus e_{34}K$,
$\mathrm{soc}(e_4 A_{q^{tr}}) = e_{42}K \oplus e_{43}K$.
This shows that the dimension of the left socle of the algebra A_q equals $\dim_K \mathrm{soc}({}_{A_q}A_q) = \dim_K \mathrm{soc}(A_{q^{tr}\,A_{q^{tr}}}) = 10$. Consequently, we have shown that $\dim_K \mathrm{soc}(A_{q\,A_q}) \neq \dim_K \mathrm{soc}({}_{A_q}A_q)$.

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