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# A survey on pairwise mutually permutable products of finite groups

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*Dedicated to Professor Leonid A. Kurdachencko on the occasion of his sixtieth birthday*

Abstract. In this paper we present some recent results about pairwise mutually permutable products and classes of groups related to the saturated formation of all supersoluble groups.

## 1. Introduction and Preliminary results

A group $G$ is said to be the product of its pairwise permutable subgroups $G_1, G_2, \ldots, G_n$ if $G = G_1G_2 \ldots G_n$ and $G_iG_j = G_jG_i$ for all integers $i$ and $j$ with $i, j \in \{1, 2, \ldots, n\}$. Groups which are product of some of its subgroups have played a significant role in the theory of groups over the past sixty years. For instance a finite group is soluble if and only if it is the product of pairwise permutable Sylow subgroups. The celebrated theorem of Kegel and Wielandt goes much further and proves that a product of two finite nilpotent groups is also soluble. Moreover an easy induction argument, applying the above result, yields that a finite group which is the product of pairwise permutable finite nilpotent subgroups is soluble. Huppert studied pairwise permutable products of finite cyclic groups and proved that they are supersoluble.

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These results give rise to the following questions:

*Let the group $G = G_1G_2 \dots G_n$ be the product of its pairwise permutable subgroups $G_1, G_2, \dots, G_n$ and suppose that the factors $G_i$, $1 \leq i \leq n$, belong to a class of groups $\mathcal{X}$. When does the group $G$ belong to $\mathcal{X}$?. How does the structure of the factors $G_i$, $1 \leq i \leq n$ affect the structure of the group $G$?.*

Obviously, if $G_i$, $1 \leq i \leq n$, are finite, then the group $G$ is finite. However not many properties carry over from the factors of a factorized group to the group itself. Two well known examples support this claim: there exist non abelian groups which are products of two abelian subgroups and every finite soluble group is the product of pairwise permutable nilpotent subgroups.

As special cases of products $G = G_1G_2 \dots G_k$ of pairwise permutable subgroups we have direct products, central products and normal products. However there is a great distance between general products and direct products. For instance, the normal product of supersoluble groups is not supersoluble in general while the direct product of supersoluble groups is always supersoluble. As a consequence formations, even saturated, are in general not closed under normal products although they are always closed under direct products and even under central products. Consequently it seems reasonable to create intermediate situations. In this context, assumptions on permutability connections between the factors turn out to be very useful. One of the most important ones is the mutual permutability introduced by Asaad and Shaalan in [1].

Two subgroups $A$ and $B$ of a group $G$ are mutually permutable if $A$ permutes with every subgroup of $B$ and $B$ permutes with every subgroup of $A$. If $G = AB$ and $A$ and $B$ are mutually permutable, then $G$ is called a mutually permutable product of $A$ and $B$. More generally, a group $G = G_1G_2 \dots G_n$ is said to be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$ if $G_i$ and $G_j$ are mutually permutable subgroups of $G$ for all $i, j \in \{1, 2, \dots, n\}$. Asaad and Shaalan ([1]) proved that if $G$ is a mutually permutable product of the subgroups $A$ and $B$ and $A$ and $B$ are finite and supersoluble, then $G$ is supersoluble provided that either $G'$, the derived subgroup of $G$, is nilpotent or $A$ or $B$ is nilpotent. This result was the beginning of an intensive study of such factorized groups (see, for instance, [1, 2, 4, 5, 7, 8, 11, 12]).

In this survey article we are concerned with the structure of groups factorized by finitely many pairwise mutually permutable subgroups. Most of the results presented here can be found in [4, 5].

*Throughout the paper all groups considered will be finite.*

The following lemmas involve some crucial properties of subgroups of pairwise mutually permutable products including the analysis of the behaviour of the soluble residual and non-abelian minimal normal subgroups.

**Lemma 1.** *[4] Let $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$. Then:*

*(i) If $U$ is a subgroup of $G$, then $(U \cap G_1)(U \cap G_2) \dots (U \cap G_n)$ is the pairwise mutually permutable product of the subgroups $U \cap G_1, U \cap G_2, \dots, U \cap G_n$.*

*(ii) If $U$ is a normal subgroup of $G$, then $(U \cap G_1)(U \cap G_2) \dots (U \cap G_n)$ is a normal subgroup of $G$.*

*(iii) $G'_i$ is a subnormal subgroup of $G$ for each $i \in \{1, 2, \dots, n\}$.*

*(iv) $\bigcap_{i=1}^{n} G_i$ is a subnormal subgroup of $G$.*

**Lemma 2.** *[4] Let the group $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$. Then the soluble residuals of the factors $G_i$ are normal subgroups of $G$ and their product is the soluble residual of $G$.*

**Lemma 3.** *[4] Let the group $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$. If $M$ is a non-abelian minimal normal subgroup of $G$, then either $M \cap G_i = 1$ or $M$ is contained in $G_i$ for every $i = 1, 2, \dots, n$. Moreover there exists $j \in \{1, 2, \dots, n\}$ such that $M$ is contained in $G_j$.*

## 2. Pairwise mutually permutable products and classes of groups

Our main objective in this section will be to prove that some relevant classes of groups are closed under the operation of forming pairwise mutually permutable products.

Using Lemma 1 we obtain an alternative proof (see [4]) of the following result proved previously by Carocca in [12].

**Theorem 1.** *Let $p$ be a prime and let $G = G_1G_2 \dots G_n$ be a group which is the product of the pairwise mutually permutable $p$-soluble subgroups $G_1, G_2, \dots, G_n$. Then $G$ is $p$-soluble.*

Our next two results are related to classes of groups defined by sets of primes which are extensions of the class of all supersoluble groups.

Let $p$ be a prime. Following [8], we say that a set of primes $\Pi$ is $p$-special if $q \notin \Pi$ whenever $p$ divides $q(q-1)$.

The following result is an extension of [8, Theorem 2].

**Theorem 2.** *[4] Let the group $G = G_1G_2 \ldots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \ldots, G_n$. If $G_1, G_2, \ldots, G_n$ are normal extensions of $p$-groups by $\Pi$-groups for a $p$-special set of primes $\Pi$, then the same is true for $G$.*

**Corollary 1.** *[4] Let the group $G = G_1G_2 \ldots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \ldots, G_n$. If $G_i$, $1 \le i \le n$, is a Sylow tower group with respect to a Sylow tower where all primes $q$ dividing $p-1$ appear on top of $p$, then $G$ is a Sylow tower group of the same form.*

Supersoluble groups belong to all of these classes of Sylow tower groups.

A mutually permutable product of $p$-nilpotent (respectively $p$-supersoluble) groups is not a $p$-nilpotent (respectively $p$-supersoluble) group. On the other hand, as we have said in the introduction, Asaad and Shaalan [1] proved that a group $G$ factorized by two supersoluble mutually permutable subgroups is supersoluble provided $G'$, the derived subgroup of $G$, is nilpotent or at least one of the factors is nilpotent. It is a natural question if these results are also true for mutually permutable products of more than two factors. Carocca [12] proved the following:

*Assume the group $G = G_1G_2 \ldots G_n$ is the pairwise mutually permutable product of the $p$-supersoluble subgroups $G_1, G_2, \ldots, G_n$. If the commutator subgroup $G'$ of $G$ is $p$-nilpotent, then $G$ is $p$-supersoluble.*

However the answer to the second question is negative as the following example shows:

**Example 1.** Let the group $T = AB$ be a mutually permutable product of two supersoluble subgroups $A$ and $B$. Suppose that $T$ is not supersoluble (see [1, Remark, p. 322]). Consider $G = T \times C$, the direct product of $T$ with a cyclic group of order $p$, $p$ a prime. Then $G = ABC$ is the product of the pairwise mutually permutable subgroups $A, B$ and $C$, $G$ is not supersoluble and $C$ is nilpotent.

However we have:

**Theorem 3.** *[4] Let the group $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$. If $G_1$ is $p$ - supersoluble and $G_i$ is $p$ - nilpotent for every $i \in \{2, \dots, n\}$, then $G$ is $p$ - supersoluble.*

**Corollary 2.** *[4] Let the group $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$. Assume that $p$ is a prime such that $G_i$ is a normal extension of a subgroup whose order is coprime to $p(p-1)$ by a $p$-group for every $i = 1, 2, \dots, n$. Then the same is true for $G$.*

Our next aim is to analyze the behaviour of pairwise mutually permutable products with respect to some classes which can be considered as generalizations of the class of all supersoluble groups.

A group $G$ is called an $SC$-group (respectively $SNAC$-group), if all chief factors (respectively all non-abelian chief factors) of $G$ are simple. These classes were introduced and studied by Robinson in [16]. Note that both of them are formations, that is, they are closed under taking homomorphic images and subdirect products.

**Theorem 4.** *[4] Let the group $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$. Then $G$ is an $SNAC$-group if and only if $G_i$ is an $SNAC$-group for every $i = 1, 2, \dots, n$.*

With similar arguments to those used in the above theorem we have:

**Theorem 5.** *[4] Let the group $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$. If $G$ is an $SC$-group, then $G_i$ is an $SC$-group for every $i \in \{1, 2, \dots, n\}$.*

However, the converse of the above theorem is not true in general: consider for instance a non-supersoluble mutually permutable product of supersoluble groups.

Some interesting results on pairwise mutually permutable products appear when the factors belong to some classes of finite groups which are defined in terms of permutability. They are the class of $PST$-groups, or finite groups $G$ in which every subnormal subgroup of $G$ permutes with every Sylow subgroup of $G$, the class of $PT$-groups, or finite groups in which every subnormal subgroup is a permutable subgroup of the group, the class of $T$-groups, or groups in which every subnormal subgroup is normal, and the class of $\mathcal{Y}$-groups, or finite groups $G$ for which for every subgroup $H$ and for all primes $q$ dividing the index $|G : H|$ there exists a subgroup $K$ of $G$ such that $H$ is contained in $K$ and $|K : H| = q$, and their corresponding local versions (see [2, 3]). Theses non-empty classes of groups have been widely studied and also considered in the context of mutually permutable products (see [2, 7, 8, 11]). Robinson [16] proves that $PST$-groups are $SC$-groups.

The converse of Theorem 5 is true if the factors are $PST$-groups.

**Theorem 6.** *[4] Let the group $G = G_1G_2 \dots G_n$ be the product of the pairwise mutually permutable $PST$-groups $G_1, G_2, \dots, G_n$. Then $G$ is an $SC$-group.*

Our main goal now is to take this studies a step further by analyzing the structure of the pairwise mutually permutable products whose factors belong to some local classes of finite groups closely related to the classes of all $T$-groups and $\mathcal{Y}$-groups.

A group $G$ satisfies property $\mathcal{C}_p$, or $G$ is a $\mathcal{C}_p$-group, if each subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in the normalizer $N_G(P)$. This class of groups was introduced by Robinson in [15] as a local version of the class of all soluble $T$-groups. In fact, he proved that a group $G$ is a soluble $T$-group if and only if $G$ is a $\mathcal{C}_p$-group for all primes $p$.

Moreover in [9] the second and third authors introduce and analyze an interesting class of groups closely related to the class of all $T$-groups. A group $G$ is a $T_1$-group if $G/Z_\infty(G)$ is a $T$-group. Here $Z_\infty(G)$ denotes the hypercenter of $G$, that is, the largest normal subgroup of $G$ having a $G$-invariant series with central $G$-chief factors. The local version of the class $T_1$ in the soluble universe is the class $\bar{\mathcal{C}_p}$ introduced and studied in [10]:

**Definition 1.** *Let $G$ be a group and let $Z_p(G)$ be the Sylow $p$-subgroup of $Z_\infty(G)$. A group satisfies $\bar{\mathcal{C}_p}$ if and only if $G/Z_p(G)$ is a $\mathcal{C}_p$-group.*

**Theorem A** ([10]) *A group $G$ is a soluble $T_1$-group if and only if $G$ is a $\bar{\mathcal{C}_p}$-group for all primes $p$.*

We analyze now the behaviour of pairwise mutually permutable products with respect to the class $\bar{\mathcal{C}_p}$.

**Theorem 7.** *[5] Let $G = G_1G_2 \dots G_k$ be the pairwise mutually permutable product of the subgroups $G_1, G_2, \dots, G_k$. If $G_i$ is a $p$-soluble $\bar{\mathcal{C}_p}$-group for every $i \in \{1, 2, \dots, k\}$, then $G$ is $p$-supersoluble.*

Combining Theorem A and Theorem 7 we have:

**Corollary 3.** *[5] Let $G = G_1G_2 \dots G_k$ be a product of the pairwise mutually permutable soluble $T_1$-groups $G_1, G_2, \dots, G_k$. Then $G$ is supersoluble.*

Another class of groups closely related to $T$-groups is the class $T_0$ of all groups $G$ whose Frattini quotient $G/\Phi(G)$ is a $T$-group. This class was introduced in [17] and studied in [13, 14, 17].

The procedure of defining local versions in order to simplify the study of global properties has also been successfully applied to the study of the classes $T_0$ ([13]) and $\mathcal{Y}$ ([3]).

**Definition 2.** *Let $p$ be a prime and let $G$ be a group.*

*(i) ([13]) Let $\Phi(G)_p$ be the Sylow $p$-subgroup of the Frattini subgroup of $G$. $G$ is said to be a $\hat{\mathcal{C}}_p$-group if $G/\Phi(G)_p$ is a $\mathcal{C}_p$-group.*

*(ii) ([3, Definition 11]) We say that $G$ satisfies $\mathcal{Z}_p$ or $G$ is a $\mathcal{Z}_p$-group when for every $p$-subgroup $X$ of $G$ and for every power of a prime $q$, $q^m$, dividing $\mid G : XO_{p'}(G) \mid$, there exists a subgroup $K$ of $G$ containing $XO_{p'}(G)$ such that $\mid K : XO_{p'}(G) \mid = q^m$.*

It is not difficult to see that the class of all $\hat{\mathcal{C}}_p$-groups is closed under taking epimorphic images and all $p$-soluble groups belonging to $\hat{\mathcal{C}}_p$ are $p$-supersoluble. Moreover:

**Theorem B** ([13]) *A group $G$ is a soluble $T_0$-group if and only if $G$ is a $\hat{\mathcal{C}}_p$-group for all primes $p$.*

The results of [6] show that the class $\mathcal{C}_p$ is a proper subclass of the class $\mathcal{Z}_p$.

Furthermore in [2, Theorem 16] it is proved that a pairwise mutually permutable product of $\mathcal{Y}$-groups is supersoluble. Moreover the authors ask if a pairwise mutually permutable product of $\mathcal{Z}_p$-groups is $p$-supersoluble. In what follows, we answer to this question affirmatively. In fact, our main purpose here is to study pairwise mutually permutable products whose factors belong to some class of groups closely related to $\mathcal{Z}_p$-groups.

**Definition 3.** *Let $p$ be a prime, let $G$ be a group and let $\Phi(G)_p$ be the Sylow $p$-subgroup of the Frattini subgroup of $G$. $G$ is said to be a $\hat{\mathcal{Z}}_p$-group if $G/\Phi(G)_p$ is a $\mathcal{Z}_p$-group.*

Our last theorem shows that pairwise mutually permutable products of $\hat{\mathcal{Z}}_p$-groups are $p$-supersoluble.

**Theorem 8.** *[5] Let $G = G_1G_2 \dots G_n$ be the pairwise mutually permutable product of the subgroups $G_1, G_2, \dots, G_n$. If $G_i$ is a $p$-soluble $\hat{\mathcal{Z}}_p$-group for every $i$, then $G$ is $p$-supersoluble.*

Since every $\mathcal{Z}_p$-group is a $\hat{\mathcal{Z}}_p$-group and by [3, Theorem 15] if $G$ is a soluble group, $G$ is a $\mathcal{Y}$-group if and only if it is a $\mathcal{Z}_p$-group for every prime $p$, we can apply Theorem 8 to obtain the following:

**Corollary 4.** *[2, Theorem 16] Let $G = G_1G_2 \dots G_n$ be a group such that $G_1, G_2, \dots, G_n$ are pairwise mutually permutable subgroups of $G$. If all $G_i$ are $\mathcal{Y}$-groups, then $G$ is supersoluble.*

Finally, applying Theorem B and Theorem 8, we have:

**Corollary 5.** *[5] Let $G = G_1G_2 \dots G_n$ be a group such that $G_1, G_2, \dots, G_n$ are pairwise mutually permutable subgroups of $G$. If all $G_i$ are soluble $T_0$-groups, then $G$ is supersoluble.*

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