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RESEARCH ARTICLE

A note on semidirect products and nonabelian tensor products of groups

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Abstract. Let G and H be groups which act compatibly on one another. In [2] and [8] it is considered a group construction $\eta(G,H)$ which is related to the nonabelian tensor product $G \otimes H$. In this note we study embedding questions of certain semidirect products $A \rtimes H$ into $\eta(A,H)$, for finite abelian H-groups A. As a consequence of our results we obtain that complete Frobenius groups and affine groups over finite fields are embedded into $\eta(A,H)$ for convenient groups A and H. Further, on considering finite metabelian groups G in which the derived subgroup has order coprime with its index we establish the order of the nonabelian tensor square of G.

Dedicated to Professor Miguel Ferrero
on occasion of his 70-th anniversary

Introduction

Let K and H be groups each of which acts upon the other (on the right),

$$K \times H \to K,\ (k,h) \mapsto k^h;\ \ H \times K \to H,\ (h,k) \mapsto h^k$$

and on itself by conjugation, in such a way that for all $k, k_1 \in K$ and $h, h_1 \in H$,

$$k^{\left(h^{k_1}\right)} = \left(\left(k^{k_1^{-1}}\right)^h\right)^{k_1} \quad \text{and} \quad h^{\left(k^{h_1}\right)} = \left(\left(h^{h_1^{-1}}\right)^k\right)^{h_1}. \qquad (1)$$

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In this situation we say that K and H act *compatibly* on each other.

An operator η in the class of (operator) groups has been introduced in [8] (see also [2] and [9]) which is defined as follows: let $K,\ H$ be as above, acting compatibly on each other, and H^{φ} an extra copy of H, isomorphic through $\varphi : H \to H^{\varphi},\ h \mapsto h^{\varphi}$, for all $h \in H$. Then we define the group

$$\eta(K,H) := \langle K, H^{\varphi} \mid [k,h^{\varphi}]^{k_1} = [k^{k_1},(h^{k_1})^{\varphi}],\ [k,h^{\varphi}]^{h_1^{\varphi}} = [k^{h_1},(h^{h_1})^{\varphi}],$$

for all $k,k_1 \in K,\ h,h_1 \in H\rangle$.

In particular we write $\nu(H)$ for $\eta(H,H)$ when all actions are conjugations (cf. [12]).

Besides its intrinsic group-theoretic interest, it follows from Proposition 1.4 in [3] that there is an isomorphism from the subgroup $[K,H^{\varphi}]$ of $\eta(K,H)$ onto the nonabelian tensor product $K \otimes H$ (as introduced by R. Brown and J.-L. Loday [1]), such that $[k,h^{\varphi}] \mapsto k \otimes h$, for all $k \in K$ and $h \in H$. It is worth mentioning that $[K,H^{\varphi}]$ is a normal subgroup of $\eta(K,H)$ and that $\eta(K,H) = ([K,H^{\varphi}] \cdot K) \cdot H^{\varphi}$, where the dots denote semidirect products.

On discussing nilpotency conditions on $\eta(K,H)$ in [10], where K and H are nilpotent groups, we observe that even in very elementary situations (in which at least one of the actions is non-nilpotent) the group $\eta(K,H)$ fails to be nilpotent. In fact, with appropriate actions $\eta(C_p,C_2)$ contains the dihedral group of order $2p$ (where p denotes an odd prime), while $\eta(V_4,C_3)$ contains the alternating group A_4 (here C_n denotes the cyclic group of order n and V_4 is the Klein four group; see [10] for details).

In this note we are interested in embedding certain split extensions $A \rtimes H$ into $\eta(A,H)$, where A is an abelian H-group acting trivially on H. It is an easy exercise to check the compatibility of these actions for any given action of H on A. In the present situation we write $\eta^*(A,H)$ for the corresponding group $\eta(A,H)$. If B is any H-subgroup of A, then $B \cdot H$ means the semidirect product of B by H. We also write $[A,H]$ for the subgroup of A generated by the set $\{a^{-1}a^{h} \mid a \in A,\ h \in H\}$.

With the above notation we can formulate

Proposition A. If $(|A|,\ |H|) = 1$ then $[A,\ H] \cdot H$ is embedded into $\eta^*(A,H)$. If, in addition, $A = [A,\ H]$ and $A \neq 1$, then $\eta^*(A,H)$ is non-nilpotent.

In order to deal with some situations involving non-coprime actions we prove

Proposition B. If A is a finite group and there is a central element $h \in H$ such that h acts fixed-point-free (f.p.f., for short) on A, then $A \rtimes H$ is embedded into $\eta^*(A,H)$.

In particular if $F = GF(q)$, the finite field with q elements, then the affine group $\mathcal{A}_n(F)$ is embedded into $\eta^*(A, \mathrm{GL}_n(F))$, where here $A \cong (F^n, +)$ is the translation subgroup.

Next we shall consider finite metabelian groups G in which the derived subgroup G' has order coprime with its index. We observe that the defining relations of $\eta(H, K)$ are externalisations of commutator relations. Thus there is an epimorphism $\kappa : [G, G^\varphi] \to G'$, $[x, y^\varphi] \mapsto [x, y]$, for all $x, y \in G$, whose kernel we denote by $J(G)$. As usual we write $\mathrm{M}(G)$ for the Schur Multiplier of G and G^{ab} for the abelianized group G/G'. Our contribution is

Proposition C. Let G be a finite metabelian group such that $|G'|$ and $|G^{ab}|$ are coprime. Then

(i) $|G \otimes G| = n|G'| \cdot |G^{ab} \otimes_{\mathbb{Z}} G^{ab}|$;

(ii) $|J(G)| = n|G^{ab} \otimes_{\mathbb{Z}} G^{ab}|$,

where n is the order of the G^{ab}-stable subgroup of $\mathrm{M}(G')$.

Notation in this note is fairly standard. For elements x, y, z in an arbitrary group G, the conjugate of x by y is $x^y = y^{-1}xy$; the commutator of x and y is $[x, y] = x^{-1}x^y$ and our commutators are left normed; in particular $[x, y, z] = [[x, y], z]$.

Throughout the paper we assume that the groups K and H act compatibly on one another.

1. Proofs

Our starting point is the embedding of $K \otimes H$ into $\eta(K, H)$ via the isomorphism $K \otimes H \cong [K, H^\varphi]$ given by $k \otimes h \mapsto [k, h^\varphi]$ for all $k \in K$, $h \in H$ (cf. [3], Proposition 1.4). By [2, Theorem 1],

$$\eta(K, H) = [K, H^\varphi]H^\varphi K \cong ((K \otimes H) \rtimes H) \rtimes K.$$

We shall use this decomposition without any further reference. This together with [1, Proposition 2.3] gives

Lemma 1. *The following relations hold in* $\eta(K, H)$ *for all* $k, x \in K$ *and* $h, y \in H$*:*

(a) $[k, h^\varphi]^{[x, y^\varphi]} = [k, h^\varphi]^{x^{-1}x^y} = [k, h^\varphi]^{(y^{-x}y)^\varphi}$;

(b) $[k, h^\varphi]^{[x, y^\varphi]^{-1}} = [k, h^\varphi]^{x^{-y}x} = [k, h^\varphi]^{(y^{-1}y^x)^\varphi}$;

(c) $[[k, h^\varphi], [x, y^\varphi]] = [k^{-1}k^h, (y^{-x}y)^\varphi]$;

(d) $\left[[k, h^\varphi], [x, y^\varphi]^{-1}\right] = [k^{-1}k^h, (y^{-1}y^x)^\varphi]$.

The above relations immediately lead to the

Corollary 1. **(a)** *If K acts trivially on H, then $[K, H^\varphi]$ is abelian;*

(b) *If K and H act trivially on each other, then $[K, H^\varphi]$ is isomorphic to the ordinary tensor product $K^{ab} \otimes_{\mathbb{Z}} H^{ab}$ of the abelianized groups.*

Proof of Proposition A. Since A is abelian and acts trivially on H, [5, Proposition 2.3] gives an isomorphism $[A, H^\varphi] \cong A \otimes_{\mathbb{Z}H} I(H)$, where $I(H)$ denotes the augmentation ideal of $\mathbb{Z}H$, such that $[a, h^\varphi] \mapsto a \otimes (h-1)$. On the other hand there is an H-epimorphism $\lambda : [A, H^\varphi] \to [A, H]$, $[a, h^\varphi] \mapsto [a, h] = a^{-1}a^h$. It folllows from [11, 11.4.2] that $\mathrm{Ker}(\lambda)$ is isomorphic to the first homology group $\mathrm{H}_1(H, A)$. Since $\gcd(|A|, |H|) = 1$ we have $\mathrm{H}_1(H, A) = 0$ (here we use additive notation in A), so that λ is an H-isomorphism. Therefore $[A, H^\varphi] \cong [A, H]$ and, consequently, the subgroup $[A, H^\varphi] \cdot H^\varphi$ of $\eta^*(A, H)$ is isomorphic to the semi-direct product $[A, H] \cdot H$. If in addition $[A, H] = A$, then certainly all terms $\gamma_i(\eta^*(A, H))$ of the lower central series of $\eta^*(A, H)$ will contain the subgroup $[A, H^\varphi] \cong A$. This finishes the proof. $\square$

We recall that a finite group G containing a proper subgroup $H \neq 1$ such that $H \cap H^g = 1$ for all $g \in G \setminus H$ is called a *Frobenius group.* The subgroup H is called a *Frobenius complement.* By a celebrated theorem of Frobenius, the set $N = G \setminus (\cup_{x \in G} (H^*)^x)$ is a normal subgroup of G (called its *Frobenius kernel*) such that $G = NH$ and $N \cap H = 1$. We have that $|H|$ divides $|N| - 1$. If $|H| = |N| - 1$, then we say that G is a *complete Frobenius group*; in this case the kernel N is an elementary abelian group (see for instance [14]).

Corollary 2. *Every finite Frobenius group with an abelian kernel A and complement H is embedded into $\eta^*(A, H)$.*

Proof of Proposition B. Let h be a central element of H such that h acts f.p.f. on A. Since A is abelian and acts trivially on H, $[A, h^\varphi] = \{[a, h^\varphi] : a \in A\}$ is a subgroup of $\eta^*(A, H)$. Further, there is a homomorphism $\alpha : [A, h^\varphi] \longrightarrow A$ such that $[a, h^\varphi] \mapsto a^{-1}a^h$. Because h is central in H, we have for all $a \in A$ and $x \in H$,

$$\alpha\left([a, h^\varphi]^x\right) = \alpha([a^x, h^\varphi]) = a^{-x}a^{xh} = a^{-x}a^{hx} = \left(a^{-1}a^h\right)^x = \left(\alpha[a, h^\varphi]\right)^x .$$

Thus α is an H-homomorphism. Further, if $A = \{a_1, \cdots, a_r\}$, then $Im(\alpha) = \{{a_1}^{-1}{a_1}^h, \cdots, {a_r}^{-1}{a_r}^h\}$. As $a^h = a$ implies $a = 1$, it follows that ${a_i}^{-1}{a_i}^h = {a_j}^{-1}{a_j}^h$ if and only if $a_i = a_j$. Hence $|Im(\alpha)| = |A|$. It is clear that $|[A, h^\varphi]| \leq |A|$. Therefore α is an H-isomorphism and $A \rtimes H$ is embedded into $\eta^*(A, H)$. □

As a consequence of Proposition B we obtain

Corollary 3. *The affine group $\mathcal{A}_n(F)$ is embedded into $\eta^*(A, \mathrm{GL}_n(F))$, where F denotes the finite field with q elements $GF(q)$ and $A \cong (F^n, +)$ denotes the translation subgroup.*

Proof. Set $h = \mu I_n$, where I_n denotes the identity matrix of order n and μ is a generator of the multiplicative group $(F\backslash\{0\}, \cdot)$. Then h is central in $\mathrm{GL}_n(F)$ and acts f.p.f. on A. Thus the corollary follows from the above result. □

Now we observe that there is a epimorphism $\kappa : [G, G^\varphi] \to G'$, $[x, y^\varphi] \mapsto [x, y]$, whose kernel is denoted by $J(G)$. Result in [1] implies that the exact sequence

$$1 \longrightarrow J(G) \longrightarrow [G, G^\varphi] \longrightarrow G' \longrightarrow 1 \tag{2}$$

yields a central extension. On denoting by $\Delta(G)$ the subgroup $\langle [g, g^\varphi] | g \in G\rangle$ of $\nu(G)$ we have that the section $J(G)/\Delta(G)$ is isomorphic to the Schur Multiplier of G (cf. [7]).

We need a couple of lemmas before the proof of Proposition C.

Lemma 2. ([12, Lemma 2.1] and [13, Lemma 3.1]) *The following relations hold in $\nu(G)$, for all $x, y, z \in G$.*

(i) $[x, y^\varphi, z] = [x, y, z^\varphi] = [x, y^\varphi, z^\varphi]$;

(ii) $[x^\varphi, y, z] = [x^\varphi, y, z^\varphi] = [x^\varphi, y^\varphi, z]$;

(iii) $[g, g^\varphi]$ *is central in* $\nu(G)$, *for all* $g \in G$;

(iv) $[g, g^\varphi] = 1$, *for all* $g \in G'$;

(v) *If* $g \in G'$ *then* $[g, h^\varphi][h, g^\varphi] = 1$, *for all* $h \in G$.

Lemma 3. *Let $G = G' \cdot H$ be a semidirect product of its subgroups G' and H. Then in $\nu(G)$,*

(i) $[H, (G')^\varphi] = [G', H^\varphi]$;

(ii) $\Delta(G) = \langle [h, h^\varphi] \mid h \in H\rangle$.

Proof. Part (i) is a consequence of the item (v) of Lemma 2. As for part (ii), let $g \in G$ be an arbitrary element. Then $g = ch$ for some elements $c \in G'$ and $h \in H$. Thus we have:

$$\begin{aligned}
[g, g^{\varphi}] &= [ch, (ch)^{\varphi}] \\
&= [c, h^{\varphi}]^{h}[c, c^{\varphi}]^{h^2}[h, h^{\varphi}][h, c^{\varphi}]^{h^{\varphi}} && \text{(by commutator identities)} \\
&= [c, h^{\varphi}]^{h}[h, h^{\varphi}][h, c^{\varphi}]^{h^{\varphi}} && \text{(by Lemma 2 (iv))} \\
&= ([c, h^{\varphi}][h, c^{\varphi}])^{h^{\varphi}}[h, h^{\varphi}] && \text{(by Lemma 2 (iii))} \\
&= [h, h^{\varphi}], && \text{(by Lemma 2 (v)).}
\end{aligned}$$

Therefore $\Delta(G) = \langle [h, h^{\varphi}] \mid h \in H \rangle$, as required. □

Proof of Proposition C. Firstly we observe that as $\gcd(|G'|, |G^{ab}|) = 1$, by Schur-Zassenhaus Theorem [14, Theorem 2.7.4], there is a subgroup H of G, with $H \cong G^{ab}$, such that $G = G' \cdot H$ is a semidirect product of G' and H. Further, the tensor squares $H \otimes H$ and $G^{ab} \otimes G^{ab}$ are isomorphic. Since G^{ab} is abelian, Corollary 1 (b) gives $G^{ab} \otimes G^{ab} \cong G^{ab} \otimes_{\mathbb{Z}} G^{ab}$. Using Lemma 3.2 in [8] we obtain an exact sequence

$$1 \longrightarrow [G', G^{\varphi}] \xrightarrow{inc} [G, G^{\varphi}] \longrightarrow G^{ab} \otimes G^{ab} \longrightarrow 1 \qquad (3)$$

where $[G, G^{\varphi}] \leq \nu(G)$. As $\gcd(|G'|, |G^{ab}|) = 1$, it follows from (2) and (3) that

$$|G'| \left| G^{ab} \otimes_{\mathbb{Z}} G^{ab} \right| \ \text{divides} \ |[G, G^{\varphi}]| . \qquad (4)$$

On the other hand, [8, Theorem 3.3] gives that

$$|[G, G^{\varphi}]| \ \text{divides} \ |G' \wedge G'| \left| G' \otimes_{\mathbb{Z}[G^{ab}]} I(G^{ab}) \right| \left| G^{ab} \otimes_{\mathbb{Z}} G^{ab} \right| \qquad (5)$$

where $G' \wedge G'$ is the exterior square of the $\mathbb{Z}$-module G'. As G' is abelian, $G' \wedge G' \cong \mathrm{M}(G')$ (cf. [7]). By [5, Proposition 5.2] we have that $G' \otimes_{\mathbb{Z}[G^{ab}]} I(G^{ab})$ is isomorphic to the subgroup $[G', (G^{ab})^{\psi}]$ of the group $\eta^*(G', G^{ab})$ (here we assume that G' acts trivially on G^{ab} and G^{ab} acts on G' induced by conjugation in G, that is, $c^{G'g} = c^g$, for all $c \in G'$ and $g \in G$). Thus, Proposition A gives

$$G' \otimes_{\mathbb{Z}[G^{ab}]} I(G^{ab}) \cong [G', G^{ab}] = [G', H]. \qquad (6)$$

Now it follows from the proof of [12, Proposition 3.5] that in $\nu(G)$

$$[G, G^{\varphi}] = [G', (G')^{\varphi}][G', H^{\varphi}][H, (G')^{\varphi}][H, H^{\varphi}]$$

where $[H, H^{\varphi}] \cong H \otimes H$. However, by Lemma 3, $[H, (G')^{\varphi}] = [G', H^{\varphi}]$ and consequently

$$[G\,,\,G^{\varphi}] = [G'\,,\,(G')^{\varphi}][G'\,,\,H^{\varphi}][H\,,\,H^{\varphi}]. \tag{7}$$

Since G' and H are abelian, we have $[G', (G')^{\varphi}][H, H^{\varphi}] \subseteq J(G)$, so that

$$[G', H] = \kappa([G', H^{\varphi}]) = \kappa([G, G^{\varphi}]) = G'.$$

This, together with (6), yields

$$G' \otimes_{\mathbb{Z}[G^{ab}]} I(G^{ab}) \cong G'. \tag{8}$$

From (4), (5) and (8) it follows that

$$|G \otimes G| = |[G, G^{\varphi}]| = n\,|G'| \cdot \left|G^{ab} \otimes_{\mathbb{Z}} G^{ab}\right| \tag{9}$$

where n is a divisor of $|\mathrm{M}(G')|$. Using (9) and sequence (2) we obtain

$$|J(G)| = n|G^{ab} \otimes_{\mathbb{Z}} G^{ab}|.$$

Let us show that $n = |\mathrm{M(G')^H}|$, where $\mathrm{M}(G')^H$ denotes the H-stable subgroup of $\mathrm{M}(G')$ (see [6] for an overview). We observe that $\mathrm{M}(G) \cong J(G)/\Delta(G)$. Now by Lemma 3 (ii), $\Delta(G) = \langle [h, h^{\varphi}] \mid h \in H\rangle \subseteq [H, H^{\varphi}]$. Considering that $[H, H^{\varphi}] \cong H \otimes H \cong G^{ab} \otimes_{\mathbb{Z}} G^{ab}$ and H is abelian, we have

$$|\mathrm{M}(G)| = n\left|\frac{[H, H^{\varphi}]}{\langle [h, h^{\varphi}] \mid h \in H\rangle}\right| = n\,|H \wedge H| = n\mathrm{M}(H). \tag{10}$$

On the other hand, since the orders of G' and H are coprimes, from [6, Corollary 2.2.6]

$$M(G) \cong \mathrm{M}(H) \times \mathrm{M}(G')^H. \tag{11}$$

The required equalities then follow by (10) and (11). □

Corollary 4. *Let G be a group as given in Proposition C. If $M(G') = 1$, then*

(i) $G \otimes G \cong G' \times (G^{ab} \otimes_{\mathbb{Z}} G^{ab})$;

(ii) $J(G) \cong G^{ab} \otimes_{\mathbb{Z}} G^{ab}$.

Proof. If $M(G') = 1$ then previous result yields $|J(G)| = \left|G^{ab} \otimes_{\mathbb{Z}} G^{ab}\right|$. But, according to the proof of Proposition C, $J(G)$ contains $[H, H^{\varphi}]$, which is isomorphic to $G^{ab} \otimes_{\mathbb{Z}} G^{ab}$. Hence

$$J(G) = [H, H^{\varphi}] \cong G^{ab} \otimes_{\mathbb{Z}} G^{ab}$$

This proves part (ii). Part (i) follows from (ii) and the central extension (2). □

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