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On arrangement of subgroups in groups and related topics: some recent developments

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To my friend Leonid Kurdachenko on the occasion of his 60-th birthday

Abstract. Investigation of groups satisfying certain conditions, related to the subgroup arrangement, enabled algebraists to introduce and describe many important classes of groups. The roots of such investigations lie in the works by P. Hall, R. Carter, J. Rose, and Z. Borevich. Numerous interesting results in this area have been obtained lately by many authors. The main goal of this survey is to reflect some important new developments in study of subgroup arrangement in infinite groups.

1. Introduction

Subgroups of a group allow a wide variety of arrangements. We can mention among them their pairwise dispositions and their dispositions relatively to the group. Investigation of groups satisfying sertain conditions, related to the subgroup arrangement, enabled algebraists to introduce and describe many important classes of groups. The roots of such investigations lie in the works by P. Hall, R. Carter, J. Rose. The pronormal, contranormal, abnormal, and Carter subgroups introduced by them play a key role here. Recall that a subgroup H of a group G is said to be *pronormal* in G if for every $g \in G$ the subgroups H and H^g are conjugate in the subgroup $\langle H, H^g \rangle$. These subgroups have been introduced by P. Hall [22]. Such important subgroups of finite (soluble) groups as

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Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups are pronormal. A subgroup H of a group G is said to be *abnormal* in G if $g \in \langle H, H^g \rangle$ for each element $g \in G$. These subgroups have been introduced by P. Hall [22], although the specific term *abnormal subgroup* is due to R. Carter [12]. Based on these definitions, J. Rose considered balanced chains of subgroups in a group and contranormal subgroups [56]. Later, Z.I. Borevich and his students in the frame of the theory of fan subgroups [1] developed by them introduced new generalizations of the above mentioned subgroups, such as polynormal, paranormal, weakly pronormal, and weakly abnormal subgroups. It is important to note that if his predecessors considered only finite groups, Z. I. Borevich did not limit his observation to the finite case. Almost all the definitions (except of the Carter subgroups) have no limitation of finiteness. However, in the infinite case, consideration of arrangement of subgroups is naturally much more diverse and complicated. Of course, there is no unique universal class of infinite groups on which the transfer of all these concepts is possible. Moreover, very frequently this extending can be only realized for some distinct and weakly related classes of infinite groups. Of course, in each particular case the choice of such a class is supposed to be determined by not only convenience, but mostly by logic.

The main goal of the current survey is to reflect some of the important new developments in study of subgroup arrangement in infinite groups.

2. Families of fan subgroups and arrangement of subgroups

Let G be a group and G_0 its subgroup. A subgroup H is called *intermediate to* G_0 if $G_0 \leq H \leq G$ [1]. If G_0 is a normal subgroup, then, by the theorem on homomorphisms, the intermediate subgroups H are in a natural bijective correspondence with subgroups of the factor group G/G_0. Z.I. Borevich and his students studied the lattices of all subgroups intermediate to a fixed subgroup G_0. Their goal was to generalize the theorem on homomorphisms on some non-normal subgroups [7, 10, 8, 18, 1]. The following concept belongs to Z.I. Borevich [1].

Suppose G is a group and G_0 a subgroup, $G_0 \leq G$. A system $\{G_\alpha\}_{\alpha \in I}$ (I is an index set) of intermediate to G_0 subgroups is called a fan for G_0 if for each intermediate subgroup H there exists an unique index $\alpha \in I$ such that

$$G_\alpha \leq H \leq N_\alpha$$

where $N_\alpha = N_G(G_\alpha)$ is the normalizer of G_α in the group G.

The factor-groups N_α/G_α are called *sections of this fan.*

Subgroups G_α are called *a basis subgroups* of the fan; they form *the basis of the fan.*

If there exists a fan G_0, then G_0 is called *a fan subgroup of* G.

Each normal subgroup G_0 of G, $G_0 \trianglelefteq G$, is a fan subgroup. In this case, the fan consists of this subgroup G_0 only and has the unique section G/G_0. An opposite example of fan subgroups is provided by a subgroup G_0 for which every intermediate subgroup H coincides with its own normalizer in G, i. e. $N_G(H) = H$. In this case, the fan is the set of all intermediate to G_0subgroups, and all sections are trivial. Abnormal subgroups are examples of such subgroups. So the definition of fan subgroups merges these two polar opposite concepts of normal and abnormal subgroups. It is not difficult to observe that pronormal subgroups are fan subgroups.

Let D be a subgroup of group G. If D is a fan subgroup in G and the intermediate subgroups to a subgroup D satisfy the minimum condition, then there exists only one fan for D [1]. In particular, the uniqueness of the fan holds for finite groups. Simple examples show that this statement is not true for infinite groups. However some subgroups such as pronormal and abnormal subgroups always have a unique fan [1]. They play an important role in arrangement of subgroups [7, 10, 8, 18, 1].

Consider now some classes of groups whose subgroups are fan subgroups.

Obviously, in a Dedekind group G all subgroups form a fan and the fan basis of any subgroup H consists of H itself.

T.A. Peng [49] described soluble finite groups whose all subgroups are pronormal. He proved that finite soluble groups whose subgroups are pronormal are exactly the finite soluble groups whose subgroups satisfy the transitivity of normality.

Recall that a group G is said to be *a* $T-$*group* if every subnormal subgroup of G is normal. A group G is said to be a $\bar{T}$ – *group*, if every subgroup of G is a $T-$group. E. Best and O. Taussky have introduced these groups in [9]. Finite soluble $T-$groups have been described by W. Gaschütz [20]. In particular, he found that every finite soluble $T-$group is a $\bar{T}-$group. Infinite soluble $T-$groups and $\bar{T}$ – groups have been studied by D.J.S. Robinson [51]. A locally soluble $\bar{T}$ – group G has the following structure. If G is not periodic, then G is abelian. If G is periodic and L is the locally nilpotent residual of G, then G/L is a Dedekind group, $\pi(L) \cap \pi(G/L) = \varnothing, 2 \notin \pi(L)$, and every subgroup of L is $G-$invariant. In particular, if $L \neq \langle 1 \rangle$, then $L = [L, G]$.

The following theorem due to N.F. Kuzennyi and I.Ya. Subbotin generalizes this Peng's result on infinite groups.

Theorem [37] *Suppose that G is a locally soluble group or a periodic locally graded group. Then the following conditions are equivalent.*

1. Every cyclic subgroup of G is pronormal in G.

2. G is a soluble T-group.

Infinite groups whose subgroups are pronormal firstly have been considered in [36]. The authors completely described such infinite locally soluble non-periodic and infinite locally graded periodic groups. The main result of that paper is the following theorem.

Theorem [36] *Let G be a group whose subgroups are pronormal, and L be a locally nilpotent residual of G.*

(i) If G is periodic and locally graded, then G is a soluble $\bar{T}$– group, in which L complements every Sylow $\pi(G/L)$– subgroup.

(ii) If G is non periodic and locally soluble, then G is abelian.

Conversely, if G has a such structure, then every subgroup of G is pronormal in G.

In the paper [54], the assertion (ii) has been extended to non – periodic locally graded groups. It was proved that in this case, such groups are also abelian.

N.F. Kuzennyi and I. Ya. Subbotin completely described locally graded periodic groups in which all primary subgroups are pronormal [39] and infinite locally soluble groups in which all infinite subgroups are pronormal [37]. They showed that in the infinite case, the class of groups whose all subgroups are pronormal is a proper subclass of the class of groups with the transitivity of normality. Moreover, it is also a proper subclass of the class of groups whose primary subgroups are pronormal. However, the pronormality condition for all subgroups can be weakened to the pronormality for only abelian subgroups [40].

We obtain a natural extension of Dedekind group if we suppose that there exists a fixed subgroup $M(G)$ such that every non-invariant subgroup $D \leq G$ has a fan whose basis is the set $\{D, M(G)\}$. In case $M = G$, the following theorem has been obtained.

Theorem [41] *A group G is a group whose each subgroup D has a fan with a basis that is a subset of the set $\{D, G\}$ if and only if G is either an infinite non-locally graded two generated group whose all proper subgroups are Dedekind groups, or G is a group of one of the following kinds:*

1) G is a Dedekind group;

2) G is a finite non-Dedekind Miller-Moreno group (i.e. a non-ableian group whose all proper subgroups are abelian);

3) G is a finite Schmidt group (i.e. a non-nilpotent group whose all proper subgroups are nilpotent) whose invariant factor is a quaternion group;

4) G is a generalized quaternion group of order 16.

In the case where $M(G) \neq G$ we can state the following result.

Theorem [41] *Suppose that a group G contains a fixed subgroup $M(G) \neq G$ such that each non-normal subgroup $D < G$ has a fan with the basis $\{D, M(G)\}$. Then G is a group of one of the following kinds:*

1) $G = (\langle a\rangle \times K) \leftthreetimes \langle b\rangle$, where K is Prüfer p-group, $a^{p^n} = b^{p^m} = 1$, $[G,G] = \langle k_0\rangle$, $k_0 \in K, k_0^p = 1$, $[b,K] = I, [b,a] = k_0$;

2) $G = (\langle a\rangle \times \langle k\rangle) \leftthreetimes \langle b\rangle$, $a^{p^n} = b^{p^m} = k^{p^r} = 1$, $[b,k] = 1$, $[a,b] = k^{p^{r-1}}$, $n \geq m \geq 1, r \geq n+1$, $expG \geq 8$;

3) $G = \langle k\rangle \leftthreetimes \langle b\rangle, b^{p^m} = k^{p^r} = 1, [k,b] = k^{p^f}, 2f \geq r, f \geq m, expG \geq 8$;

4) $G = (\langle a\rangle \times \langle b\rangle) \langle k\rangle$, $a^{p^n} = b^{p^m} = k^{p^r} = 1, [a,b] = a^{p^{n-1}}, [k,a] = [b,k] = 1, n \geq 2, m \geq 1, r \geq m+1, expG \geq 8$.

The converse statement is also true.

Another extreme case was considered in the following theorem.

Theorem [41] *Each non-invariant subgroup of a group G has a fan whose basis consists of all intermediate subgroups if and only if one of the following statements is true:*

1) G is an infinite group whose every non-identity subgroup is self-normalizing;

2) G is an infinite p-group with non-trivial cyclic center whose all non-central subgroups are selfnormalizing and all non-invariant cyclic subgroups contain the center $\zeta(G)$ of G as a maximal subgroup;

3) G is a Dedekind group;

4) G is a periodic group $G = A \leftthreetimes \langle b\rangle$ where A is an abelian Hall subgroup whose every subgroup is normal in $G, 2 \notin \pi(A), A = [G,G]$, $\langle b\rangle$ is a cyclic p-subgroup, $\zeta(G) \geq \langle b^p\rangle$.

It is important to note that Theorems 28.1 and 31.8 from [48] provide us with sophisticated examples of groups of types 1) and 2) respectively.

Consider some other examples of fan subgroups.

Let G be a group and D be its subgroup. An intermediate subgroup $F, D < F < G$, is called *a complete intermediate subgroup* if the normal closure D^F of D in F coincides with F.

A subgroup D is called *a polynormal* subgroup in a group G if for any $x \in G$ the subgroup $D^{<x>} = \langle D^x \mid x \in \langle x\rangle\rangle$ is a complete intermediate subgroup [1].

From the fan point of view, these concepts could be characterized in the following way [1].

A subgroup D is polynormal in group G if and only if it is a fan subgroup and all complete intermediate subgroups form a system of basis subgroups of its fan.

A subgroup D is abnormal in group G if and only if

1) D is a fan subgroup and its fan basis consists of all intermediate subgroups, and

2) any two intermediate conjugate subgroups coincide.

A subgroup D is pronormal in G if and only if

1) D is a fan subgroup and its fan basis consists of D and all subgroups of group G, which strictly contain the normalizer $N_G(D)$; and

2) any such two conjugate subgroups coincide.

The subgroups mentioned above and their generalizations are very useful in finite group theory. As usual, the situation in infinite groups is significantly different from the situation in the corresponding finite case. In infinite groups, these subgroups gain some properties they cannot posses in the finite case. For example, it is well-known that every finite p–group has no proper abnormal subgroups. Nevertheless, A.Yu. Olshanskii has constructed a series of examples of infinite finitely generated p–groups saturated with abnormal subgroups. Specifically, for a sufficiently large prime p there exists an infinite $p-$group G whose all proper subgroups have prime order p [48, Theorem 28.1].

In a certain sense, abnormal subgroups are some antipodes of normal subgroups. Thus, in finite soluble groups, abnormality is tightly bound to self - normalizing. For example, D. Taunt has shown that *a subgroup H of a finite soluble group G is abnormal if and only if every intermediate subgroup for H coincides with its normalizer in G*; that is, such a subgroup is self-normalizing (see, for example, [52, 9.2.11]).

The following theorem extends this result to the radical groups.

Theorem [33] *Let G be a radical group and let H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

The following results are straightforward consequences of this theorem.

Corollary [19] *Let G be a hyperabelian group and let H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

Corollary *Let G be a soluble group and let H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

We recall that a subgroup H of a group G is said to have *the Frattini property,* if given two intermediate subgroups K and L for H such that $K \trianglelefteq L$, we have $L \leq N_G(H)K$ (in this case, it is also said that H is *weakly pronormal* in G). T.A. Peng in his paper [50] characterized pronormal subgroups in finite soluble groups. He has proved that *a subgroup H of a finite soluble group G is pronormal if and only if H is weakly pronormal.* This Peng's characterization of pronormal subgroups could be extended in the following way.

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is said to be *a hyper–$\mathfrak{X}$–group* if G has an ascending series of normal subgroups whose factors are $\mathfrak{X}$–groups.

Theorem [24] *Let G be a hyper–N–group. Then a subgroup H of G is pronormal in G if and only if H is weakly pronormal in G.*

This result has two immediate corollaries.

Corollary [19] *Let G be a hyperabelian group and let H be a subgroup of G. Then H is pronormal in G if and only if H is weakly pronormal in G.*

Corollary *Let G be a soluble group and let H be a subgroup of G. Then H is pronormal in G if and only if H is weakly pronormal in G.*

Carter subgroups are abnormal subgroups. In the finite group theory, these subgroups have been introduced by R. Carter [12] as the self-normalizing nilpotent subgroups. Some attempts of extending the notion of a Carter subgroup to infinite groups were made by S.E. Stonehewer [58, 59], A.D. Gardiner, B. Hartley and M.J. Tomkinson [17], and M.R. Dixon [15]. In [33], this concept have been extended to the class of nilpotent–by–hypercentral (not necessary periodic) groups.

Taking in account that we may define a Carter subgroup of a finite metanilpotent group as a minimal abnormal subgroup, the first logical step here is to consider artinian–by–hypercentral groups whose locally nilpotent residual is nilpotent. Here we have the following results.

Theorem [33] *Let G be an artinian–by–hypercentral group and suppose that its locally nilpotent residual K is nilpotent.*

1. *G has a minimal abnormal subgroup L. Moreover, L is maximal hypercentral subgroup, and it includes the upper hypercenter of G. In particular, $G = KL$.*

2. *Two minimal abnormal subgroups of G are conjugate.*

Theorem [33] *Let G be an artinian–by–hypercentral group and suppose that its locally nilpotent residual K is nilpotent.*

1. *G has a hypercentral abnormal subgroup L. Moreover, L is a maximal hypercentral subgroup, and it includes the upper hypercenter of G. In particular, $G = KL$; and*

2. *Two hypercentral abnormal subgroups of G are conjugate.*

Thus, given an artinian–by–hypercentral group with a nilpotent hypercentral residual, *a subgroup L is called a Carter subgroup of a group G if H is a hypercentral abnormal subgroup of G or equivalently, if H is a minimal abnormal subgroup of G.*

A Carter subgroup of a finite soluble group can be also characterized as *a covering subgroup for the formation of nilpotent groups*. In the paper [33], this definition was extended to the class of artinian–by–hypercentral groups with a nilpotent locally nilpotent residual.

Let G be a group, H be a subgroup of G and X be a subset of G. Put

$$H^X = \langle h^x = x^{-1}hx \mid h \in H, x \in X \rangle .$$

In particular, H^G (the *normal closure* of H in G) is the smallest normal subgroup of G containing H. Following J.S. Rose [RJ1968], a subgroup H of a group G is called *contranormal*, if $H^G = H$.

A subgroup H of a group G is called nearly pronormal if $N_K(H)$ is contranormal in every subgroup $K \geq H$, including H. In the paper [29], the groups whose subgroups are nearly pronormal have been considered.

Theorem [29] *Let G be a locally radical group.*

(i) If every cyclic subgroup of G is nearly pronormal, then G is a $\bar{T}$–group.

(ii) If every subgroup of G is nearly pronormal, then every subgroup of G is pronormal in G.

It seems logical to describe the groups whose all proper non-normal subgroups are abnormal. It means that all proper subgroups of such groups form two classes with an empty intersection: the class of normal subgroups and the class of abnormal subgroups. Following [31], we will call normal and abnormal subgroups U-normal (from "union" and "U-turn"). Finite groups with only U-normal subgroups have been considered in [16]. Locally soluble (in the periodic case locally graded) infinite groups with U-subgroups have been studied in [60]. In [31], the groups with all U-normal subgroups and the groups with transitivity of U-normality were completely described.

It is a natural question regarding the structure of *groups whose U-normal subgroups form a lattice.* These groups are denoted as *$\#U$-groups* [35]. It is easy to see that the groups with no abnormal subgroups are $\#U$-groups. In particular, all locally-nilpotent groups have this property [38].

Observe that a union of any two U-normal subgroups is U-normal. However, the similar assertion is obviously false for intersections.

It is easy to see that in a soluble group an abnormal subgroup R is exactly the subgroup that is contranormal in all subgroups containing R [13]. The condition "every contranormal subgroup is abnormal" (the CA-property) is an amplification of the transitivity of abnormality (the TA-property). Some simple examples show that the class of TA-groups is wider then the class of CA-groups and does not coincide with the class of $\#U$-groups.

Here is the description of soluble CA-groups having $\#U$-property.

Theorem [35] *Let a soluble CA-group G containing an abnormal proper subgroup be a $\#U$-group. Then $G = [G, G] \langle b\rangle$ and one of the following assertions holds.*

(1) b is an element of order $p^n, n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, and $\zeta(G) \leq \langle b^p\rangle \lhd G$.If the center $\zeta(G)$ is identity, then $b^p \in [G, G]$, i.e. $[G, G]$ has index p in G. If the center $\zeta(G)$ is non-identity and G is periodic, then $\zeta(G) = \langle b^p\rangle$.

(2) $|b| = \infty$ and there are a prime number p and a natural number n such that $b^{p^{n-1}} \in [G, G]$, but $b^{p^n} \notin [G, G]$; $\left\langle b^{p^n}\right\rangle$ is a normal subgroup of G defining the factor-group G^* with the property $p \notin \pi([G^*, G^*])$, and $\zeta(G) \leq \langle b^p\rangle \lhd G$.

If *the center $\zeta(G)$ is identity, then at $p \neq 2$, $b^p \in [G, G]$, and at $p = 2$, $|G : [G, G]| \leq 4$.*

Theorem [35] *A soluble periodic CA-group G containing an abnormal proper subgroup is a $\#U$-group if and only if $G = [G, G] \langle b\rangle$, every abnormal subgroup B of G intersects $[G, G]$ by a normal in $[G, G]$ subgroup, b is an element of order $p^n, n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, $\zeta(G) \leq \langle b^p\rangle \lhd G$.*

Moreover, the following assertions hold.

(1) If the center $\zeta(G)$ is identity, then $b^p \in [G, G]$, i.e. $[G, G]$ has index p in G.

(2) If the center $\zeta(G)$ is non-identity then $\zeta(G) = \langle b^p\rangle$.

In both mentioned cases $|G : [G, G]\zeta(G)| = p$.

If G is a finite group, then for each subgroup H there is a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots . \leq H_{n-1} \leq H_n = G$$

such that H_j is maximal in H_{j+1}, $0 \leq j \leq n-1$. Generalizing this, J. Rose has arrived at the *balanced chain* connecting a subgroup H to a group G, that is, a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots . \leq H_{n-1} \leq H_n = G$$

such that for each j, $0 \leq j \leq n-1$, either H_j is normal in H_{j+1}, or H_j is abnormal in H_{j+1}; the number n is *the length* of this chain. He refers appropriately to two consecutive subgroups $H_j \leq H_{j+1}$ as forming a *normal link* or an *abnormal link* of this chain [55]. In a finite group, every subgroup can be connected to the group by some balanced chain.

It is natural to consider the case when all of these balanced chains are short, i.e. their lengths are bounded by a small number. If these lengths are ≤ 1, then every subgroup is either normal or abnormal in a group. Such finite groups were studied in [16]. Infinite groups of this kind and some of their generalizations were described in [60] and [13]. Observe that in the groups in which the normalizer of any subgroup is abnormal and in the groups in which every subgroup is abnormal in its normal closure, the mentioned lengths are ≤ 2. It is logical to choose these groups as the subject for investigation.

It is interesting to observe that if G is a soluble $\overline{T}$-group, then every subgroup of G is abnormal in its normal closure. As we mentioned above, for any pronormal subgroup H of a group G, the normalizer $N_G(H)$ is an abnormal subgroup of G. So the subgroups having abnormal normalizers make a generalization of pronormal subgroups. There are examples showing that this generalization is non-trivial.

The article [30] initiated the study of groups whose subgroups are connected to a group by balanced chains of length at most 2. As we recently mentioned, such groups are naturally related to the T-groups. Perhaps, the following simple but important characterization of T-groups is one of the reasons for this: *a group G is a T-group if and only if for every $x \in G$ the equation $x^{x^G} = x^G$ is true* [51].

The following new characterizations of $\overline{T}$-groups are obtained in this passing.

Theorem [30] *Let G be a radical group. Then G is a $\overline{T}$-group if and only if every cyclic subgroup of G is abnormal in its normal closure.*

Theorem [30] *Let G be a periodic soluble group. Then G is a$\overline{T}$–group if and only if its locally nilpotent residual L is abelian and the normalizer of each cyclic subgroup of G is abnormal in G.*

The following theorem is a new interesting and useful characterization of groups with all pronormal subgroups.

Theorem [30] *Let G be a periodic soluble group. Then every subgroup of G is pronormal if and only if its locally nilpotent residual L is abelian and the normalizer of every subgroup of G is abnormal in G.*

For the non-periodic case, there exist non-periodic non-abelian groups, in which normalizers of all subgroups are abnormal [30]. On the other hand, the non-periodic locally soluble groups in which all subgroups are pronormal are abelian [36]. So, in the non-periodic case we cannot count on a characterization, similar to above. However, we have the following result.

Theorem [30] *Let G be a non-periodic group with the abelian locally nilpotent residual L. If the normalizer of every cyclic subgroup is abnormal and for each prime $p \in \Pi(L)$ the Sylow p-subgroup of L is bounded, then G is abelian.*

3. Fan subgroups and transitivity of some subgroup properties

As we have seen above, pronormality is strongly connected to transitivity of normality. In this passing, it is worthy to mention that groups with transitivity of pronormality, abnormality and other connected to these subgroup properties have been studied by L.A. Kurdachenko, I.Ya. Subbotin, and J. Otal (see, [32], [31], and [24]).

We denote groups, in which pronormality is transitive by TP-groups and a group in which all subgroups are $TP-$groups by a $\bar{T}P-$group.

Theorem [32] *Locally soluble $\bar{T}P$-groups become exhausted with the locally soluble groups in which all subgroups are pronormal.*

Theorem [32] *A periodic soluble group G is a TP-group if and only if*

1. *G decomposes into a semidirect product $G = A \leftthreetimes (B \times P)$ where A and B are abelian Hall subgroups in G;*
2. $2 \notin \pi(A)$;
3. *P either an identity group or a 2-T-group;*
4. *the derived subgroup G' is an abelian quasicentral subgroup of G and decomposes into a direct product of A and the derived subgroup P' of P;*
5. *any Sylow $\pi(B \times P)-$subgroup of G complements A in G.*

Corollary [32] *Periodic soluble $TP-$groups become exhausted by groups of the following types.*

I. $G = A \leftthreetimes B$, where A is an abelian quasicentral Hall $2'$-subgroup, B is a Dedekind group, $G' = A \times B'$, and any Sylow $\pi(B)-$subgroup of G complements A in G.

II -V. $G = G_1 \leftthreetimes P$ where $G_1 = A \leftthreetimes B$ is a Hall normal subgroup of type I, P is a 2-group of one of the following types of groups:

1. $G = (C \leftthreetimes < z >) \times D$ where C is a divisible non-trivial abelian group, $expD \leq 2$, $z^2 = 1$ or $z^4 = 1$, $z^2 \in \zeta(G)$, $c^z = c^{-1}$ for each $c \in C$;

2. $G = (C \leftthreetimes < z, s >) \times D$ where C and D are same groups as in 1, $< z, s >$ is a quaternion group, $c^z = c^{-1}, c^s = s$, for each $c \in C$;

3. $G = (C \leftthreetimes K < s >) \times D$ where D is the same group as in 1, C is a divisible abelian group, K is a Prüfer 2-group, $z^4 = 1$, $z^2 \in K, [K, C] = 1$, $c^z = c^{-1}$, $k^z = k^{-1}$ for each $c \in C$ and $k \in K$.

In addition, $[P, B] = 1, G' = A \times P'$ and any Sylow $\pi(B \times P)$-subgroup of G complements A in G.

The following two theorems complete the description of soluble TP-groups.

Theorem [32] *Let G be a soluble group with a non-periodic centralizer $C = C_G(G')$. Then G is a TP-group if and only if it is a T-group.*

Theorem [32] *Let G be a soluble non-periodic group with a periodic centralizer of the derived group. The group G is a $TP-$group if and only if it is a hypercentral $T-$group.*

In this setting, it is interesting to mention the following, most general yet, result on transitivity of abnormal subgroups.

Theorem [33] *Let G be a group and suppose that A is a normal subgroup of G such that G/A has no proper abnormal subgroups. If A satisfies the normalizer condition, then abnormality is transitive in G.*

Recall the following interesting property of pronormal subgroups:

Let G be a group, H, K be the subgroups of G and $H \leq K$. If H is a subnormal and pronormal subgroup in K, then H is normal in K.

We say that a subgroup H of a group G *is transitively normal* if H is normal in every subgroup $K \geq H$ in which H is subnormal [34]. In [47], these subgroups have been introduced under a different name. Namely, a subgroup H of a group G is said to satisfy *the subnormalizer condition* in G if for every subgroup K such that H is normal in K we have $N_G(K) \leq N_G(H)$.

We say that a subgroup H of a group G is *strong transitively normal*, if HA/A is transitively normal for every normal subgroup A of the group G [34]. Since the homomorphic image of pronormal subgroup is pronormal, we can conclude that every pronormal subgroup is a strong transitively normal subgroup.

Theorem [34] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble subgroup R such that G/R is hypercentral. If H is strong transitively normal in G and R satisfies $Min - H$, then H is a pronormal subgroup of G.*

Corollary [34] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble Chernikov subgroup R such that G/R is hypercentral. If H is strong transitively normal in G, then H is a pronormal subgroup of G. In particular, if G is a soluble Chernikov group and H is a hypercentral strong transitively normal subgroup of G, then H is pronormal in G.*

Corollary [34] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble subgroup R such that G/R is hypercentral. If H is a polynormal subgroup of G and R satisfies $Min - H$ (in particular, if R is Chernikov), then H is pronormal in G.*

Corollary [47] *Let G be a soluble finite group, H be a nilpotent subgroup of G. If H is a polynormal subgroup of G, then H is a pronormal subgroup of G.*

A subgroup H is said to be *paranormal* in a group G if H is contranormal in $\langle H, H^g\rangle$ for all elements $g \in G$ (M.S. Ba and Z.I. Borevich [1]). Every pronormal subgroup is paranormal, and every paranormal subgroup is polynormal [1]. Thus we have

Corollary [34] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble subgroup R such that G/R is hypercentral. If H is a paranormal subgroup of G and R satisfies $Min-H$ (in particular, if R is a Chernikov group), then H is pronormal in G.*

Corollary [34] *Let G be a soluble finite group, H be a nilpotent subgroup of G. If H is a paranormal subgroup of G, then H is a pronormal subgroup of G.*

Theorem [34] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal nilpotent subgroup R such that G/R is hypercentral. If H is transitively normal in G and R satisfies $Min - H$ (in particular, if R is Chernikov), then H is a pronormal subgroup of G.*

Corollary [34] *Let G be a nilpotent – by – hypercentral Chernikov group, H be a hypercentral subgroup of G. If H is transitively normal in G, then H is a pronormal subgroup of G.*

Corollary [50] *Let G be a nilpotent – by – abelian finite group, H be a nilpotent subgroup of G. If H is transitively normal in G, then H is a pronormal subgroup of G.*

A subgroup H of a group G is called *weakly normal* if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ (K.H. Müller [46]). We note that every pronormal subgroup is weakly normal [2], every weakly normal subgroup satisfies the subnormalizer condition [2], and hence it is transitively normal in G. Thus we have

Corollary [34] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal nilpotent subgroup R such that G/R is hypercentral. If H is weakly normal in G and R satisfies* $\mathbf{Min} - H$ *(in particular, if R is a Chernikov group), then H is a pronormal subgroup of G.*

A subgroup H of a group G is called *an $\mathfrak{H}$– subgroup* if $N_G(H) \cap H^g \leq H$ for all elements $g \in G$ [6]. Note that every $\mathfrak{H}$– subgroup is transitively normal [6]. Therefore, we obtain

Corollary [34] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal nilpotent subgroup R such that G/R is hypercentral. If H is an $\mathfrak{H}$– subgroup of G and R satisfies* $\mathbf{Min} - H$ *(in particular, if R is a Chernikov group), then H is a pronormal subgroup of G.*

Some properties of transitively normal subgroups (under another name) in $FC-$groups have been considered in the paper [14], which in particular, contains the following result.

Theorem [14] *Let G be an $FC-$group, H be a transitively normal subgroup of G. If H is a $p-$subgroup for some prime p, then H is a pronormal subgroup of G.*

A subgroup H of a group G is said to be *permutable* in G (or quasinormal in G), if $HK = KH$ for every subgroup K of G. This concept arises as a generalization of the normal subgroup. The study of the properties of the permutable subgroups is presented in the book [57]. According to a well – known theorem by E. Stonehewer, permutable subgroups are always ascendant. Therefore, it is natural to consider the opposite case: the groups whose ascendant subgroups are permutable. A group G is said to be *an $AP-$ group* if every ascendant subgroup of G is permutable in G. These groups are very close to the groups in which the property "to be a permutable subgroup" is transitive. A group G is said to *be a $PT-$group if permutability is a transitive relation in* G, that is, if K is a permutable subgroup of H and H is a permutable subgroup of G, then K is a permutable subgroup of G. The description of finite soluble $PT-$groups has been given by G. Zacher [61]. It looks close to the description of finite soluble $T-$groups due to W. Gaschütz [20]. Namely, if G is finite soluble group and L is a nilpotent residual of G, then every subgroup of G/L is permutable, $\pi(L) \cap \pi(G/L) = \varnothing, 2 \notin \pi(L)$ and every subgroup of

L is $G-$invariant. The soluble infinite $PT-$groups have been described by F. Menegazzo [44, 45].

Obviously, a finite group G is a $PT-$group if and only if every subnormal subgroup of G is permutable. In [53, Lemma 4], it is claimed that in an arbitrary $PT-$group every ascendant subgroup is permutable. A simple counterexample shows that this statement is incorrect. Let $G = A \rightthreetimes \langle b \rangle$ be a semidirect product of a Prüfer $2-$group A and a group $\langle b \rangle$ of order 2 that acts on A by $a^b = a^{-1}$ for each $a \in A$ (an infinite dihedral group). Clearly, G is hypercentral, and in particular, every subgroup of G is ascendant. However, it is not hard to see, that the subgroup $\langle b \rangle$ is not permutable. It means that for infinite groups the classes of $AP-$groups and $PT-$groups do not coincide.

The paper [4] initiated the study of infinite $AP-$groups.

Theorem [4] *Let G be a radical hyperfinite $AP-$group. Then the following assertions hold:*

(i) G is metabelian;

(ii) if R is the locally nilpotent radical of G, then $R = L \times Z$, where L is the locally nilpotent residual of G and Z is the upper hypercenter of G;

(iii) $\pi(L) \cap \pi(G/L) = \varnothing, 2 \notin \pi(L)$;

(iv) L is abelian and every subgroup of L is $G-$ invariant; and

(v) every subgroup of G/L is permutable (in particular, G/L is nilpotent).

Moreover, if the factor – group G/L is countable, then G splits over L.

On the other hand, if G is a periodic group having a normal abelian subgroup L that satisfies the conditions (iii) - (v), then G is an $AP-$group.

Note that this theorem describes a much wider class of groups. The following result justifies this.

Theorem [4] *Let G be a periodic $AP-$ group. If G is a hyper – N – group, then G is hyperfinite. In particular, G is a hypercyclic metabelian $AP-$ group.*

Corollary [4] *Let G be a periodic $AP-$ group. If G is a hyper – Gruenberg – group, then G is a hypercyclic metabelian $AP-$group.*

In particular, if G is a countable radical group, then G is a hypercyclic metabelian $AP-$ group.

Corollary [4] *Let G be a periodic $AP-$group. If G is residually soluble, then G is a hypercyclic metabelian $AP-$group.*

Let G be a group and let p be a prime. We say that G *belongs to the class* $\mathfrak{B}p$ if each Sylow $p-$subgroup P of G satisfies the following condition:(i) every subgroup of P is permutable in P; (ii) every normal subgroup of P is pronormal in G.

The following result from [5] shows the role of pronormal subgroups in the theory of $AP-$groups.

Theorem [5] *Let G be a periodic locally soluble group. If G belongs to the class $\mathfrak{B}p$ for all primes p, then G is a hypercyclic $AP-$ group. Moreover, if L is the locally nilpotent residual, then L has a complement in G.*

4. Pronormal, abnormal, contranormal subgroups and criteria of nilpotency

The following well-known characterizations of finite nilpotent groups are tightly bound to abnormal and pronormal subgroups.

A finite group G is nilpotent if and only if G has no proper abnormal subgroups.

A finite group G is nilpotent if and only if its every pronormal subgroup is normal.

Note that since the normalizer of a pronormal subgroup is abnormal, *the absence of abnormal subgroups is equivalent to the normality of all pronormal subgroups.* Recall also that if G *is a locally nilpotent group, then G has no proper abnormal subgroups and every pronormal subgroup of G is normal* [38]. However, we do not know whether or not the converse of this result holds.

In the paper [23], the following generalization of the well-known nilpotency criterion was obtained.

Theorem [23] *Let G be a generalized minimax group. If every pronormal subgroup of G is normal, then G is hypercentral.*

Let G be a group, A a normal subgroup of G. We say that A satisfies the condition Max–G (respectively Min–G) if A satisfies the maximal (respectively the minimal) condition for G–invariant subgroups. A group G is said to be a generalized minimax group, if it has a finite series of normal subgroups

$$\langle 1 \rangle = H_0 \leq H_1 \leq \ \cdots \ \leq H_n = G,$$

every factor of which is abelian and either satisfies Max–G or Min–G.

Every soluble minimax group is obviously generalized minimax. However, the class of generalized minimax groups is significantly wider than the class of soluble minimax groups.

Let G be a group. Then the set

$$FC(G) = \{x \in G \mid x^G \text{ is finite}\}$$

is a characteristic subgroup of G which is called *the FC–center of G.* Note that a group G is an FC–group if and only if $G = FC(G)$. Starting from the FC–center, we construct *the upper FC–central series of a group G*

$$\langle 1\rangle = C_0 \leq C_1 \leq \ \cdots \ \leq C_\alpha \leq C_{\alpha+1} \leq \ \cdots C_\gamma$$

where $C_1 = FC(G)$, $C_{\alpha+1}/C_\alpha = FC(G/C_\alpha)$ for all $\alpha < \gamma$, and $FC(G/C_\gamma) = \langle 1\rangle$.

The term C_α is called *the α–FC–hypercenter of G,* while the last term C_γ of this series is called *the upper FC–hypercenter of G.* If $C_\gamma = G$, then the group G is called *FC–hypercentral,* and, if γ is finite, then G is called *FC–nilpotent.*

The following criteria of hypercentrality have been obtained in [28].

Theorem [28] *Let G be a group whose pronormal subgroups are normal. Then every $FC-$*hypercenter *of G having finite number is hypercentral.*

Theorem [28] *Let G be an FC–nilpotent group. If all pronormal subgroups in G are normal, then G is hypercentral.*

Theorem [28] *Let G be a group whose pronormal subgroups are normal. Suppose that H be an FC- hypercenter of G having finite number. If C is a normal subgroup of G such that $C \geq H$ and C/H is hypercentral, then C is hypercentral.*

For periodic groups, the above results were obtained in [31].

Observe that abnormal subgroups are an important particular case of contranormal subgroups: abnormal subgroups are exactly the subgroups that are contranormal in each subgroup containing them. Recall that abnormal subgroups are also a particular type of pronormal subgroups.

Pronormal subgroups are connected to contranormal subgroups in the following way. If H is a pronormal subgroup of a group G and $H \leq K$, then its normalizer $N_K(H)$ in K is an abnormal and hence contranormal subgroup of K.

Starting from the normal closure of H, we can construct the *normal closure series of H in G*

$$H^G = H_0 \geq H_1 \geq ...H_\alpha \geq H_{\alpha+1} \geq ...H_\gamma$$

by the following rule: $H_{\alpha+1} = H^{H_\alpha}$ for every $\alpha < \gamma$, $H_\lambda = \bigcap\limits_{\mu<\gamma} H_\mu$ for a limit ordinal λ. The term H_α of this series is called the *α-th normal* *closure of* H *in* G and will be denoted by $H^{G,\alpha}$. The last term H_γ of this series is called *the lower normal closure of* H *in* G and will be denoted by $H^{G,\infty}$. Observe that every subgroup H is contranormal in its lower normal closure.

In finite groups, the subgroup $H^{G,\infty}$ is called *the subnormal closure* of H in G. The rationale for this is the following. In a finite group G, the normal closure series of every subgroup H is finite, and $H^{G,\infty}$ is the smallest subnormal subgroup of G containing H. A subgroup H *is called descendant in* G if H coincides with its lower normal closure $H^{G,\infty}$. An important particular case of descendant subgroups are subnormal subgroups. A subnormal subgroup is exactly a descending subgroup having finite normal closure series. These subgroups strongly affect structure of a group. For example, it is not hard to prove that if every subgroup of a locally (soluble – by – finite) group is descendant, then this group is locally nilpotent. If every subgroup of a group G is subnormal, then, by a remarkable result due to W. Möhres [42], G is soluble. Subnormal subgroups have been studied very thoroughly for quite a long period of time. We are not going to consider this topic here since it has been excellently presented in the survey of C. Casolo [11]. However, we need to admit that, with the exception of subnormal subgroups, we have no significant information regarding descendant subgroups. The next results connect the conditions of generalized nilpotency to descendant subgroups.

Theorem [31] *Let* G *be a group, every subgroup of which is descendant. If* G *is* FC*–hypercentral, then* G *is hypercentral.*

Theorem [24] *Let* G *be a generalized minimax group. Then every subgroup of* G *is descendant if and only if* G *is nilpotent.*

Theorem [3] *Let* G *be a group, every subgroup of which is descendant. If* G *is a radical group with Chernikov Sylow* p *– subgroups for all primes* p*, then* G *is hypercentral and the center of* G *includes the divisible part of* G*.*

If every subgroup of a group G is descendant, then G does not include proper contranormal subgroups, and, in particular, proper abnormal subgroups. On the other hand, if G is a locally nilpotent group, then G does not include proper abnormal subgroups [37]. As we mentioned above, some classes of groups without abnormal subgroups have been described (see the survey [25]).

The study of groups without contranormal subgroups is the next logical step. We observe that every non – normal maximal subgroup of an arbitrary group is contranormal. Since a finite group whose maximal subgroups are normal is nilpotent, we come to the following criterion of nilpotency of finite groups in terms of contranormal subgroups:

A finite group G is nilpotent if and only if G does not include proper contranormal subgroups.

However, in the general case, this property cannot serve as a characterization of contranormal subgroups. There exist non – nilpotent groups whose every subgroup is subnormal [21]. Note that simple examples show that there exist some locally nilpotent groups with contranormal subgroups. However, for some classes of infinite groups the absence of contranormal subgroups implies nilpotency of a group. Some of these classes have been considered in the recent articles [26, 27].

Theorem [26] *Let G be group and H be a normal soluble – by – finite subgroup such that the factor – group G/H is nilpotent. Suppose that H satisfies the minimal condition on $G-$invariant subgroups $(Min-G)$. If G has no proper contranormal subgroups, then G is nilpotent.*

We observe that an analog of this Theorem for the maximal condition on G-invariant subgroups (the condition $\mathbf{Max} - G$) is not valid. In the paper [26], a corresponding counterexample has been constructed.

Let G be a group and let A be an infinite normal abelian subgroup of G. We say that A is a $G-$*quasifinite subgroup*, if every proper $G-$invariant subgroup of A is finite. This means that either A includes a proper finite $G-$invariant subgroup B such that A/B is $G-$simple, or A is an union of all finite proper $G-$invariant subgroups.

Corollary [26] *Let G be a polynilpotent group satisfying minimal condition for normal subgroups. If G has no proper contranormal subgroups, then G is nilpotent.*

Corollary [27] *Let G be a group and H be a normal Chernikov subgroup. Suppose that G/H is nilpotent. If G has no proper contranormal subgroups, then G is nilpotent.*

Corollary [26] *Let G be group and H be a normal subgroup such that the factor – group G/H is nilpotent. Suppose that H has a finite series of $G-$invariant subgroups*

$$\langle 1\rangle = C_0 \leq C_1 \leq ... \leq C_n = H$$

whose factors $C_j/C_{j-1}, 1 \leq j \leq n$, satisfy one of the following conditions:

(i) C_j/C_{j-1} is finite;

(ii) C_j/C_{j-1} is hyperabelian and minimax;

(iii) C_j/C_{j-1} is hyperabelian and finitely generated;

(iv) C_j/C_{j-1} is abelian and satisfies $Min - G$.

If G has no proper contranormal subgroups, then G is nilpotent.

Corollary [27] *Let G be a group and let C be a normal subgroup of G such that G/C is nilpotent. Suppose that C is a hyperabelian finitely generated subgroup. If G has no proper contranormal subgroups, then G is nilpotent.*

In particular, if G is hyperabelian finitely generated group without proper contranormal subgroups, then G is nilpotent.

Recall that a group G has *finite section rank* if every elementary abelian $p-$ section of G is finite for all prime p.

Theorem [27] *Suppose that the group G includes a normal $G-$ minimax subgroup C such that G/C is a nilpotent group of finite section rank. If G has no proper contranormal subgroups, then G is nilpotent.*

Following A.I. Maltsev [43], we say that a group G is *a soluble A_3- group* if it has a finite series of normal subgroups whose factors are abelian and either are Chernikov or torsion – free groups of finite $0-$rank.

Generalizing this notion we say that a group G is *a generalized A_3- group* if G has a finite series of normal subgroups

$$\langle 1\rangle = H_0 \leq H_1 \leq ... \leq H_n = G$$

every infinite factor H_j/H_{j-1} of which is abelian and satisfies one of the following conditions:

H_j/H_{j-1} is a torsion – free group of finite $0-$rank;

H_j/H_{j-1} satisfies the condition **Min** – G;

H_j/H_{j-1} satisfies the condition **Max** – G.

Corollary [27] *Let G be a generalized A_3-group. If G has no proper contranormal subgroups, then G is a nilpotent A_3-group.*

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