Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 4. **(2009).** pp. 78 – 96

On a question of Wiegold and torsion images of Coxeter groups

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Dedicated to Leonid Kurdachenko on the occasion of his 60th birthday

Abstract. We answer positively a question raised by Wiegold in Kourovka Notebook and show that every Coxeter group that is not virtually abelian and for which all labels in the corresponding Coxeter graph are powers of 2 or infinity can be mapped onto uncountably many infinite 2-groups which, in addition, may be chosen to be just-infinite, branch groups of intermediate growth.

1. Introduction

One of the most outstanding problems in Algebra known as the Burnside Problem (on periodic groups) was formulated by Burnside in 1902 and was later split into three branches: the General Burnside Problem, the Bounded Burnside Problem, and the Restricted Burnside Problem. The General Burnside Problem was asking if there exists an infinite finitely generated torsion group. It was answered positively by Golod in 1964 [13] based on Golod-Shafarevich Theorem [14]. The Bounded Burnside Problem was solved by S. P. Novikov and S. I. Adjan [37, 1]. The Restricted Burnside Problem was solved by E. Zelmanov [48, 49] as a corollary of his fundamental results on Lie and Jordan algebras. The problem of Burnside inspired a lot of activity and new directions of research. For solution of these problems, various constructions, and surveys we recommend [1,

The author expresses his thanks to T. Januszkiewicz, V. Nekrashevych, Z. Šunić, and I. Subbotin for their valuable remarks

2010 Mathematics Subject Classification: *20F50,20F55,20E08.*

Key words and phrases: *torsion groups, Coxeter groups, just-infinite groups, branch groups, groups of intermediate growth.*

2, 45, 38, 22, 16, 25, 28, 48, 49, 27, 30, 24, 47, 26, 6, 20, 4, 40, 11, 41] which contain further information on this topic.

In 2006, J. Wiegold raised the following question in Kourovka Notebook [32, 16.101]. Do there exist uncountably many infinite 2-groups that are quotients of the group

$$\Delta = \langle x, y \| \ x^2, y^4, (xy)^8 \rangle?$$

The problem is motivated by the following comment by J.Wiegold "There certainly exists one, namely the subgroup of finite index in Grigorchuk's first group generated by b and ad; see (R.I. Grigorchuk, *Functional Anal. Appl.*, **14** (1980), 41–43)."

Immediately after the appearance we informed one of the Editors of Kourovka Notebook, I. Khukhro, that the answer to the question is positive, and that the results of [16] can be easily used to provide a justification. Unfortunately, it took some time for the author to write the corresponding text, and he is finally presenting his arguments, but considering the question of Wiegold from a more general perspective. Different argument has been used recently in the article [33] and the authors were notified of the approach given here (they acknowledgment this fact at the end of Section 2).

A few other groups that deserve to be considered in the context of the question of Wiegold are groups

$$\Lambda = \langle x, y \| \ x^4, y^4, (xy)^4 \rangle,$$

$$\Xi = \langle a, c, d \| \ a^2, c^2, d^2, (cd)^2, (ad)^4, (ac)^8 \rangle,$$
$$\Phi = \langle x, y, z \| \ x^2, y^2, z^2, (xy)^4, (xz)^4, (yz)^4 \rangle$$

(the latter two groups are the "smallest" Coxeter groups which can be mapped onto an infinite 2-group).

Observe that the group

$$\langle x, y \| \ x^2, y^4, (xy)^4 \rangle$$

is virtually $\mathbb{Z}^2$ (it contains a subgroup isomorphic to

$$\langle a, b, c \| \ a^2, b^2, c^2, (abc)^2 \rangle$$

which is the group generated by the rotations by π around the middle points of the sides of an isosceles right triangle. Therefore this group cannot have infinite 2-torsion quotients. Also one cannot make any power of the product of generators in the presentation of Ξ or Φ smaller without making the group virtually abelian (see for instance [[36], Proposition 4.7] and the discussion in the Section 4 about Coxeter groups below). Three more "critical" Coxeter groups are Υ, Π and Γ and they are considered in Section 4.

Let $\mathcal{G}$ be the group constructed in [22]. It is defined in [22] as a group generated by four interval exchange transformations a, b, c, d of order 2 acting on the interval $[0,1]$ from which the diadic rational points are removed (later we will recall the definition). As shown by I. Lysënok [29] $\mathcal{G}$ can be described by the following presentation

$$\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \alpha^n((ad)^4), \alpha^n((adacac)^4),\ n \geq 0\rangle, \qquad (1.1)$$

where α is the substitution $\alpha : a \to aca, b \to d, c \to b, d \to c$. The group $\mathcal{G}$ is not finitely presented, and it is shown in [17] that the relators given in (1.1) are independent (i.e., none of them can be deleted from the set of relators without changing the group). The relation $bcd = 1$ implies that the group $\mathcal{G}$ is 3-generated, but it is usually convenient to work with the generating set $\mathcal{A} = \{a, b, c, d\}$, because together with the identity element it constitutes the so called nucleus of the group, the important tool in study of self-similar groups [35].

Let $\mathcal{L} \leq \mathcal{G}$ be the subgroup generated by $x = b$ and $y = ad$. Calculations show that $x^2 = y^4 = (xy)^8 = 1$ in $\mathcal{L}$, and therefore $\mathcal{L}$ is a quotient of Δ (this was observed by Wiegold in his comment). It has index 2 in $\mathcal{G}$. Note that there are three 2-generated subgroups of index 2 in $\mathcal{G}$, namely $\langle b, ac\rangle$, $\langle c, ad\rangle$, and $\langle d, ab\rangle$ (see [19]).

Let $\mathcal{S}$ be the subgroup of $\mathcal{G}$ generated by the elements ad and $(ac)^2$. The relators of Λ are also relators of $\mathcal{S}$ with respect to the map

$$x \to ad, \qquad y \to (ac)^2,$$

and therefore $\mathcal{S}$ is a quotient of Λ.

Let $\mathcal{Q}$ be the subgroup of $\mathcal{G}$ generated by the elements a, d, and cac. The relators of Φ are also relators of $\mathcal{Q}$ with respect to the map

$$x \mapsto a, \quad y \mapsto d, \quad z \mapsto cac,$$

and therefore $\mathcal{Q}$ is a quotient of Φ.

The fact that the indicated elements from $\mathcal{G}$ satisfy the defining relations of $\mathcal{L}$, $\mathcal{S}$, and $\mathcal{Q}$ can be easily checked by using the branch algorithm, described later in the text.

Using the Reideimeister-Schreier process (and the fact that $\mathcal{L}$, $\mathcal{S}$,and $\mathcal{Q}$ have finite index in $\mathcal{G}$, which will be explained later) one can rewrite the presentation (1.1) into presentations for $\mathcal{L}$, $\mathcal{S}$, and $\mathcal{Q}$.

Theorem 1.1. *(i) There are uncountably many infinite 2-groups that are quotients of Δ. These quotient groups can have the following additional properties: be residually finite, just-infinite, branch, and of intermediate growth.*

(ii) Moreover, for every finitely presented group Γ that can be mapped onto one of the groups $\mathcal{G}, \mathcal{L}, \mathcal{S}, \mathcal{Q}$, there are uncountably many infinite 2-groups that are quotients of Γ and satisfy the properties listed in (i).

(iii) Claims analogous to (i) hold also for the groups Λ, Ξ and Φ.

(iv) Moreover, claim analogous to (i) holds for any Coxeter group that is not virtually abelian and has defining relations of the form $(x_i x_j)^{m_{i,j}} = 1$, where $m_{i,j}$ are powers of 2.

Remark 1.2. From the proof it will be clear that not only we obtain uncountably many quotients, but we also obtain uncountably many quotients up to quasi-isometry because of different growth degrees of these quotients.

Recall that a group is just-infinite if it is infinite but has only finite proper quotients.

The definition of a branch group is more involved and we direct the reader to [19, 18, 4] for more information on branch groups. A group G is a branch group if it has a strictly decreasing sequence $\{H_n\}_{n=0}^{\infty}$ of normal subgroups of finite index with trivial intersection, satisfying the following properties:

$$[H_{n-1} : H_n] = m_n \in \mathbb{N},$$

for $n = 1, 2, \ldots$, there is a decompositions of H_n into the direct product of $N_n = m_1 m_2 \ldots m_n$ copies of a group L_n such that the decomposition for H_{n+1} refines the decomposition for H_n (in the sense that each factor of H_n contains the product of m_{n+1} factors of the decomposition of H_{n+1}), and for each n the group G acts transitively by conjugation on the set of factors of H_n. Branch groups constitute one of three classes into which the class of just-infinite groups naturally splits and they appear in various situations [35, 5, 3, 19]. The natural language to work with branch groups is via their actions on regular rooted trees as described in [19, 21, 4]. Observe that the groups $\mathcal{G}, \langle b, ac\rangle, \langle c, ad\rangle, \langle d, ab\rangle$ and $G_\omega, \omega \in \Omega_1$ discussed below are branch, just-infinite groups [16, 19].

A finitely generated group has intermediate growth if the growth function $\gamma(n)$, counting the number of elements of length at most n, grows faster than any polynomial but slower than any exponential function λ^n, for $\lambda > 1$. We use Milnor's equivalence on the set of growth functions of finitely generated groups: $\gamma_1(n) \sim \gamma_2(n)$ if there is $C \in \mathbb{N}$ such that $\gamma_1(n) \leq \gamma_2(Cn)$ and $\gamma_2(n) \leq \gamma_1(Cn)$, for $n = 0, 1, 2, \ldots$. For a given finitely generated group the class of equivalence of its growth function does not depend on the choice of a finite generating set and this class is called the growth degree of the group. It is an invariant of a group up to quasi-isometry [10]. It is shown in [16] that there are uncountably many growth degrees of finitely generated groups and, moreover, the partially ordered set of growth degrees of finitely generated groups contains both chains and antichains of continuum cardinality.

Finally, the well known terminology related to Coxeter groups will be recalled Section 4. Part (*iv*) Theorem 1.1 will be stated there in a more detailed form, along with a proof.

A proof of parts (*i*), (*ii*), (*iii*) of Theorem 1.1 is provided in Section 3.

2. Preliminary facts

For the proof of Theorem 1.1 we will use the construction of an uncountable family of groups G_ω, where $\omega \in \Omega = \{0, 1, 2\}^{\mathbf{N}}$ described in [16]. The group G_ω is generated by the set of elements $\mathcal{A}_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ of order 2, with $b_\omega, c_\omega, d_\omega$ commuting and generating the Klein 4-group (i.e. $b_\omega c_\omega d_\omega = 1$) (so indeed the groups G_ω are 3-generated).

Recall the definition of G_ω. Originally these groups were defined by actions on $[0,1] \backslash \{ \frac{k}{2^n} \mid n = 0, 1, 2, 3, \ldots, \; k = 0, 1, \ldots, 2^n \}$. An alternative definition may be given via the language of actions by automorphisms on a binary rooted tree T or via the language of isometric actions on the space of sequences over the alphabet $X = \{0, 1\}$. One can either use the set of finite sequences X^* identifying them with the vertices of T, or the set $\partial T = X^{\mathbb{N}}$ of infinite sequences which can be identified with the boundary of the tree T. Let us use at the moment the language of actions on sequences (the original action on $[0,1]$ automatically translates to this language via the identification of irrational diadic points from $[0,1]$ with the corresponding diadic sequences of symbols of the alphabet $\{0,1\}$).

Denote by τ the shift in the space of sequences Ω,

$$\tau(\omega_1 \omega_2 \omega_3 \ldots) = \omega_2 \omega_3 \ldots .$$

The transformation a changes the first symbol x of each sequence $w \in \{0,1\}^{\mathbb{N}}$ to the other symbol $\bar{x}$, while $b_\omega, c_\omega, d_\omega$ do not change the first symbol and act according to the following recursive formulas:

$$\begin{array}{rclll rclll}
b_{0\omega}(xw) &=& a(w) & \text{if } x=0, && b_{0\omega}(xw) &=& b_\omega(w) & \text{if } x=1,\\
b_{1\omega}(xw) &=& a(w) & \text{if } x=0, && b_{1\omega}(xw) &=& b_\omega(w) & \text{if } x=1,\\
b_{2\omega}(xw) &=& w & \text{if } x=0, && b_{2\omega}(xw) &=& b_\omega(w) & \text{if } x=1,\\[1ex]
c_{0\omega}(xw) &=& a(w) & \text{if } x=0, && c_{0\omega}(xw) &=& c_\omega(w) & \text{if } x=1,\\
c_{1\omega}(xw) &=& w & \text{if } x=0, && c_{1\omega}(xw) &=& c_\omega(w) & \text{if } x=1,\\
c_{2\omega}(xw) &=& a(w) & \text{if } x=0, && c_{2\omega}(xw) &=& c_\omega(w) & \text{if } x=1,\\[1ex]
d_{0\omega}(xw) &=& w & \text{if } x=0, && d_{0\omega}(xw) &=& d_\omega(w) & \text{if } x=1,\\
d_{1\omega}(xw) &=& a(w) & \text{if } x=0, && d_{1\omega}(xw) &=& d_\omega(w) & \text{if } x=1,\\
d_{2\omega}(xw) &=& a(w) & \text{if } x=0, && d_{2\omega}(xw) &=& d_\omega(w) & \text{if } x=1,
\end{array}$$

for each $\omega \in \Omega$ and each $w \in \{0,1\}^{\mathbb{N}}$. The group $\mathcal{G}$ is a particular case of this construction and corresponds to the periodic sequence $\varsigma = 012012\ldots$. For any group G acting on the space $\{0,1\}^{\mathbb{N}}$ the subgroup consisting of the elements that do not change the first symbol in any sequence is denoted by $st_G(1)$ (this corresponds to the stabilizer of the first level of the tree when we consider the action on the rooted binary tree). Thus $b_\omega, c_\omega, d_\omega \in st_{G_\omega}(1)$.

Let $G_n = G_{\tau^{n-1}\omega}$. The sequence $\{G_n\}_{n=1}^{\infty}$ is called the linking class of G_ω and its members are called the accompanying groups of G_ω . In the case ω is periodic, the sequence $\{G_n\}_{n=1}^{\infty}$ is periodic as well. Moreover, in the case $\omega = \varsigma = (012)^\infty$, the linking sequence is constant and its members are isomorphic to $\mathcal{G}$ (meaning that the group $\mathcal{G}$ is self-similar [35]). This can be seen from the fact tat the set of generators of the accompanying groups of $\mathcal{G}$ is the same as of $\mathcal{G}$ (just written in a different cyclic order, i.e., renamed). The group G_n contains a subgroup $H_n = H_{\tau^{n-1}\omega}$ of index 2 consisting of the elements of G_n that do not change the first symbol (equivalently, H_n can be defined as the stabilizer of the first level of the tree when we realize the group through an action on the binary tree). The group H_n embeds into $G_{n+1} \times G_{n+1}$ via the maps $\psi_n, \psi_n(g) = (\varphi_0^{(n)}(g), \varphi_1^{(n)}(g))$, where $\varphi_0^{(n)}(g)$ and $\varphi_1^{(n)}(g))$ are the projection maps, and the following relations hold (with $\psi = \psi_1$):

$$\psi(b_{0\omega}) = (a, b_\omega), \quad \psi(b_{1\omega}) = (a, b_\omega), \quad \psi(b_{2\omega}) = (1, b_\omega),$$

$$\psi(c_{0\omega}) = (a, c_\omega), \quad \psi(c_{1\omega}) = (1, b_\omega), \quad \psi(a_{2\omega}) = (a, b_\omega),$$

$$\psi(d_{0\omega}) = (1, b_\omega), \quad \psi(d_{1\omega}) = (a, d_\omega), \quad \psi(d_{2\omega}) = (a, d_\omega).$$

Observe that if $\psi(g) = (g_0, g_1)$, then $\psi(aga) = (g_1, g_0)$, and that

$$\{b_\omega, c_\omega, d_\omega, ab_\omega a, ac_\omega a, ad_\omega a\}$$

is a generating set for H_ω. Keeping in mind the fact that the linking class of $\mathcal{G}$ consists of a single group, the above relations for this group may be rewritten in the following form

$$\psi(b) = (a, c), \quad \psi(c) = (a, d), \quad \psi(d) = (1, b) \tag{2.1}$$

more suitable for calculations by hand.

Since the elements $b_\omega, c_\omega, d_\omega$ have order 2, commute, and their product $b_\omega c_\omega d_\omega$ is the identity element, the group

$$\Theta = \langle a, b, c, d \| \ a^2, b^2, c^2, d^2, bcd \rangle$$

covers each G_ω via the map

$$a \mapsto a, \quad b \mapsto b_\omega, \quad c \mapsto c_\omega, \quad d \mapsto d_\omega.$$

The group Θ is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, where the first factor is the group of order 2 generated by a, while the second factor is the Klein group consisting of the elements $1, b, c, d$. Thus each element of Θ can be presented by a reduced word of the form

$$*a * a \cdots * a*, \tag{2.2}$$

where each star denotes an element from the set $\{b, c, d\}$, and the first and the last star may also denote the empty symbol. Consequently we can represent the elements in G_ω in reduced form of type (2.2), only this time the stars $*$ denote elements from the set $\{b_\omega, c_\omega, d_\omega\}$. We consider the length $|g|$ of elements in G_ω with respect to the generating set $\mathcal{A}_\omega$. If w is a shortest word in the alphabet $\mathcal{A}_\omega$ representing the corresponding element of G_ω, then it must be in reduced form. The linking class $\{G_n\}_{n=1}^\infty$ has the following contracting property. If $g \in H_n$, and $\psi(g) = (g_0, g_1), g_i \in G_{n+1}, i = 0, 1$, then

$$|g_i| \leq \frac{|g| + 1}{2}, i = 0, 1. \tag{2.3}$$

The elements g_i, $i = 0, 1$, are called the projections. The inequality (2.3) shows that the length of each projection is strictly less than the length of g if $|g| > 1$, and that the linking class $\{G_n\}_{n=1}^\infty$ has the contracting property with coefficient $1/2$. The contracting property is the basis for the algorithm with oracle ω (i.e. the algorithm which uses the symbols of the sequence ω in its work) for decision of the word problem in G_ω. We call this algorithm branch algorithm because of its branching nature. A short description follows.

To check if a given word w over the alphabet given by the generating set $\mathcal{A}_\omega$ represents the identity element, the algorithm checks if

$w \in H_\omega$ (i.e. if w contains even number of symbol a) and declares that $w \neq 1$ if $w \notin H_\omega$. If $w \in H_\omega$ the algorithm reduces w to the form (2.2), computes the projections w_0, w_1 using the rewriting rules for ψ, and reduces them. Because ψ is an embedding, $w = 1$ in G_ω if and only if $w_i = 1, i = 0, 1$ in $G_{\tau(\omega)}$. In the second step of the algorithm, it replaces w by the set $\{w_0, w_1\}$ of projection words and operates with each of the projections in the same way as it did with w. Continuing its work, the algorithm either finds that one of the projections appearing in the procedure is not in H_n for some n, and therefore $w \neq 1$, or all 2^n projections $w_{i_1 i_2 \dots i_n}, i_1, i_2, \dots, i_n \in \{0, 1\}$ are the identity in G_n, and then $w = 1$ in G_ω. As the lengths of the projections strictly decrease whenever they are longer than 1, the process stops after at most $\log_2(|w|) + 1$ steps.

It is obvious that the described procedure is an actual algorithm if the sequence ω is recursive, and in fact the converse is also true. Moreover, the word problem in G_ω is decidable if and only if the sequence ω is recursive (see §5 in[16]).

Let $\Omega_0 \subset \Omega$ be the subset consisting of sequences ω which contain each symbol $0, 1, 2$ infinitely many times, $\Omega_1 \subset \Omega$ be the set of sequences which contain at least two symbols from $\{0, 1, 2\}$ infinitely many times, and $\Omega_2 = \Omega \setminus \Omega_1$ be the set of sequences $\omega = \omega_1 \omega_2 \dots \omega_n \dots$ such that $\omega_n = \omega_{n+1} = \omega_{n+2} = \dots$ starting with some coordinate n. Observe that all sets $\Omega_0, \Omega_1, \Omega_2$ are invariant with respect to the shift τ in the space of sequences. The groups G_ω are virtually abelian for $\omega \in \Omega_2$, while the groups G_ω, for $\omega \in \Omega_1$ are just infinite, branch groups of intermediate growth. Additionally, the groups G_ω, for $\omega \in \Omega_0$ are 2-groups. The proofs of these facts are provided by Theorems 2.1, 2.2, 8.1, and Corollary 3.2 in [16] (observe that the term branch group is not used in [16] as at the time of writing of the paper there was no definition of this class of groups, but the proof of Theorem 2.2 provided there implies the branch property for G_ω, $\omega \in \Omega_1$). One of important facts that will be used in the proof of the Theorem 1.1 is that the set of growth degrees of groups $G_\omega, \omega \in \Omega_1$ has uncountable cardinality, the growth of G_ω is equal to the square of growth of $G_{\tau(\omega)}$, and therefore the set of squares of growth degrees of groups $G_\omega, \omega \in \Omega_1$ has the cardinality of continuum. In the next section we will also use the fact, proved in [16], that the group G_ω, for $\omega \in \Omega_1$ is abstractly commensurable to the direct product of two copies of $G_{\tau(\omega)}$, and therefore its growth degree is equal to the square of the growth degree of $G_{\tau(\omega)}$.

In the second part of the article [16] the construction of the groups $G_\omega, \omega \in \Omega$ is slightly modified. Namely the groups G_ω, $\omega \in \Omega_2$ are replaced by virtually metabelian groups (we keep the same notation G_ω for the new groups and $a, b_\omega, c_\omega, d_\omega$ for their new generators) in such a way that the new set of groups $\{G_\omega \mid \omega \in \Omega\}$ becomes a closed subset of the compact space $\mathcal{X}_4$ of 4-generated groups. The spaces $\mathcal{X}_m$ of m-generated marked groups (consisting of the pairs (G, A), where G is an m-generated group and $A = \{a_1, \dots a_m\}$ is an ordered set of generators) were introduced in [16] as a tool in the study of the groups from the set $\{G_\omega \mid \omega \in \Omega\}$. Two groups in $\mathcal{X}_m$ are close if they have isomorphic Cayley graphs in the balls of large radius around the identity element (the larger the radius is where the groups have isomorphic Cayley graphs the closer the groups are). This topology was used in different ways in the study of the set $\{G_\omega \mid \omega \in \Omega\}$. Now it plays important role in many studies [8, 7, 43, 36, 46, 44].

One of facts proven in [16] is that the ball of radius 2^n around the identity in the Cayley graph of G_ω with the system of generators $\mathcal{A}_\omega$ is determined by the first $n+1$ symbols of the sequence ω. In particular, two Cayley graphs $\mathcal{C}(G_\omega, \mathcal{A}_\omega)$ and $\mathcal{C}(G_\eta, \mathcal{A}_\eta)$ have isomorphic subgraphs in the neighborhoods of identity of radius 2^n if the first $n+1$ terms of the sequences ω and η coincide. This implies that if the sequence $\{\eta_n\}_{n=1}^\infty$ of sequences converges to ω in the Tychonoff topology on the space Ω, then the sequence of groups $\{G_{\eta_n}\}_{n=1}^\infty$ converges to G_ω in $\mathcal{X}_4$. The converse is also true and the proof of this fact is based on the construction of testing words (denoted X_N^4, Y_N^4, Z_N^4 , see Proposition 6.2 and §5 in [16]). The proof of topological properties of the class $\{G_\omega, \omega \in \Omega\}$ we listed here is heavily based on the branch algorithm with oracle ω described earlier. Observe also that the set of relators of G_ω of length $\leq 2^n$ is determined by the first $n-1$ symbols of ω.

3. The proof of Theorem 1.1 $(i), (ii), (iii)$

Proof. We are now ready to prove the first three parts of Theorem 1.1.

Let us start with proving part (iii) for the group Ξ. The relators of Ξ are of length ≤ 16 and the group $\mathcal{G}$ which is isomorphic to the group G_ς has the same relators with respect to the map

$$\kappa : \ a \to a_\varsigma, \ c \to c_\varsigma, \ d \to d_\varsigma$$

(this can be verified directly by application of the branch algorithm and use of the relations in (2.1)).

The relators of length $\leq 2^n$ in G_ω are determined by the first $n-1$ symbols of ω, so $a^2, b^2, c^2, (cd)^2, (ad)^4, (ac)^8$ are among the relators in each group G_ω with $\omega = 012\omega_4\dots$. Let $\Omega_3 \subset \Omega_0$ be the subset consisting of the sequences in Ω_0 that begin with the prefix 012. Then, for each $\omega \in \Omega_3$, the map κ can be extended to a surjective homomorphism $\Xi \to G_\omega$.

Each group $G_\omega, \omega \in \Omega_3$ is residually finite, just-infinite, branch 2-group of intermediate growth.

In [16] it is proven that for each $\omega \in \Omega$ there exist at most countably many $\eta \in \Omega$ with $G_\omega \simeq G_\eta$. In [35, Subsection 2.10] this result is strengthened and it is shown that $G_\omega \simeq G_\eta$ if and only if the sequences ω and η can be obtained from each other by a permutation of the symbols in the alphabet $\{0, 1, 2\}$ (therefore for each ω there exist at most 6 groups $G_\eta, \eta \in \Omega$, isomorphic to G_ω). From this we conclude that the set $\{G_\omega \mid \omega \in \Omega_3\}$ contains uncountably many pairwise non isomorphic groups. This proves Theorem 1.1(iii) for Ξ.

Recall that $\mathcal{L}$ is a subgroup of $\mathcal{G}$ generated by the elements $x = b$ and $y = ad$ and is a quotient of Λ. With respect to the canonical generators of $\mathcal{G}$ the relators $x^2, y^4, (xy)^8$ have length $\leq 24 < 2^5$. Let $L_\omega \leq G_\omega$ be the subgroup of G_ω of index 2 generated by b_ω and ad_ω. Then, if ω begins with 0120 (the first four symbols of ς), the group L_ω has $x^2, y^4, (xy)^8$ among its relators with respect to the map

$$\mu: \; x \mapsto b_\omega, \; y \mapsto ad_\omega$$

and is therefore a homomorphic image of Δ.

Let $\Omega_4 \subset \Omega_3$ be the set of sequences which begin with 0120. We claim that the set $\{L_\omega \mid \omega \in \Omega_4\}$ contains uncountably many non isomorphic groups. In order to prove this we may use the same type of argument used in [16, Theorem 5.1] to prove that for each $\omega \in \Omega$ there exists at most countably many groups $G_\eta, \eta \in \Omega$ isomorphic to G_ω. Namely, testing words X_n^4, Y_n^4, Z_n^4 are constructed in [16], which allow one to reconstruct (if we know which of these test words represent the identity element and which do not) the nth symbol of the sequence ω. As these elements are the fourth powers of some elements they represent elements in the subgroup L_ω of index 2 in G_ω. Application of the same arguments as in the proof of Lemma 5.2 and at the end of proof of [16, Theorem 5.1] leads to a proof that there are uncountably many groups in the set $\{L_\omega \mid \omega \in \Omega_4\}$.

Calculations show that both projections of the stabilizer $st_{L_\omega}(1)$ are isomorphic to $G_{\tau(\omega)}$. From this we conclude that the groups L_ω, for $\omega \in \Omega_4$ are branch groups, and hence are just-infinite because every proper quotient of a branch group is virtually abelian ([19, Theorem 4] and [16, Theorem 2.1(3)]) and L_ω is a 2-group. Obviously each group L_ω, for $\omega \in \Omega_4$, is a residually finite group and has intermediate growth according to [16].

Observe that we could also conclude that the set $\{L_\omega \mid \omega \in \Omega_4\}$ contains uncountably many non isomorphic groups using growth arguments. Namely, using the same type of arguments that are used in proof of [16, Theorem 7.1] one can show that the set of growth degrees of the groups $G_\omega, \omega \in \Omega_4$ has uncountable cardinality. Therefore the set of growth degrees of groups in $L_\omega, \omega \in \Omega_4$ has uncountable cardinality (subgroups of finite index have the same growth degree as the group). This finishes the proof of Theorem 1.1(i) and answers Wiegold's question.

Recall that $\mathcal{S}$ is the subgroup of $\mathcal{G}$ generated by the elements ad and $(ac)^2$ and is a quotient of Λ. The relators of Λ have length ≤ 24 with respect to the canonical generating set $\mathcal{A}$ of $\mathcal{G}$. Computations show that $\psi(st_{\mathcal{S}}(1))$ is a subgroup in $\mathcal{G} \times \mathcal{G}$ generated by the pairs $(b,b), (da, ad), (bac, da), (badac, (da)^2)$, and the projections of this subgroup on each factor is the group $\langle b, ac\rangle = \langle b, ad\rangle$, which has index 2 in $\mathcal{G}$. The element $(dabac, 1)$ belongs to $\psi(st_{\mathcal{S}}(1))$, and as $\langle b, ac\rangle$ is just-infinite branch group, the group $\psi(st_{\mathcal{S}}(1))$ contains the direct product $\mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ is the normal closure of $dabac$ in $\langle b, ac\rangle$. This shows that $\mathcal{S}$ is commensurable with $\mathcal{G} \times \mathcal{G}$.

Let S_ω, for $\omega \in \Omega_4$ be the subgroups of G_ω generated by ad_ω and $(ac_\omega)^2$. Then the relators of Λ are also relators of S_ω with respect to the map

$$\nu: \; x \to ad_\omega, \; y \to (ac_\omega)^2.$$

The image $\psi(st_{S_\omega}(1))$ is a subgroup in $G_{\tau(\omega)} \times G_{\tau(\omega)}$, with the projections on each factor being subgroups of index 2 in the group $G_{\tau(\omega)}$. Moreover $\psi(st_{S_\omega}(1))$ contains the direct product $B_\omega \times B_\omega$, where B_ω is a subgroup of finite index in $G_{\tau(\omega)}$. This shows that $\mathcal{S}$ is commensurable with $G_{\tau(\omega)} \times G_{\tau(\omega)}$ and the growth degree of S_ω is equal to the square of the growth degree of $G_{\tau(\omega)}$. As there are uncountably many growth types of the group $G_{\tau(\omega)}$, for $\omega \in \Omega_4$, we conclude that the set $\{S_\omega \mid \omega \in \Omega_4\}$ contains uncountably many non isomorphic groups. Each group S_ω, for $\omega \in \Omega_4$, is a branch group because it acts transitively on the levels of the tree and contains $B_\omega \times B_\omega$. It is of intermediate growth since so is $G_{\tau(\omega)}$. This proves Theorem 1.1(iii) for Λ.

Consider now the case of the group Φ and its homomorphic image $Q = \langle a, d, cac\rangle \le \mathcal{G}$. Observe that the $\mathcal{A}$-length of the relators of Q inherited from Φ is ≤ 16. The stabilizer $st_{\mathcal{Q}}(1)$ has both projections equal to the subgroup $\langle b, da\rangle$ of index 2 in $\mathcal{G}$ and its ψ-image contains $\mathcal{P} \times \mathcal{P}$, where $\mathcal{P}$ is the normal closure of b in $\langle b, da\rangle$ as $\psi(d) = (1, b)$. Therefore $\mathcal{Q}$ is abstractly commensurable with $\mathcal{G} \times \mathcal{G}$.

Let Q_ω, for $\omega \in \Omega_3$ be the subgroup of G_ω generated by the elements a, d_ω, and $c_\omega a c_\omega$. Then the relators of Φ are the relators of Q_ω with respect to the map

$$x \to a, \; y \to d_\omega, \; z \to c_\omega a c_\omega,$$

and therefore each Q_ω, for $\omega \in \Omega_3$ is a quotient of Φ. Both projections of $st_{Q_\omega}(1)$ are equal to the subgroup $\langle d_{\tau(\omega)}, ac_{\tau(\omega)}\rangle$ which is of index 2 in $G_{\tau(\omega)}$, branch and just-infinite. Also the ψ-image of $st_{Q_\omega}(1)$ contains the subgroup $D_\omega \times D_\omega$, where D_ω is the normal closure of $d_{\tau(\omega)}$ in $\langle d_{\tau(\omega)}, ac_{\tau(\omega)}\rangle$. Therefore the group Q_ω is abstractly commensurable with $G_{\tau\omega} \times G_{\tau\omega}$ and its growth degree is equal to the square of the growth degree of $G_{\tau\omega}$. Using the fact that the set of growth degrees of the groups in $\{Q_\omega \mid \omega \in \Omega_3\}$ is uncountable we complete the proof of Theorem 1.1(iii).

Let us now prove Theorem 1.1(ii). Let $\Gamma = \langle \mathcal{A} \| \mathcal{R}\rangle$ be a 4-generated group which can be mapped onto $\mathcal{G}$ (and therefore, whose relators $r \in \mathcal{R}$ are also the relators of $\mathcal{G}$). Let k be the maximum of the lengths of the relators $r \in \mathcal{R}$ and suppose $k \leq 2^n$. Let $\Omega_5 \subset \Omega_0$ be the set of sequences which have the same prefix of length n as ς. Then according to the branch algorithm with oracle ω the relators of Γ are at the same time relators in G_ω, for $\omega \in \Omega_5$ (for the corresponding set of generators). The arguments used in [16] to justify the results about growth allow us to prove that the set of growth degrees of the groups in $\{G_\omega \mid \omega \in \Omega_5\}$ is uncountable. Together with the arguments used before to prove the properties of the corresponding subgroups in G_ω we obtain the claim. The proof of Theorem 1.1(ii) for the groups $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{Q}$ is similar and we omit it.

This completes the proof of Theorem 1.1(i),(ii),(iii). □

Observe that Ξ and Φ are examples of hyperbolic triangular groups $T^*_{2,4,8}$ and $T^*_{4,4,4}$ acting by isometries on the hyperbolic plane $\mathbb{H}^2$ and generated by reflections with respect to sides of a triangle with corresponding interior angles. The subgroups $T_{2,4,8}$ and $T_{4,4,4}$ of $T^*_{2,4,8}$ and $T^*_{4,4,4}$ respectively, consisting of orientation-preserving isometries, have index 2, and have presentations similar to the presentations of groups Λ and Δ ([10], page 136). Therefore in the proof of Theorem 1.1 we could consider only the cases of groups Ξ and Φ. Instead we preferred to deal with all four cases because it provides more information about the quotients in view of part (ii) of Theorem 1.1.

4. Torsion quotients of Coxeter groups

In this section we discuss the question which Coxeter groups have infinite torsion quotients and how many of such quotients they could have. First, we specify the question by imposing that the quotients have to be 2-groups and require the additional properties in the spirit of the statement of Theorem 1.1.

Recall that a Coxeter group can be defined as a group with presentation

$$\mathcal{C} = \langle x_1, x_2, \ldots, x_n \| \ x_i^2, (x_i x_j)^{m_{ij}}, 1 \le i < j \le n \rangle,$$

where $m_{i,j} \in \mathbb{N} \cup \{\infty\}$ (the case $m_{i,j} = \infty$ essentially means that there is no defining relator involving x_i and x_j). For instance the groups Ξ and Φ are examples of Coxeter groups.

If $m_{ij} = 2$ this means that x_i and x_j commute. A Coxeter group can be described by a Coxeter graph $\mathcal{Z}$. The vertices of the graph are labeled by the generators of the group $\mathcal{C}$, the vertices x_i and x_j are connected by an edge if and only if $m_{i,j} \ge 3$, and an edge is labeled by the corresponding value m_{ij} whenever this value is 4 or greater. If a Coxeter graph is not connected, then the group $\mathcal{C}$ is a direct product of Coxeter subgroups corresponding to the connected components. Therefore we may focus on the case of connected Coxeter graphs. If we are interested in 2-torsion quotients of $\mathcal{C}$, then one has to assume that m_{ij} are powers of 2 or infinity. In order for $\mathcal{C}$ to have infinite torsion quotients it has to be infinite and not virtually abelian. The list of finite and virtually abelian Coxeter groups with connected Coxeter graphs is well known. A comprehensive treatment of Coxeter groups can be found in M. Davis' book [9].

Theorem 4.1. *Let $\mathcal{C}$ be a non virtually abelian Coxeter group defined by a connected Coxeter graph $\mathcal{Z}$ with all edge labels m_{ij} being powers of 2 or infinity. If $\mathcal{Z}$ is not a tree or is a tree with ≥ 4 vertices, or is a tree with two edges with one label ≥ 4 and the other ≥ 8, then the group $\mathcal{C}$ has uncountably many 2-torsion quotients. Moreover these quotients can be chosen to be residually finite, just-infinite, branch 2-groups of intermediate growth.*

Observe that all cases of connected Coxeter graphs that are excluded by the statement of Theorem 4.1 are related to finite or virtually abelian crystallographic groups. For instance, the case when $\mathcal{Z}$ consist of one edge, or has two edges labeled by 4 correspond to the dihedral group D_4 and to the crystallographic group $\langle x, y, z \| \ x^2, y^2, z^2, (yz)^2, (xy)^4, (xz)^4 \rangle$, respectively. On the other hand, there are 5 "critical" Coxeter groups Ξ, Φ, Υ, Π and Γ that satisfy the requirements of Theorem 4.1 and play a crucial role in the proof. Their Coxeter graphs are depicted in Figure 1.

Corollary 4.2. *A Coxeter group $\mathcal{C}$ given by a Coxeter graph $\mathcal{Z}$ with all labeles m_{ij} being powers of 2 can be mapped onto an infinite 2-group if and only it has at least one connected component that satisfies the conditions of Theorem 4.1. In the case $\mathcal{Z}$ has such a component, $\mathcal{C}$ has uncountably many quotients that are residually finite, just-infinite, branch 2-groups of intermediate growth.*

Proof. Assume that the graph $\mathcal{Z}$ is not a tree, so it contains a cycle of length ≥ 3 consisting of vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ for some $3 \leq k \leq n$. Taking the quotient of $\mathcal{C}$ by the normal subgroup generated by the generators x_j which do not belong to this cycle, we can pass to the case when the graph $\mathcal{Z}$ is a cycle. Taking the quotient by the relation $x_{i_1} = 1$ (if the length n of the cycle is greater than 4) we make it to look like a "segment" (all vertices are of degree ≤ 2), or, what is the same, as a connected part of the one-dimensional lattice with $n-1 \geq 4$ vertices. The case when the graph $\mathcal{Z}$ is a tree will be considered below. If the length of the cycle is 3, then making further factorization by replacing the numbers $m_{i,j} \geq 8$ by $m_{i,j} = 4$, we map $\mathcal{C}$ onto Φ, and then apply Theorem 1.1, finishing the argument in this case. The case of cycle of length 4 will be considered later.

If $\mathcal{Z}$ is a tree, passing to an appropriate quotient (by making some "leaf" verices identity) reduces the situation to the case when the graph $\mathcal{Z}$ looks like a "segment" with 3 or 4 vertices, and labeling of edges given by the set $\{4,8\}$ or $\{4,4,4\}$) respectively, or like a tripod "Y" (i.e., is a tree with four vertices, one of degree 3 and three leaves) with all edges labeled by 4.

Let $\mathcal{Z}$ be a "segment" whose edges are colored by $\{4,8\}$. Then $\mathcal{C} = \Xi$ and we can apply Theorem 1.1.

Now let $\mathcal{Z}$ be a segment whose edges are colored by $\{4,4,4\}$. The corresponding group is

$$\Upsilon = \langle a,b,c,d \| \; a^2, b^2, c^2, d^2, (ac)^2, (ad)^2, (bd)^2, (ab)^4, (bc)^4, (cd)^4 \rangle.$$

Let $\bar{\Lambda}$ be any of the 2-quotients of Λ given by the statement of Theorem 1.1 (and of its proof, so the group acts on the binary rooted tree), and let x, y be the set of generators of $\bar{\Lambda}$ which are the images of the generators of Λ (we keep the same notation). Consider the group $\bar{\Lambda}_1$, acting on binary rooted tree T, generated by the element a of order two (permutation of two subtrees T_0, T_1 with roots at the first level) and the elements $b = (1, y), c = (x, x), d = (y, y)$, where x, y and the identity element act on the left or right subtree respectively (in a same way they act on the whole tree; here we use the self-similarity of the binary tree). Then a commutes with c and d, b commutes with d, and $(ab)^4 = (bc)^4 = (cd)^4 = 1$, so the group is a quotient of Υ. The ψ-image of stabilizer of the first level of $\bar{\Lambda}_1$ is a subdirect product of $\bar{\Lambda} \times \bar{\Lambda}$ and contains the group $D \times D$ where D is the normal closure of y in $\bar{\Lambda}$. As $\bar{\Lambda}$ is just-infinite, D has finite index in Λ. Therefore the growth of $\bar{\Lambda}_1$ is equal to the square of the growth of $\bar{\Lambda}$. It is clear that $\bar{\Lambda}$ satisfies all the other requirements and that there are uncountably many groups of this type (because of the growth argument).

Now let $\mathcal{Z}$ the tripod "Y" with labeling $\{4,4,4\}$. Let $G = G_\omega$, for $\omega \in \Omega_0$ be the 2-group from the main construction whose generators will be denoted, for simplicity, by a, b, c, d instead of by $a, b_\omega, c_\omega, d_\omega$. Recall that a acts by permutation of the two subtrees T_0, T_1 of the binary tree with roots on the first level. Consider the group $V = \langle a, \bar{a}, \bar{b}, \bar{c}\rangle$, where $\bar{a}, \bar{b}, \bar{c}$ are automorphisms of the tree fixing the vertices of the first level whose ψ-images are $(a,1), (1,b), (1,c)$ respectively (here we use again the self-similarity of binary rooted tree identifying T with T_0, T_1). Then the generators $a, \bar{a}, \bar{b}, \bar{c}$ are of order 2, $\bar{a}, \bar{b}, \bar{c}$ commute, and $(a\bar{a})^4 = (a\bar{b})^4 = (a\bar{c})^4 = 1$, so the group is a homomorphic image of the "tripod" group

$$\Pi = \langle a, b, c, d \| \ a^2, b^2, c^2, d^2, (bc)^2, (bd)^2, (cd)^2, (ab)^4, (ac)^4, (ad)^4\rangle.$$

with respect to the map

$$a \mapsto a, \quad b \mapsto \bar{b}, \quad c \mapsto \bar{c}, \quad d \mapsto \bar{a}.$$

The ψ-image of $st_V(1)$ is a subdirect product of $G \times G$ and contains $A \times A$, where A is the normal closure of a in G (A has finite index in G, as G is just-infinite). Therefore V is abstractly commensurable with $G \times G$. It is a 2-group, and satisfies all the other requirements. The growth argument shows that there are uncountably many such quotients of Π.

Finally, let us consider the case when the graph $\mathcal{Z}$ is a cycle of legth 4. Again, passing to a quotient, we may assume that $m_{i,j} = 4$ for non-commuting generators, so that Γ become a quotient of $\mathcal{C}$. To get uncountably many quotients of Γ we consider the subgroup of the permutational wreath product of the Klein group K acting on the set $X = \{1,2,3,4\}$ with the group G_ω, $\omega \in \Omega_0$ (the Klein group is active). For simplicity, we denote the generators of G_ω by a, b, c, d. Let $\bar{\Gamma}$ be the subgroup of $G_\omega \wr_X K$ generated by elements $x_1 = (12)(34), x_3 = (14)(23), x_1, x_3 \in K$, $x_2 = (1, a, 1, d), x_4 = (a, 1, c, 1)$. Then $(x_1x_3)^2 = (x_2x_4)^2 = 1$, and $(x_1x_2)^2 = (a,a,d,d), (x_1,x_4)^2 = (a,a,c,c), (x_3x_2)^2 = (d,a,a,d), (x_3x_4)^2 = (a,c,c,a)$. Therefore $(x_1x_2)^4 = (x_1x_4)^4 = (x_3x_2)^4 = (x_3x_4)^4 = 1$ and $\bar{\Gamma}$ is a quotient of Γ. It is straightforward to check that all four projections of the base group of $\bar{\Gamma}$ are isomorphic to the group G_ω and that $\bar{\Gamma}$ is commensurable with G_ω^4. Applying the growth arguments we get uncountably many quotients $\bar{\Gamma}$ of Γ. This quotients are branch and just-infinite by the same type of arguments as above. This finishes the proof of the theorem. □

Observe that one can construct infinite torsion quotients of Coxeter groups which can be mapped onto non-elementary hyperbolic groups (in

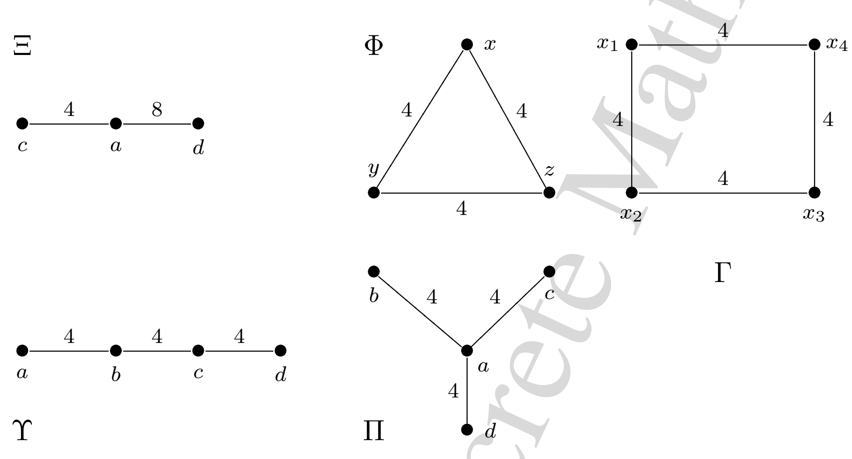

Figure 1: Coxeter graphs corresponding to Ξ, Φ, ϒ, Π, and Γ

Gromov sense [23]), or which are "large" groups in the sense of S. Pride [42, 12] (a group is "large" if it has a subgroup of finite index that can be mapped onto a free group of rank 2), using the results from [26, 39, 11, 33].

The criterion for a Coxeter group given by a connected Coxeter graph to be non-elementary hyperbolic, given by G. Moussong in [34], requires that each Coxeter subgroup generated by a subset $\{x_i, x_j, x_k\}$ of three generators is a hyperbolic triangular group, i.e. a group isomorphic to the group $T^*_{m,n,q}$ with

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{q} < 1.$$

The groups Ξ and Φ are non-elementary hyperbolic and, as was indicated by T. Januszkiewicz, the groups ϒ, Π and Γ can be mapped onto non-elementary hyperbolic groups. Therefore all these groups have uncountably many homomorphic torsion images of bounded degree according to [33].

Indeed, all Coxeter groups which are not virtually abelian are "large", which is a particular case of the results by G. Margulis and E. Vinberg from [31]. This fact was also proved independently by C. Gonciulea, as is indicated in the A. Lubotzki's review [MR1748082 (2001h:22016)] to [31], but published only in a weaker form [15]. Therefore in view of the results from [39, 33], for any prime number p and any Coxeter group $\mathcal{C}$ that is not virtually abelian, there is $2^{\aleph_0}$ pairwise non isomorphic quotients of $\mathcal{C}$ which are virtually p-groups.

It is pointed out by T .Januszkiewicz that it is possible that every Coxeter group that is not virtually abelian has a non-elementary hyperbolic quotient (perhaps this is known fact). If this is the case, then every

Coxeter group that is not virtually abelian has uncountably many torsion quotients of bounded exponent [33].

A preliminary version of this paper submitted to arXiv:0912.2758 has an inaccuracy in the proof of main result. This inaccuracy is corrected here in the proof of theorem 4.1 by including into the consideration the case of group Γ.

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Received by the editors: 25.11.2009
and in final form 15.12.2009.