Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 9 **(2010)**. Number 1. pp. 86 – 102

Finite groups as groups of automata with no cycles with exit

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Communicated by V. I. Sushchansky

Dedicated to Professor I. Ya. Subbotin on the occasion of his 60-th birthday

Abstract. Representations of finite groups as automata groups over a binary alphabet are investigated. The subclass of groups of automata with no cycles with exit is studied.

1. The class of automata groups contains many interesting and complicated groups. Its thorough investigation was started with automata with small number of states. In the case of two states groups of automata over a binary alphabet were completely described in [3]. The great work has been done in the case of 3-state automata over a binary alphabet [2].

Another approach is to investigate representations of some classes of groups as automata groups. Abelian groups generated by automata over a binary alphabet were considered in [5]. Finite automata that generate some finitely presented solvable groups were constructed in [1]. Note interesting examples of finite automata that generate free group of rank 2 ([4]) and free products of finite number of cyclic groups of order 2 ([8]).

In this paper we investigate embedding of finite groups in the class of automata groups over a binary alphabet. We study the subclass of groups of automata with no cycles with exit. All groups from this class are finite. We prove that this class is closed under wreath product with some permutation groups and direct powers. It is proved that the action

2000 Mathematics Subject Classification: *20E08.*
Key words and phrases: *automaton, automaton group, finite group, wreath product.*

of the elements of the group of an automaton with no cycles with exit with n states is uniquely defined by the action on words of length n. This property simplifies calculations in such groups. Some presentations of finite groups by automata with no cycle with exit are obtained. Using GAP system the complete list of groups of automata with no cycles with exit with 2–5 states over a binary alphabet is calculated.

2. Let X be a finite nonempty set. This set is called an alphabet and its elements are called letters. Let us denote by $S(X)$ the symmetric permutation group on alphabet X.

An automaton over the alphabet X is a tuple $A = \langle X, Q, \varphi, \lambda \rangle$, where Q denotes the set of states of A, $\varphi : Q \times X \to Q$ is the transition function and $\lambda : Q \times X \to X$ is the output function.

A cycle in the automaton A is a sequence of pairwise different states $q_1, q_2, \ldots, q_n \in Q$, $n \geq 1$, such that there exists a sequence of letters $x_1, x_2, \ldots, x_n \in X$ which satisfies equalities $\varphi(q_i, x_i) = q_{i+1}$, $1 \leq i < n$, and $\varphi(q_n, x_n) = q_1$. This cycle is called a cycle with exit if there exist i, $1 \leq i \leq n$, and $x \in X$ such that $\varphi(q_i, x) \notin \{q_1, q_2, \ldots, q_n\}$. In other case this cycle is called a cycle without exit. A state $q \in Q$ is cyclic if it belongs to some cycle of the automaton.

Consider the set $X^* = \bigcup_{n \geq 1} X^n \cup \{\Lambda\}$ of all words over X. The symbol Λ denotes the empty word. On this set one can define the operation of concatenation. The set X^* is a vertex set of a rooted tree T_X in which two words are connected by an edge if and only if they are of the form v and vx, where $v \in X^*$, $x \in X$. The empty word Λ is the root of the tree T_X.

The transition and output functions of the automaton A can be extended to the set $Q \times X^*$ by the next formulas. For all $q \in Q$, $w \in X^*$ and $x \in X$

$$\begin{aligned} \varphi(q, wx) &= \varphi(\varphi(q, w), x), & \varphi(q, \Lambda) &= q, \\ \lambda(q, wx) &= \lambda(q, w)\lambda(\varphi(q, w), x), & \lambda(q, \Lambda) &= \Lambda. \end{aligned}$$

Every state $q \in Q$ defines an endomorphism $f_q = \lambda(q, \cdot) : X^* \to X^*$ of the tree T_X. We will usually identify the state q with the endomorphism f_q. The automaton A is called invertible if all these maps are automorphisms.

Definition 1 ([4, section 1.5.4])**.** *The group of an invertible automaton $A = \langle X, Q, \varphi, \lambda \rangle$ is the group generated by the set $\{f_q : q \in Q\}$.*

Hereafter all automata are invertible.

One can describe the automaton A by labeled directed graph with the set of vertices Q. Each vertex $q \in Q$ is labeled by the permutation

$\lambda(q,\cdot) \in S(X)$. The transition function defines the set of edges. For every pair $(q,x) \in Q \times X$ there exists an edge with label x from vertex q to vertex $\varphi(q,x)$.

Let G be the group generated by an automaton A over an alphabet X. For element $g \in G$ and word $w \in X^*$ there exists an automorphism $g|_w : X^* \to X^*$ such that $g(wu) = g(w)g|_w(u)$ for every $u \in X^*$. This automorphism $g|_w$ is called the restriction of g in w. If all restrictions of g in all words of length m coincide we will also denote them by $g|_m$. In calculations in the group G it is useful to describe an element $g \in G$ by its action on words of length 1 and restrictions in all words of length 1:

$$g = (g|_{x_1}, g|_{x_2}, \ldots, g|_{x_n})\pi,$$

where $x_1 < x_2 < \ldots < x_n$ is some ordering of the alphabet X and $\pi = g|_X : X \to X$ is restriction of g on words of length 1.

Theorem 1 ([6, theorem 2])**.** *The group of a finite automaton with no cycles with exit is finite.*

Theorem 2 ([7, theorem 4])**.** *Let $A = \langle X, Q, \varphi, \lambda\rangle$ be an automaton with no cycles with exit over a binary alphabet, $|Q| = n$ and G be the group generated by this automaton. Then $|G| \le 2^{2^{n-1}}$.*

Let $P < S(X)$ be a permutation group, G be a group and $N \lhd G$. Consider the subgroup $H < G^X$ that contains only elements $h : X \to G$ satisfying condition $h(x)(h(y))^{-1} \in N$ for all $x, y \in X$. The subwreath product of the permutation group P with the group G by the normal subgroup N is the semidirect product $P \ltimes H$ with the natural action of group P on the subgroup H. Let us denote it by $P \wr_N G$. If $N = G$ then $P \wr_N G \simeq P \wr G$.

3. We start with some constructions of automata to produce certain classes of groups.

Theorem 3. *Let G be a group generated by (finite) automaton (with no cycles with exit) $A = \langle X, Q, \varphi, \lambda\rangle$ over an alphabet X, $P < S(X)$ and for every state $q \in Q$ the permutation $\lambda(q,\cdot)$ belongs to the group P. Then the group $P \wr G$ is generated by (finite) automaton (with no cycles with exit) over an alphabet X.*

Proof. Suppose that the set $\{p_1, p_2, \ldots, p_k\}$ is a generating set of the group P. Let us construct a new automaton $P \wr A = \langle X, \tilde{Q}, \tilde{\varphi}, \tilde{\lambda}\rangle$ that generates the group $P \wr G$. Consider the set

$$\tilde{Q} = Q \sqcup \{1\} \sqcup \{\hat{p}_i : 1 \le i \le k\} \sqcup (Q \times X).$$

The transition and output functions of this automaton are defined on the set $\tilde{Q} \times X$ as follows. Put $\tilde{\varphi}|_{Q\times X} = \varphi$, $\tilde{\lambda}|_{Q\times X} = \lambda$ and for $x \in X$

$$\begin{aligned}
\tilde{\varphi}(1,x) &= 1, & \tilde{\lambda}(1,x) &= x; \\
\tilde{\varphi}(\hat{p}_i,x) &= 1, & \tilde{\lambda}(\hat{p}_i,x) &= p_i(x), \quad 1 \leq i \leq k; \\
\tilde{\varphi}((q,x_0),x) &= q, x = x_0, & & \\
\tilde{\varphi}((q,x_0),x) &= 1, x \neq x_0, & \tilde{\lambda}((q,x_0),x) &= x, \qquad (q,x_0) \in Q \times X.
\end{aligned}$$

Consider the map $\psi : P \wr G \to \operatorname{Aut} T_X$ defined by

$$[p; g_1, g_2, \dots, g_n] \mapsto (g_1, g_2, \dots, g_n)p,$$

where $p \in P$ and $g_1, g_2, \dots, g_n \in G$. It is a monomorphism. The elements

$$\psi^{-1}(f_{\hat{p}_i}) = [p_i; 1, 1, \dots, 1], \quad 1 \leq i \leq k,$$

and

$$\psi^{-1}(f_{(q,x)}) = [1; 1, \dots, f_q, \dots, 1], \quad q \in Q, x \in X,$$

are generators of the group $P \wr G$. Therefore, the elements $f_{\hat{p}_i}$, $1 \leq i \leq k$, and $f_{(q,x)}$, $q \in Q$, $x \in X$ are generators of the group $\psi(P \wr G) \simeq P \wr G$.

If the group P acts transitively on the set X then for every state $q \in Q$ only one state (q,x) is required for some $x \in X$.

For every state $q \in Q$ we have

$$\psi^{-1}(f_q) = [\lambda(q,\cdot); f_{\varphi(q,x)}, x \in X] \in P \wr G,$$

since $\lambda(q,\cdot) \in P$. Thus, $f_q \in \psi(P \wr G)$ and the automaton $P \wr A$ generates the group $P \wr G$.

If the automaton A is finite or does not contain cycles with exit then the constructed one has the same property. □

Further we give alternative proof of the proposition 2.9.3 from [4] that shows that the class of groups of automata with no cycles with exit is closed under direct powers too and describe construction of an automaton that generates direct power explicitly.

Theorem 4. *Let G be a group generated by (finite) automaton (with no cycles with exit) over an alphabet X. Then for every positive integer n the group $\bigoplus_{i=1}^{n} G$ is generated by (finite) automaton (with no cycles with exit) over an alphabet X.*

Proof. Suppose that the automaton $A = \langle X, Q, \varphi, \lambda \rangle$ generates the group G. Let us construct a new automaton $A^{[n]}$ that generates $\bigoplus_{i=1}^{n} G$. We denote by M_n the set $\{0, 1, \ldots, n-1\}$. Consider the automaton $A^{[n]} = \langle X, Q \times M_n, \varphi_n, \lambda_n \rangle$ with the transition and output functions for $q \in Q$ and $x \in X$ defined by equalities

$$\begin{aligned} &\varphi_n((q,k),x) = (q,k-1), k \neq 0, && \lambda_n((q,k),x) = x, k \neq 0, \\ &\varphi_n((q,0),x) = (\varphi(q,x), n-1), && \lambda_n((q,0),x) = \lambda(q,x). \end{aligned}$$

To prove that $A^{[n]}$ generates the group $\bigoplus_{i=1}^{n} G$ let us extend functions φ_n and λ_n to the set $(Q \times M_n) \times X^*$. Then

$$\begin{aligned} &\varphi_n((q,k), x_1x_2\ldots x_n) = (\varphi(q, x_{k+1}), k), \\ &\lambda_n((q,k), x_1x_2\ldots x_n) = x_1 \ldots x_k \lambda(q, x_{k+1}) x_{k+2} \ldots x_n \end{aligned}$$

and

$$f_{(q,k)}(x_1x_2\ldots x_n u) = x_1 \ldots x_k \lambda(q, x_{k+1}) x_{k+2} \ldots x_n f_{(\varphi(q,x_{k+1}),k)}(u).$$

The set $\{f_{(q,k)} : q \in Q, k \in M_n\}$ is a generating set of the group of the automaton $A^{[n]}$.

Consider a word $x_1x_2\ldots x_n x_{n+1} \ldots x_{mn} \in X^{mn}$, $m \geq 1$. For every $k \in M_n$ all elements of the set $\{f_{(q,k)} : q \in Q\}$ preserve letters in positions l satisfying condition $l - 1 \neq k \pmod{n}$. Transformation of the subword $x_{k+1}x_{n+k+1}\ldots x_{(m-1)n+k+1}$ by $f_{(q,k)}$ coincide with the action of f_q on words of length m. Therefore, the set $\{f_{(q,k)} : q \in Q\}$ generates the group G for every $k \in M_n$.

Let us prove that $f_{(q_1,k_1)}$ and $f_{(q_2,k_2)}$ commute for $k_1 \neq k_2$. It is enough to consider the case $k_1 < k_2$. We will prove by induction on m that for $u \in X^{mn}$ the equality

$$f_{(q_1,k_1)}(f_{(q_2,k_2)}(u)) = f_{(q_2,k_2)}(f_{(q_1,k_1)}(u)) \tag{1}$$

holds. The case $m = 0$ is trivial. For any letters $x_1, x_2, \ldots, x_n \in X$ and any word $u \in X^{mn}$ be induction hypothesis we have

$$\begin{aligned} &f_{(q_1,k_1)}(f_{(q_2,k_2)}(x_1x_2\ldots x_n u)) = \\ &\quad = f_{(q_1,k_1)}(x_1 \ldots x_{k_2} \lambda(q_2, x_{k_2+1}) x_{k_2+2} \ldots x_n f_{(\varphi(q_2,x_{k_2+1}),k_2)}(u) = \\ &\quad = x_1 \ldots x_{k_1} \lambda(q_1, x_{k_1+1}) x_{k_1+2} \ldots x_{k_2} \lambda(q_2, x_{k_2+1}) \\ &\qquad\qquad x_{k_2+2} \ldots x_n f_{(\varphi(q_1,x_{k_1+1}),k_1)}(f_{(\varphi(q_2,x_{k_2+1}),k_2)}(u)) = \\ &\quad = f_{(q_2,k_2)}(x_1 \ldots x_{k_1} \lambda(q_1, x_{k_1+1}) x_{k_1+2} \ldots x_n f_{(\varphi(q_1,x_{k_1+1}),k_1)}(u) = \end{aligned}$$

$$= f_{(q_2,k_2)}(f_{(q_1,k_1)}(x_1x_2\dots x_nu)).$$

Therefore, equality (1) holds for any word $u \in X^{(m+1)n}$.

Thus, the automaton $A^{[n]}$ generates $\bigoplus_{i=1}^{n} G$.

If the automaton A is finite or does not contain cycles with exit then the constructed one has the same property. □

Corollary 1. *Let G be a group generated by (finite) automaton (with no cycles with exit) over a binary alphabet. Then $\mathbb{Z}_2 \wr G$ and $\bigoplus_{i=1}^{n} G$ are generated by (finite) automata (with no cycles with exit) over a binary alphabet.*

In general case the fact that the groups G_1 and G_2 are generated by automata over an alphabet X does not imply that the group $G_1 \times G_2$ is generated by an automaton over the same alphabet X. For example there is no automaton over a binary alphabet that generates the group $\mathbb{Z} \oplus \mathbb{Z}_2$ ([5, proposition 3.4]), but the groups $\mathbb{Z}$ and $\mathbb{Z}_2$ are generated by automata over a binary alphabet.

If one does not require that the alphabet is the same then it is easy to construct an automaton that generates the group $G_1 \times G_2$. We assume that automata generating G_1 and G_2 are defined over alphabets X_1 and X_2 respectively and $X_1 \cap X_2 = \varnothing$. The automaton generating the group $G_1 \times G_2$ is constructed over $X_1 \cup X_2$. It contains two disjoint parts: the first acts on X_1 as automaton for G_1 and trivially on X_2 and the second acts on X_2 as automaton for G_2 and trivially on X_1.

4. Here we show how to distinguish elements of the group of automata that have no cycles with exit.

Lemma 1. *Let A be an n-state automaton with no cycles with exit and G be the group generated by the automaton A. Suppose that all states of A are cyclic. If elements $w_1, w_2 \in G$ act equally on words of length n, then $w_1 = w_2$.*

Proof. Let us denote by k the number of cycles in the automaton A and prove the statement of the lemma by induction on k. It is true for $k = 1$, since $w_i(uu') = w_i(u)w_i(u')$, $i = 1, 2$, for all words $u \in X^n, u' \in X^*$.

Suppose that statement is true for automata with $k-1$ cycles. Let us choose a cycle C in the automaton A. Generators of the group G pairwise commute as all restrictions in words of length 1 coincide for every state. Therefore, elements w_1 and w_2 can be written in the form $w_i = u_i w_i'$, $i = 1, 2$, where u_1 and u_2 contain all multipliers defined by states of the cycle C. The product $u_1^{-1}u_2 w_1'^{-1} w_2'$ acts trivially on words of length n. Hence, elements $u_2^{-1}u_1$ and $w_1'^{-1}w_2'$ act equally on words of length n. Let us denote by m the length of the cycle C. Then $(u_1^{-1}u_2)\big|_m = u_1^{-1}u_2$. On words of length $n-m$ the equality

$$(w_1'^{-1}w_2')\big|_m = (u_2^{-1}u_1)\big|_m = u_2^{-1}u_1 = w_1'^{-1}w_2'$$

holds. Elements $(w_1'^{-1}w_2')\big|_m$ and $w_1'^{-1}w_2'$ belong to the group generated by an automaton with $k-1$ cycles. Therefore, by induction hypothesis, we have $(w_1'^{-1}w_2')\big|_m = w_1'^{-1}w_2'$ and

$$(u_1^{-1}u_2w_1'^{-1}w_2')\big|_m = (u_1^{-1}u_2)\big|_m(w_1'^{-1}w_2')\big|_m = u_1^{-1}u_2w_1'^{-1}w_2'.$$

As the product $u_1^{-1}u_2w_1'^{-1}w_2'$ acts trivially on words of length m, the previous equality implies $w_1^{-1}w_2 = u_1^{-1}u_2w_1'^{-1}w_2' = 1$. □

Theorem 5. *Let A be an n-state automaton with no cycles with exit and G be the group generated by the automaton A. If elements $w_1, w_2 \in G$ act equally on words of length n, then $w_1 = w_2$.*

Proof. Let us choose the smallest $m \geq 0$ such that for all words $v \in X^m$ elements $w_1\big|_v$ and $w_2\big|_v$ are products of generators defined by cyclic states of the automaton A or their inverses. Let $v \in X^m$ be arbitrary word. It follows from conditions of the theorem that $w_1\big|_v = w_2\big|_v$ on words of length $n-m$. Number of cyclic states of the automaton A is less than $n-m$. Therefore, lemma 1 implies $w_1\big|_v = w_2\big|_v$. Elements w_1 and w_2 act equally on first m letters. Hence, $w_1 = w_2$. □

5. Let us describe some series of groups generated by automata with no cycles with exit over a binary alphabet.

Proposition 1. *The group of the automaton shown in figure 1 is isomorphic to the group $\bigoplus_{i=1}^n \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \wr \mathbb{Z}_2)$, $n \geq 1$.*

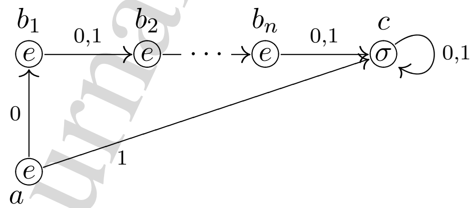

Figure 1

Proof. The generators of the group satisfy the equalities

$$a^2 = c^2 = 1, \quad b_i^2 = 1, \quad b_ib_j = b_jb_i, \quad b_ic = cb_i, \quad 1 \leq i,j \leq n.$$

Therefore equalities $ab_i = (b_1, c)(b_{i+1}, b_{i+1}) = (b_{i+1}, b_{i+1})(b_1, c) = b_i a$, $1 \leq i < n$ and $ab_n = (b_1, c)(c, c) = (c, c)(b_1, c) = b_n a$ hold. So, for the group of the automaton we get

$$\langle a, c, b_i, 1 \leq i \leq n\rangle \simeq \langle b_i, 1 \leq i \leq n\rangle \oplus \langle a, c\rangle \simeq \bigoplus_{i=1}^{n} \mathbb{Z}_2 \oplus \langle a, c\rangle,$$

since the generators $\{b_i, 1 \leq i \leq n\}$ pairwise commute and their orders are equal to 2. Let us find the order of the product $ac = (b_1, c)(c, c)\sigma = (b_1 c, 1)\sigma$:

$$\begin{aligned}
(ac)^2 &= (b_1c, 1)\sigma(b_1c, 1)\sigma = (b_1c, b_1c);\\
(ac)^4 &= (b_1c, b_1c)(b_1c, b_1c) = (b_1cb_1c, b_1cb_1c) = (1, 1) = 1.
\end{aligned}$$

This implies that $\langle a, c\rangle \simeq \mathbb{Z}_2 \wr \mathbb{Z}_2$. $\square$

Proposition 2. *The group of the automaton shown in figure 2 is isomorphic to the group $\mathbb{K}_4 \wr \mathbb{Z}_2$.*

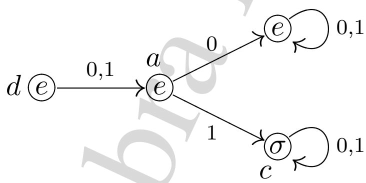

Figure 2

Proof. The generators of the group satisfy the equalities

$$\begin{aligned}
a^2 = c^2 &= d^2 = 1,\\
(ca)^4 &= ((c, c)\sigma(1, c))^4 = ((1, c)\sigma(1, c)\sigma)^2 = (c, c)^2 = 1,\\
(ad)^4 &= ((1, c)(a, a))^4 = (a^4, (ca)^4) = 1,\\
(ada \cdot c)^2 &= ((1, c)(a, a)(1, c)(c, c)\sigma)^2 = ((ac, ca)\sigma)^2 = 1.
\end{aligned}$$

Since $(ada)^2 = 1$, we get $\langle c, ada\rangle \simeq \mathbb{K}_4$ and

$$\langle a, c, d\rangle \simeq \langle c, ada, a\rangle \simeq \langle c, ada\rangle \ltimes \langle a, cac, dad, cdadc\rangle.$$

The action of the generators c, ada on the set $\{a, cac, dad, cdadc\}$ is shown on the diagram.

$$\begin{tikzcd}
a \arrow[r, leftrightarrow, "c"] \arrow[d, Leftrightarrow, "ada"'] & cac \arrow[d, Leftrightarrow, "ada"] \\
dad \arrow[r, leftrightarrow, "c"] & cdadc
\end{tikzcd}$$

The equalities $(ac)^4 = (ad)^4 = 1$ imply that the generator a commutes with the generators cac and dad. Since

$$\begin{aligned}(a\cdot cdadc)^2 &= (a\cdot c\cdot adada\cdot c)^2 = (a\cdot ada\cdot c\cdot da\cdot c)^2 = \\ &= (dac)^4 = ((ac,a)\sigma)^4 = ((ac,a)(a,ac))^2 = \\ &= (aca,c)^2 = 1,\end{aligned}$$

the generators a and $cdadc$ commute and the generators cac and dad commute as well. It remains to check that the generator $cdadc$ commutes with the generators cac and dad:

$$\begin{aligned}(cac\cdot cdadc)^2 &= (cadadc)^2 = c(ad)^4c = 1;\\ (dad\cdot cdadc)^2 &= (dadc)^4 = (adada\cdot c)^4 = \\ &= (ad\cdot c\cdot ada)^4 = ad(ca)^4da = 1.\end{aligned}$$

Thus $\langle a,c,d\rangle \simeq \mathbb{K}_4 \wr \mathbb{Z}_2 \simeq \langle a,b,c\,|\,a^2,b^2,c^2,(ab)^4,(ac)^4,(bc)^2,(abc)^4\rangle$, where $b = ada$. □

Proposition 3. *The group of the automaton shown in figure 3 is isomorphic to the group* $\bigoplus_{i=1}^{n}\mathbb{Z}_2 \oplus (\mathbb{K}_4 \wr \mathbb{Z}_2)$, $n \geq 1$.

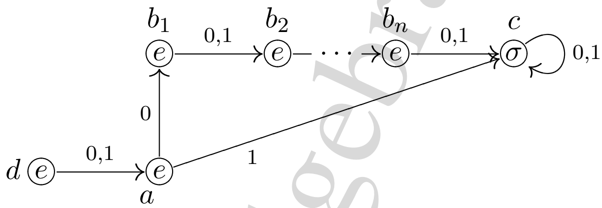

Figure 3

Proof. One can obtain this automaton from the automaton shown in figure 1 by adding the state d. Therefore we have

$$a^2 = c^2 = d^2 = 1, \quad b_i^2 = 1,$$
$$ab_i = b_ia, \quad cb_i = b_ic, \quad b_ib_j = b_jb_i, \quad 1 \leq i,j \leq n, \quad (ac)^4 = 1.$$

It follows that

$$db_i = (a,a)(b_{i+1},b_{i+1}) = (b_{i+1},b_{i+1})(a,a) = b_id, \quad 1 \leq i < n,$$

and for the group of the automaton we get

$$\begin{aligned}\langle a,c,d,b_i, 1\leq i\leq n\rangle &\simeq \langle b_i, 1\leq i<n\rangle \oplus \langle a,c,d,b_n\rangle \simeq \\ &\simeq \bigoplus_{i=1}^{n-1}\mathbb{Z}_2 \oplus \langle a,c,d,b_n\rangle.\end{aligned}$$

Let as show that $b_n(ac)^2$ commutes with a, c and d:

$$\begin{aligned}
a\cdot b_n(ac)^2 &= b_n a(ac)^2 = b_n(ac)^2\cdot a,\\
c\cdot b_n(ac)^2 &= b_n c(ac)^2 = b_n(ac)^2\cdot c,\\
d\cdot b_n(ac)^2 &= (a,a)(c,c)(b_1c,b_1c) = (a,a)(b_1,b_1) = (b_1,b_1)(a,a)\\
&= (c,c)(b_1c,b_1c)(a,a) = b_n(ac)^2\cdot d,
\end{aligned}$$

which implies

$$\langle a,c,d,b_n\rangle \simeq \langle b_n(ac)^2\rangle \oplus \langle a,c,d\rangle \simeq \mathbb{Z}_2 \oplus \langle a,c,d\rangle.$$

It remains to prove that $\langle a,c,d\rangle \simeq \mathbb{K}_4 \wr \mathbb{Z}_2$. Since $(ab_1)^2 = (ac)^4 = 1$, we have

$$\begin{aligned}
(ada\cdot c) &= (b_1,c)(a,a)(b_1,c)(c,c)\sigma = (ac,ca)\sigma,\\
(ada\cdot c)^2 &= (ac,ca)\sigma(ac,ca)\sigma = (ac,ca)(ca,ac) = 1,\\
(ad)^4 &= ((b_1,c)(a,a))^4 = ((b_1a)^4,(ca)^4) = 1.
\end{aligned}$$

As $c^2 = (ada)^2 = 1$, we get $\langle c, ada\rangle \simeq \mathbb{K}_4$ and

$$\langle a,c,d\rangle \simeq \langle c, ada, a\rangle \simeq \langle c, ada\rangle \ltimes \langle a, cac, dad, cdadc\rangle.$$

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dad \arrow[r, leftrightarrow, "c"] & cdadc
\end{tikzcd}$$

The equalities $(ac)^4 = (ad)^4 = 1$ imply that the generator a commutes with the generators cac and dad. Since

$$\begin{aligned}
(a\cdot cdadc)^2 &= (a\cdot c\cdot adada\cdot c)^2 = (a\cdot ada\cdot c\cdot da\cdot c)^2 =\\
&= (dac)^4 = ((ab_1c,a)\sigma)^4 = ((ab_1c,a)(a,ab_1c))^2 =\\
&= (ab_1ca, b_1c)^2 = 1,
\end{aligned}$$

the generators a and $cdadc$ commute and the generators cac and dad commute as well. The proof that the generator $cdadc$ commutes with the generators cac and dad is the same as in the proof of the proposition 2. Thus $\langle a,c,d\rangle \simeq \mathbb{K}_4 \wr \mathbb{Z}_2$. $\square$

Proposition 4. *The group of the automaton shown in figure 4 is isomorphic to the group* $\mathbb{Z}_2 \oplus \wr_{i=1}^{n} \mathbb{Z}_2$, $n \geq 1$

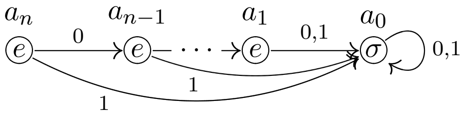

Figure 4

Proof. The generators of the group satisfy the equalities

$$a_i^2 = (a_{i-1}, a_0)(a_{i-1}, a_0) = (a_{i-1}^2, a_0^2) \text{ for } i > 0 \text{ and } a_0^2 = (a_0^2, a_0^2)$$

which implies $a_i^2 = 1$, $0 \leq i \leq n$. Let us choose another set of generators in the group G of the automata

$$G = \langle a_i, 0 \leq i \leq n \rangle \simeq \langle a_0, a_i a_{i-1}, 1 \leq i \leq n \rangle.$$

Consider the subgroup $H = \langle a_i a_{i-1}, 1 \leq i \leq n \rangle$. For its generators we get

$$\begin{aligned} a_i a_{i-1} &= (a_{i-1}, a_0)(a_{i-2}, a_0) = (a_{i-1} a_{i-2}, 1), \\ (a_i a_{i-1})^2 &= ((a_{i-1} a_{i-2})^2, 1), \quad i > 1; \\ a_1 a_0 &= (a_0, a_0)(a_0, a_0)\sigma = (1, 1)\sigma, \\ (a_1 a_0)^2 &= 1. \end{aligned}$$

Thus

$$H = \langle a_i a_{i-1}, 1 \leq i \leq n \rangle \simeq \wr_{i=1}^{n} \mathbb{Z}_2.$$

Let us show that the generator a_0 can be replaced by another one that commutes with all generators of the subgroup H. For a word $w = a_{i_1} a_{i_2} \ldots a_{i_k}$ we denote by $r(w)$ the word $a_{i_1+1} a_{i_2+1} \ldots a_{i_k+1}$. Consider the sequence

$$w_0 = a_1, \quad w_i = r(w_{i-1}) a_0 r(w_{i-1}) a_0 a_1, \quad 1 \leq i \leq n-1.$$

All words of this sequence contain odd number of letters. We will identify them with the corresponding products in the group G. So, we have

$$w_0 = (a_0, a_0), \quad w_i = (w_{i-1}, w_{i-1}), \quad 1 \leq i \leq n-1.$$

Let us prove by induction on i that w_{i-1} commutes with all elements a_j, $0 \leq j \leq i$. Case $i = 1$ is trivial. The equalities

$$w_i a_0 = (w_{i-1}, w_{i-1})(a_0, a_0)\sigma = \\ = (w_{i-1}a_0, w_{i-1}a_0)\sigma = (a_0 w_{i-1}, a_0 w_{i-1})\sigma = \\ = (a_0, a_0)\sigma(w_{i-1}, w_{i-1}) = a_0 w_i$$

are equivalent to the equality $w_{i-1}a_0 = a_0 w_{i-1}$ which implies from the induction hypothesis. For $1 \leq j \leq i+1$ by induction hypothesis we get

$$w_i a_j = (w_{i-1}, w_{i-1})(a_{j-1}, a_0) = \\ = (w_{i-1}a_{j-1}, w_{i-1}a_0) = (a_{j-1}w_{i-1}, a_0 w_{i-1}) = \\ = (a_{j-1}, a_0)(w_{i-1}, w_{i-1}) = a_j w_i.$$

Hence w_{n-1} commutes with all elements a_i, $0 \leq i \leq n$, and, consequently, with all generators $a_i a_{i-1}$, $1 \leq i \leq n$, of the subgroup H.

Let us prove that the generator a_0 can be replaced by the generator w_{n-1}. A product of even number of elements a_i, $0 \leq i \leq n$, belongs to the subgroup H, since

$$a_i a_j = a_i a_{i-1} \cdot a_{i-1} a_{i-2} \cdots a_{j+1} a_j \text{ and } a_j a_i = (a_i a_j)^{-1} \text{ for } i > j.$$

The element $a_0 w_{n-1}$ is the product of even number of elements a_i, $0 \leq i \leq n$, and, consequently, belongs to the subgroup H. Hence the element a_0 can be represented as a product of w_{n-1} and the generators of the subgroup H. Thus

$$G \simeq \langle w_{n-1}, a_i a_{i-1}, 1 \leq i \leq n \rangle \simeq \\ \simeq \langle w_{n-1} \rangle \oplus \langle a_i a_{i-1}, 1 \leq i \leq n \rangle \simeq \mathbb{Z}_2 \oplus \mathop{\wr}_{i=1}^{n} \mathbb{Z}_2,$$

since the order of the element w_{n-1} is equal to 2. □

Proposition 5. *The group of the automaton shown in figure 5 is isomorphic to the group* $\mathbb{Z}_2 \wr_{\langle a,b,b^c \rangle} \langle a, b, c \,|\, a^2, b^2, c^2, (ab)^2, (ac)^2, (bc)^4 \rangle$.

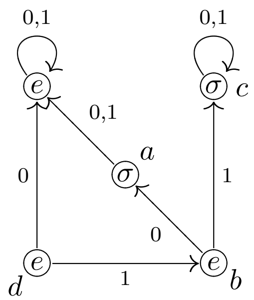

Figure 5

Proof. Consider the subgroup $H = \langle a, b, c\rangle$. It is obvious that $a^2 = b^2 = c^2 = 1$ and $ac = ca$. Let us prove that ac and b commute:

$$ac \cdot b = \sigma(c,c)\sigma(a,c) = (c,c)(a,c) = (ca,1),$$
$$b \cdot ac = (a,c)\sigma(c,c)\sigma = (a,c)(c,c) = (ac,1) = (ca,1).$$

Hence $H \simeq \langle ac\rangle \oplus \langle b, c\rangle \simeq \mathbb{Z}_2 \oplus \langle b, c\rangle$. It remains to calculate the order of the product bc.

$$bc = (a,c)(c,c)\sigma = (ac,1)\sigma,$$
$$(bc)^2 = (ac,1)\sigma(ac,1)\sigma = (ac,ac),$$
$$(bc)^4 = (ac,ac)(ac,ac) = (1,1) = 1.$$

Thus, $H \simeq \mathbb{Z}_2 \oplus \mathbb{D}_4$.

For the group $G = \langle a, b, c, d\rangle$ of the automaton we have $G < \mathbb{Z}_2 \wr H$. Elements of the group G can be represented by tables $[\pi; h_1, h_2]$, where $\pi \in \mathbb{Z}_2$ and $h_1, h_2 \in H$. Every element of the group G is a product of elements from the list

$$d = (1,b),$$
$$cdc = (c,c)\sigma(1,b)(c,c)\sigma = (c,c)(b,1)(c,c) = (cbc,1),$$
$$ada = \sigma(1,b)\sigma = (b,1),$$
$$acdca = \sigma(cbc,1)\sigma = (1,cbc),$$
$$bca = (a,c)(c,c)\sigma\sigma = (ac,1),$$
$$cba = (c,c)\sigma(a,c)\sigma = (1,ca) = (1,ac)$$

and the element $a^m c^n$, $0 \le m, n \le 1$. All elements from the list above generate subgroup $N \times N$, $N = \langle ac\rangle \oplus \langle b\rangle \oplus \langle cbc\rangle \simeq \mathbb{Z}_2^3$. The parameter m allows to choose any element $\pi \in \mathbb{Z}_2$. It does not change elements h_1 and h_2. The parameter n allows to choose any pair of elements h_1, h_2 from coset N or from coset Nc. Thus,

$$G \simeq \mathbb{Z}_2 \wr_{\langle \bar a, b, b^c\rangle} \langle \bar a, b, c \,|\, \bar a^2, b^2, c^2, (\bar a b)^2, (\bar a c)^2, (bc)^4\rangle,$$

where $\bar a = ac$. □

Proposition 6. *The group of the automaton shown in figure 6 is isomorphic to the group* $\mathbb{Z}_2 \wr_{\langle c, c^a, c^{ab}\rangle} \langle a, b, c \,|\, a^2, b^2, c^2, (ab)^4, (ac)^4, (bc)^2, (abc)^4\rangle$.

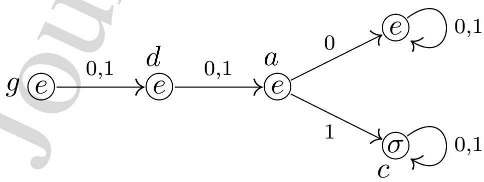

Figure 6

Proof. One can get this automaton from the automaton shown in figure 2 by adding the state g. In proposition 2 we prove that $a^2 = c^2 = d^2 = (ac)^4 = 1$ and

$$H = \langle a, c, d\rangle \simeq \langle a, b, c \mid a^2, b^2, c^2, (ab)^4, (ac)^4, (bc)^2, (abc)^4\rangle \simeq \mathbb{K}_4 \wr \mathbb{Z}_2,$$

where $b = ada$. For the group $G = \langle a, c, d, g\rangle$ of the automaton we have

$$\begin{aligned} aca &= (1, c)(c, c)\sigma(1, c) = (c, 1)(c, 1)\sigma = \sigma, \\ aca \cdot a \cdot aca &= \sigma(1, c)\sigma = (c, 1), \\ aca \cdot d \cdot aca &= \sigma(a, a)\sigma = (a, a) = d, \\ aca \cdot g \cdot aca &= \sigma(d, d)\sigma = (d, d) = g. \end{aligned}$$

Hence $G \simeq \langle aca\rangle \ltimes \langle a, a', d, g\rangle$, where $a' = acaca$. One can assume that elements a, a', d, g belong to the group $H \times H$, and aca acts by permuting components.

Consider the subgroup $\bar{H} = \langle a, a', d, g\rangle < H \times H$. Let us denote by N the normal closure of the set $\{c\}$ in the group H. Restrictions in words of length 1 of every generator a, a', d, g belong to the same coset of the subgroup N. Therefore components of every element of the group $\bar{H}$ belong to the same coset of the subgroup N. Let us prove that $\bar{H}$ contains all such elements. Using generators a, d, g one can construct a pair with any element of the group H in the second component. In the first component one can put any element from the coset of the subgroup N with second component as representative, since normal closure of the element a' in the group $\langle a', d, g\rangle$ is equal to $N \times \{1\}$.

Thus, we get $G \simeq \mathbb{Z}_2 \wr_N H$. It remains to calculate the subgroup N:

$$\begin{aligned} dcd &= (a, a)(c, c)\sigma(a, a) = (aca, aca)\sigma, \\ dacad &= (a, a)\sigma(a, a) = \sigma = aca, \\ adcda &= (1, c)(aca, aca)\sigma(1, c) = (acac, caca)\sigma = \\ &= (acac, acac)\sigma = (aca, aca)(c, c)\sigma = dcd \cdot aca \cdot c, \\ cdcdc &= (c, c)\sigma(aca, aca)\sigma(c, c)\sigma = (cacac, cacac)\sigma = \\ &= (aca, aca)\sigma = dcd. \end{aligned}$$

Hence $N = \langle c, aca, dcd\rangle = \langle c, aca, bacab\rangle = \langle c\rangle \oplus \langle aca\rangle \oplus \langle dcd\rangle \simeq \mathbb{Z}_2^3$. $\square$

6. The class of groups of automata with no cycles with exit even over a binary alphabet is very large. So, the following questions arise. What groups are there in this class? If a group G is in this class then what is the minimal number of states in an automaton that generates G?

Figure 7

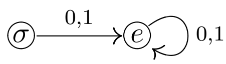

Figure 8

Using construction from the proof of the theorem 4 one can obtain from the automaton shown in figure 7 an automaton with n states that generates the group $\bigoplus_{i=1}^{n} \mathbb{Z}_2$. But a minimal generating system of this group contains n elements. Thus, the minimal number of states in an automaton that generates the group $\bigoplus_{i=1}^{n} \mathbb{Z}_2$ is equal to n.

The construction from the proof of the theorem 3 applied to the automaton shown in figure 8 produces an automaton with $n+1$ states that generates the group $\wr_{i=1}^{n} \mathbb{Z}_2$. This group has 2^{2^n-1} elements. By theorem 2 the group of an automaton with no cycles with exit with n states has at most $2^{2^{n-1}}$ elements. But $2^{2^{n-1}} < 2^{2^n-1}$ for $n \geq 2$. Thus, the minimal number of states in an automaton with no cycles with exit that generates the group $\wr_{i=1}^{n} \mathbb{Z}_2$, $n \geq 2$, is equal to $n+1$.

Group	2 states	3 states	4 states
$\mathbb{Z}_2$	2	25	642
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	2	35	838
$\mathbb{Z}_2 \wr \mathbb{Z}_2$	—	16	720
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	—	10	294
$\mathbb{Z}_2 \oplus (\mathbb{Z}_2 \wr \mathbb{Z}_2)$	—	16	880
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	—	—	66
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \wr \mathbb{Z}_2)$	—	—	200
$\mathbb{K}_4 \wr \mathbb{Z}_2$	—	—	80
$\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}_2)$	—	—	320
$\mathbb{Z}_2 \oplus (\mathbb{K}_4 \wr \mathbb{Z}_2)$	—	—	80
$\mathbb{Z}_2 \oplus (\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}_2))$	—	—	320
Total	4	102	4440

Table 1. List of the groups of automata with no cycles with exit.

The number of states in the automata shown in figures 1 and 3 is equal to the minimal number of generators of the groups generated by them. For the groups of the automata shown in figures 2 and 4 the theorem 2 implies that the number of states is minimal.

All groups mentioned above are the complete list of the groups of automata with no cycles with exit with the number of states less or equal to 4. This was checked by the GAP system; results are shown in the table 1.

The table 2 contains the complete list of groups of automata with no cycles with exit with 5 states. Is was found by the GAP system too. The groups are marked with a star if the estimation of the number of states in an automaton by theorem 2 is not precise. The following notation is used in the table:

$$G_1 = \langle a, b, c \,|\, a^2, b^2, c^2, (ab)^2, (ac)^2, (bc)^4 \rangle \simeq \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \wr \mathbb{Z}_2),$$
$$G_2 = \langle a, b, c \,|\, a^2, b^2, c^2, (ab)^4, (ac)^4, (bc)^2, (abc)^4 \rangle \simeq \mathbb{K}_4 \wr \mathbb{Z}_2,$$
$$G_3 = \langle a, b, c \,|\, a^2, b^2, c^2, (ab)^4, (ac)^4, (bc)^4, (acbc)^2 \rangle \simeq \mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}_2).$$

Group	Number of direct factors $\mathbb{Z}_2$					
	0	1	2	3	4	5
$\{1\}$	—	31436	37926	9850	2796*	492*
$\mathbb{D}_4$	37352	47576	14616	2376*	—	—
$\mathbb{Z}_2 \wr \mathbb{K}_4$	5040*	3600*	—	—	—	—
$\mathbb{K}_4 \wr \mathbb{Z}_2$	6000	6120	600*	—	—	—
$\mathbb{D}_4 \oplus \mathbb{D}_4$	840*	600*	—	—	—	—
$\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}_2)$	32640	33360	3600	—	—	—
$\mathbb{Z}_2 \wr_{\langle a,b,b^c \rangle} G_1$	2688*	1920	—	—	—	—
$\mathbb{Z}_2 \wr_{\langle a,bc \rangle} G_1$	2352*	1680	—	—	—	—
$\mathbb{Z}_2 \wr_{\langle c,c^a,c^{ab} \rangle} G_2$	480	480	—	—	—	—
$\mathbb{Z}_2 \wr_{\langle b,c,b^a,c^a \rangle} G_2$	1920	1920	—	—	—	—
$\mathbb{Z}_2 \wr_{\langle ab,c \rangle} G_2$	960	960	—	—	—	—
$\mathbb{Z}_2 \wr_{\langle b,c,b^a,c^a \rangle} G_3$	1920	1920	—	—	—	—
$\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}_2))$	11520	11520	—	—	—	—

Table 2. List of the groups of automata with no cycles with exit with 5 states.

We consider automata with no cycles with exit over the binary alphabet $X = \{0, 1\}$ with the set of states $Q = \{1, 2, \ldots, n\}$ such that there exists k, $1 \leq k \leq n-1$, and for $q \in Q$ the equality $\lambda(q, \cdot) = 1$ holds if and only if $q \leq k$. The tables show the number of automata that generate the specified group.

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Received by the editors: 22.09.2009
and in final form 18.05.2010.