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Abelian-by-FC-hypercentral groups

Группы, являющиеся расширением абелевых посредством FC-гиперцентральных групп

A splitting Theorem for FC-hypercentral group G and ZG-module of finite rank is obtained. According to this Theorem, under certain conditions, every extension E of the ZG-module A by the group G is split and all the complements to A in E are conjugate in E.

Для FC-гиперцентральной группы G и ZG-модуля конечного ранга получена теорема о расщеплении. Согласно этой теореме при определенных условиях каждое расширение E ZG-модуля A посредством группы G расщепляемо и все дополнения к A в E сопряжены в E.

Для FC-гіперцентральної групи G и $\mathbf{Z}G$-модуля скінченного рангу одержана теорема про розщеплення. Згідно з цією теоремою при певних умовах кожне розширення E $\mathbf{Z}G$-модуля A за допомогою групи G є розщеплюваним і всі доповнення до A в E спряжені в E.

In recent years a number of results have been obtained which say that, under certain conditions, every extension E of the $\mathbf{Z}G$-module A by the group G split and all the complements to A in E are conjugate in E. We say that E splits conjugately over A. Most of these results concern a group G which is generalized nilpotent (or supersoluble) and a module A which has no factors which are G-trivial (or cyclic as abelian groups). Results of this type may be found in [1—6] and may also be found in [7—9] in a cohomological setting where, in particular, it is shown that $H^2(G, A) = H^1(G, A) = 0$.

In [10, 11] D. I. Zaĭcev considered these splitting theorems for a hyperfinite group G and a module A which is either artinian or noetherian. In [14] he began to consider modules over FC-hypercentral groups showing that such a module with a finite composition series has a decomposition into a direct sum of a finite submodule and a submodule with no nonzero finite factors. Here we consider a splitting theorem in which G is FC-hypercentral and the $\mathbf{Z}G$-module A has finite rank as an abelian group. (By the rank of the abelian group A we mean Mal'cev special rank or Prufer rank.) We prove the following.

Theorem. *Let G be a locally soluble FC-hypercentral group and let A be a $\mathbf{Z}G$-module which has finite rank as an abelian group.*

i). *A has no nonzero G-hyperfinite images if and only if A has no nonzero finite G-factors.*

ii). *If A has no nonzero G-hyperfinite image then every extension E of A by G splits conjugately over A.*

A G-hyperfinite image of A is a $\mathbf{Z}G$-homomorphic image of A which has an ascending chain of $\mathbf{Z}G$-submodules in which the factors are finite. A G-factor of A is a factor B/C where B and C are $\mathbf{Z}G$-submodules of A.

It follows from i) that if A has no nonzero G-hyperfinite images then A must be torsion-free and divisible and so (as an abelian group) A is just the direct sum of finitely many copies of $\mathbf{Q}$, the additive group of the rationals. In particular, $G/C_G(A)$ is a $\mathbf{Q}$-linear group and so is soluble. The simplest examples in which the hypotheses of part ii) of the theorem occur and in which we have splitting is when G acts faithfully on A so that G is a soluble $\mathbf{Q}$-linear group and A has a finite G-composition series in which the irreducible factors are all infinite.

However, in the statement of part ii) of the theorem there is no assumption that G acts faithfully on A and the simplest way to construct examples of non-split extensions in which A has finite factors is to include an extension of A by $C_G(A)$ which is non-splitting.

For example, let $M = A \oplus B$ be a sum of two copies of the rationals with an isomorphism $\varphi : A \to B$. Let x be the automorphism of infinite order which fixes each element of A and maps $b \in B$ to $b + \varphi(b)$. Form E, the split extension of M *by* $\langle x \rangle$, and let $G = \bar{E}/A \cong \mathbf{Q} \oplus \mathbf{Z}$. Let C be any submodule of M not contained in A so that C contains an element $a + b$ with $a \in A$, $b \in B$ and $b \neq 0$. Then $(a + b)x = a + b + \varphi(b) \in C$ and so $\varphi(b) \in C$. Hence $A \cap C \neq 0$ and so A is not a direct summand of M. Thus E does not split over A.

It should be noted that there is no point in considering extensions by hyperfinite groups. For, if G is hyperfinite, then its irreducible modules are elementary abelian p-groups [1] and so, if A has finite rank, its irreducible G-factors are all finite.

We begin by proving part i) of the theorem.

Let G be a locally soluble FC-hypercentral group and let A be a $\mathbf{Z}G$-module which has finite rank as an abelian group. Then A has a nonzero G-hyperfinite image if and only if it has a nonzero finite G-factor.

Proof. We may assume that G acts faithfully on A. It is clear that if A has a nonzero G-hyperfinite image then it has a nonzero finite G-factor. So, conversely, we assume that A has a finite G-factor U/V which may be taken to be irreducible and so is a finite elementary abelian p-group.

Choose a submodule X of A maximal subject to $X \cap U = V$. Replacing A by A/X we may assume that A has a unique minimal submodule U, and U

is a finite elementary abelian p-group. Let T be the torsion part of A; then T is a nonzero p-group and, since it has finite rank, T is G-hyperfinite. So we may assume that A/T is nonzero.

We now prove by induction on $r = r(A/T)$, the rank of A/T, that A has a nonzero G-image which is a (hyperfinite) p-group. If A/T is not rationally irreducible then there is a submodule B/T of A/T such that $r(A/B) < r$ and $r(B/T) < r$. By induction, B has a nonzero G-hyperfinite p-image B/C. So the torsion part of A/C is B/C and, by induction again, A/C has a nonzero G-hyperfinite p-image. Therefore we may assume that A/T is rationally irreducible.

We claim that A/T is faithful for G. If not, then $C_G(A/T) \neq 1$ and so there is a nontrivial element $x \in C_G(A/T) \cap \Delta(G)$, where $\Delta(G)$ denotes the FC-centre ot G. Let $F = \langle x^G \rangle$; then F is generated by finitely many conjugates of x and, if $K = C_G(F)$, we have $|G/K| < \infty$.

For each $y \in F - 1$, the mapping $\varphi_y \colon a \to a(y-1)$ is a $\mathbf{Z}K$- homomorphism of A into T and, since G acts faithfully on A, φ_y is a nonzero homomorphism. Thus $A/C_A(y) \cong_{\mathbf{Z}K} A(y-1)$ is a nonzero K-image of A which is a p-group. Suppose that $F = \langle x^{t_1}, \ldots, x^{t_n} \rangle$; then $C_A(F) = \bigcap_{i=1}^{n} C_A(x^{t_i})$ and so $A/C_A(F)$ is a p-group. But $C_A(F)$ is a $\mathbf{Z}G$-submodule of A and, since A has finite rank, $A/C_A(F)$ is therefore a nonzero G-hyperfinite p-image.

Thus we may assume that A/T is faithful for G so that G is an irreducible $\mathbf{Q}$-linear soluble group and so is abelian-by finite [13] (Theorem 3.24). Let H be an abelian normal subgroup of finite index in G and let $H_1 = C_H(U)$ so that $|G/H_1| < \infty$. Now H_1 is abelian and A has a nonzero H_1-trivial p-submodule U. By Lemma 2.8 of [1], A has a nonzero H_1-hypertrivial p-image A/D. If $\{s_1, \ldots, s_m\}$ is a transversal to H_1 in G then $D_0 = \bigcap_{j=1}^{m} Ds_j$ is a $\mathbf{Z}G$-submodule of A and A/D_0 is a p-group. Since A has finite rank, A/D_0 is a nonzero G-hyperfinite p-image of A.

L e m m a 1. *Let G be an FC-hypercentral group and let A be a $\mathbf{Z}G$-module which has finite rank as an abelian group. Let B be a submodule of A such that B has no nonzero finite G-factors and G induces a finite group of automorphisms on A/B. Then there is a unique submodule C of A such that $A = B \oplus C$.*

P r o o f. We may assume that G acts faithfully on A and we proceed by induction on $r = r(B)$ to show that B has a complement in A. If B is not rationally irreducible then it has a submodule B_1 such that $r(B/B_1) < r$ and $r(B_1) < r$. Then A/B_1 contains a submodule C_1 such that $A/B_1 = (B/B_1) \oplus (C_1/B_1)$. Now $C_1/B_1 \cong_{\mathbf{Z}G} A/B$ and so G induces a finite group of automorphisms on C_1/B_1. Again by induction C_1 has a submodule C such that $C_1 = B_1 \oplus C$ and hence $A = B \oplus C$. So we may assume that B is rationally irreducible. This means that every proper G-image of B is torsion. But since A has finite rank, any nonzero torsion factor will have nonzero finite G-factors. Thus B has no proper nonzero G-images and so is actually irreducible as a $\mathbf{Z}G$-module.

Since G is FC-hypercentral there is a nontrivial element $x \in C_G(A/B) \cap \Delta(G)$. Let $F = \langle x^G \rangle$ and $L = C_G(A/B) \cap C_G(F)$, so that G/L is finite. Then L acts trivially on A/B and B has no finite L-factors [14] (Proposition 2). For each $y \in F$, $A(y-1)$ is a $\mathbf{Z}L$-submodule of B and so has no finite L-factors. Also $A/C_A(y) \cong_{\mathbf{Z}L} A(y-1)$ and so $A/C_A(y)$ has no finite L-factors. If $F = \langle x^{t_1}, \ldots, x^{t_n} \rangle$, then $C_A(F) = \bigcap_{i=1}^{n} C_A(x^{t_i})$ and so $A/C_A(F)$ has no finite L-factors. It follows that $C_A(F) + B = A$.

But $C_A(F)$ is a $\mathbf{Z}G$-submodule and so $C_A(F) \cap B$ is equal to either 0 or B. If $C_A(F) \geqslant B$ then, since $C_A(F) + B = A$, we have $C_A(F) = A$, contrary to G acting faithfully on A. Therefore $C_A(F) \cap B = 0$ and $A = C_A(F) \oplus B$.

Now suppose that $A = B \oplus C = B \oplus C_0$. Then G induces a finite group of automorphisms on each of C and C_0 and hence also on $C + C_0$. Therefore every irreducible G-factor of $C + C_0$ is finite and so $B \cap (C + C_0) = 0$. It follows that $C = C + C_0 = C_0$.

L e m m a 2. *Let G be a locally soluble FC-hypercentral group and let A be a $\mathbf{Z}G$-module which has finite rank as an abelian group and such that A has no nonzero finite G-factors. Let E be an extension of A by G and let $N = C_E(A)$. Then there is a normal subgroup M of E such that $N = A \times M$ and M is contained in all supplements to A in E.*

P r o o f. We show first that there is a normal subgroup M such that $N = A \times M$.

Choose M to be a normal subgroup of E maximal with respect to $M \leqslant N$ and $M \cap A = 1$. By considering E/M, we may assume

(*) if S is a nontrivial normal subgroup of E contained in N, then $S \cap A \neq 1$. We show that, under this assumption, N must be egual to A.

Suppose that $N \neq A$; then, since E/A is FC-hypercentral, there is a nontrivial normal subgroup L/A of E/A with $L \leqslant N$ and $L/A \leqslant \Delta(E/A)$. But then L/A is a locally soluble FC-group and so contains a nontrivial characteristic abelian subgroup K/A. (If $Z(L/A) = 1$ then L/A is periodic and we may take K/A to be the socle of L/A.) Let $x \in K - A$ and $F = \langle x^E \rangle$. Then F/A is abelian and $E/C_E(F/A)$ is finite. Let $C = C_E(F/A)$; then $[F, C, F] \leqslant [A, F] = 1$, since $F \leqslant N = C_E(A)$. Also $[C, F, F] \leqslant [A, F] = 1$ and so, by the Three Subgroup Lemma [13] (Lemma 2.13), $[F', C] = 1$. Therefore F' is centralized by C and so G induces a finite group of automorphisms on F'. It follows that any irreducible G-factors of F' are finite. By the hypothesis on A it follows thet $F' = 1$ and so F is abelian. We may therefore consider F as a $\mathbf{Z}G$-module. By Lemma 1 there is a normal subgroup C of E such that $F = A \times C$. But this is contrary to (*) and so we have $N = A$.

This completes the proof that $N = A \times M$. Now let E_1 be a supplement to A in E so that $E = AE_1$ and $N = N \cap AE_1 = A(N \cap E_1)$. Note that $N \cap E_1 \lhd E_1$ and $[N \cap E_1, A] = 1$ so that $N \cap E_1 \lhd E$.

Now $N/N \cap E_1 \cong_{ZG} A/A \cap E_1$ has no nonzero finite G-factors. But

$$M(N \cap E_1)/N \cap E_1 \cong_{ZG} M/M \cap E_1 \cong_{ZG} N/A(M \cap E_1)$$

and, since E/A is FC-hypercentral, it follows that $M(N \cap E_1) = N \cap E_1$, and so $M \leqslant E_1$.

Let G be a locally soluble FC-hypercentral group and let A be a $\mathbf{Z}G$-module which has finite rank as an abelian group. If A has no nonzero G-hyperfinite image then every extension E of A by G splits conjugately over A.

P r o o f. It follows from that A has no non-zero finite G-factors and, in particular, A is torsion-free. By induction on the rank of A we may assume that A is rationally irreducible (and, as in Lemma 1, A is irreducible as a $\mathbf{Z}G$-module).

Let $N = C_E(A)$; then E/N is an irreducible $\mathbf{Q}$-linear soluble group and so is abelian-by-finite [13] (Theorem 3.24). Let H/N be an abelian normal subgroup of finite index in E/N. By Proposition 2 of [14], A has no nonzero finite H-factors and, in particular, $[A, H] = A$.

By Lemma 2 $N = A \times M$ for some $M \lhd E$ and each supplement to A in E contains M. Let $\bar{A} = N/M \cong A$ as a $\mathbf{Z}\bar{G}$-module, where $\bar{G} = E/N$. If $\bar{H} = H/N$, then $[\bar{A}, \bar{H}] = \bar{A}$ and so, by Theorem B of [1] E/M splits conjugately over N/M.

Let K/M be a complement to N/M. Then $KA = E$ and $K \cap A = K \cap C \cap A = M \cap A = 1$ so that K is a complement to A in E.

If K_1 is any other complement to A then $K_1 \geqslant M$ and so K_1/M is a complement to N/M and hence is conjugate to K/M.

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