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Rings of functions on non-abelian groups

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Abstract. For several classes of finite nonabelian groups we investigate the structure of the ring of functions, $\mathcal{R}(C)$, determined by the cover C of maximal abelian subgroups. We determine the Jacobson radical $J(\mathcal{R}(C))$ and the semisimple quotient ring $\mathcal{R}(C)/J(\mathcal{R}(C))$.

1. Introduction

Let $G = \langle G, +\rangle$ be a group written additively but not necessarily abelian, with identity element 0, and let $C := \{A_1, A_2, \ldots, A_N\}$ be a cover of G by abelian subgroups, i.e., each A_i is an abelian subgroup of G and $\bigcup_{i=1}^{N} A_i = G$. Define $\mathcal{R}(C) := \{\sigma\colon\ G \to G \mid \sigma_{|A_i} \in \mathrm{End}(A_i), \text{ for all } i\}$. Then $\mathcal{R}(C)$ is a ring of functions on G called the ring determined by the cover C. Note that the identity function, *id.*, and the zero function, 0, are in $\mathcal{R}(C)$ and we require them to be in all of our rings of functions.

On the other hand, suppose R is a ring of functions on G. Define $\mathcal{C}(R) := \{B \subseteq G | B \text{ is an abelian subgroup of } G \text{ and } R_{|B} \subseteq \mathrm{End}(B)\}$. Then $\mathcal{C}(R)$ is a cover of G by abelian subgroups. These correspondences were initiated in [2] and were shown to form a Galois correspondence. One of the goals of this investigation is to determine structural properties of the ring $\mathcal{R}(C)$ in terms of the cover C. For additional background and results, we refer the reader to [2].

Suppose $C := \{A_1, \ldots, A_N\}$ is a cover of the finite group G by abelian subgroups. Define $\psi\colon\ \mathcal{R}(C) \longrightarrow \bigoplus_{i=1}^{N} \mathrm{End}(A_i)$ by $\psi(\sigma) = (\sigma_1, \ldots, \sigma_N)$

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where $\sigma \in \mathcal{R}(C)$ and $\sigma_i = \sigma_{|A_i}$. Then ψ is a monomorphism and one wishes to identify Im ψ in $\bigoplus_{i=1}^{N} \mathrm{End}(A_i)$. We note if C is a partition then ψ is surjective and $\mathcal{R}(C) \cong \bigoplus_{i=1}^{N} \mathrm{End}(A_i)$. However, when there are nontrivial intersections among the cells of C, the identification of Im ψ becomes more difficult.

As in [2], let $\mathcal{J}(C)$ denote the intersection semilattice determined by the cells of C, including the cells of C, so $\mathcal{J}(C)$ is a cover by abelian subgroups. For $A_i \in C$, let $\mathcal{J}(A_i) = \{A_i \cap B |$ for each $B \in \mathcal{J}(C)\}$. Then $\mathcal{J}(A_i \cap A_j) = \mathcal{J}(A_i) \cap \mathcal{J}(A_j)$.

Theorem A. *With the notation as above,* Im $\psi = \{(\sigma_1, \ldots, \sigma_N) | \sigma_{i|W} = \sigma_{j|W}$ *for each* $W \in \mathcal{J}(A_i \cap A_j)$, $1 \le i, j \le N\}$.

Proof. Let $T := \{(\sigma_1, \ldots, \sigma_N | \sigma_{i|W} = \sigma_{j|W}$, for each $W \in \mathcal{J}(A_i \cap A_j)$, $1 \le i, j \le N\}$. For $\sigma \in \mathcal{R}(C)$, $\psi(\sigma) = (\sigma_1, \ldots, \sigma_N)$ and $\sigma_{i|W} = \sigma_{j|W}$, $W \in \mathcal{J}(A_i \cap A_j)$. Thus $\sigma \in T$. For the reverse inclusion, take $(\rho_1, \ldots, \rho_N) \in T$ and define $\rho\colon\ G \to G$ by $\rho(x) = \rho_i(x)$ if $x \in A_i$. By the definition of T, ρ is a well-defined function in $\mathcal{R}(C)$ and we note that $\psi(\rho) = (\rho_1, \ldots, \rho_N)$. Hence $T \subseteq$ Im ψ as desired. □

We note that, using the above theorem, we again see that when C is a partition, ψ is surjective since in this case $\mathcal{J}(A_i \cap A_j) = \{0\}$ for $i \neq j$.

In this paper we continue the work of [2]. We restrict our attention to a particular type of cover, namely the cover, C, by maximal abelian subgroups and, for the most part, to special classes of finite nonabelian groups. We then investigate the image $\psi(\mathcal{R}(C))$ or more specifically the associated semisimple ring, $\mathcal{R}(C)/J(\mathcal{R}(C))$.

Conventions: All groups, G, in this paper will be finite and, unless stated otherwise C will always denote the cover of G by its maximal abelian subgroups. By maximal we always mean proper. If the order of the group G, denoted by $|G|$, is at most 3 then G has no cover by maximal abelian subgroups, so we take $|G| \ge 4$.

2. The symmetric group S_n

We note first that we take $n \ge 4$. For if $n = 2$, $S_2 \cong \mathbb{Z}_2$ which has no cover by maximal abelian subgroups. For S_3 we see from [2] that $\mathcal{R}(C) \cong \mathbb{Z}_3 \oplus (\mathbb{Z}_2)^3$ and thus $J(\mathcal{R}(C)) = \{0\}$. The main tool for our investigation of the symmetric group is the characterization of the maximal abelian subgroups of S_n given by Reinhard Winkler in [4]. We summarize his results which are relevant to our work.

Let $M = \{1, 2, \ldots, n\}$ and let S_n be the symmetric group on M. Let $\mathcal{P}$ be any partition of M and for every $K \in \mathcal{P}$, let $+_K$ be an abelian group operation on K. For every choice $a = (a_K)_{K\in\mathcal{P}}$, $a_K \in K$, put $f_a(b) = a_K +_K b$ for $b \in K$. Define

$$J_{\mathcal{P},(+_K)_{K\in\mathcal{P}}} := \{f_a | a = (a_K)_{K\in\mathcal{P}}, a_K \in K\}.$$

Theorem 2.1 ([4]). (i) *$H = H_{\mathcal{P},(+_K)_{K\in\mathcal{P}}}$ is an abelian subgroup of S_n and is maximal with respect to this property if and only if $\mathcal{P}$ does not contain more than one singleton class.*

(ii) *Every maximal abelian subgroup H of S_n is of this form, i.e., there is a partition $\mathcal{P}$ of M containing not more than one singleton class and a family $(+_K)$ of abelian group operations $+_K$ on K for every $K \in \mathcal{P}$ such that $H = H_{\mathcal{P},(+_K)_{K\in\mathcal{P}}}$.*

We remark that we use the cycle notation for the elements in S_n and denote the operation (composition) with the addition symbol "+." Before going into the general situation we consider the specific example S_4 which will illustrate some of the techniques.

Example 2.2. For $n = 4$ we have the partitions $4 + 0, 3 + 1, 2 + 2$ in which there is at most one singleton. For the partition $\{1, 2, 3, 4\}$ we have the cyclic groups $\langle(1\ 2\ 3\ 4)\rangle$, $\langle(1\ 2\ 4\ 3)\rangle$ and $\langle(1\ 3\ 2\ 4)\rangle$. There are other abelian group structures on $\{1, 2, 3, 4\}$ but these are "picked up" in the $2 + 2$ cases. For $3 + 1$ we get the cyclic groups $\langle(1\ 2\ 3)\rangle$, $\langle(1\ 2\ 4)\rangle$, $\langle(1\ 3\ 4)\rangle$, $\langle(2\ 3\ 4)\rangle$ and for $2 + 2$ we get $\langle(1\ 2), (3\ 4)\rangle$, $\langle(1\ 3), (2\ 4)\rangle$, $\langle(1\ 4), (2\ 3)\rangle$ so we have groups generated by 4-cycles, 3-cycles and 2-cycles. If c is a 4-cycle or a 3-cycle we get for $\sigma \in \mathcal{R}(C)$, $\sigma(c) \in \langle c\rangle$. Suppose $\sigma(1\ 2\ 3\ 4) = k(1\ 2\ 3\ 4)$. Then $2\sigma(1\ 2\ 3\ 4) = k2(1\ 2\ 3\ 4)$ or $\sigma((1\ 3) + (2\ 4)) = k(1\ 3) + k(2\ 4)$. On the other hand, $\sigma(1\ 3) = x_1(1\ 3) + x_2(2\ 4)$ and $\sigma(2\ 4) = y_1(1\ 3) + y_2(2\ 4)$. From this we find $x_1 + y_1 \equiv k \equiv x_2 + y_2 \bmod 2$. If $k \equiv 0 \bmod 2$ then $x_1 = y_1$ and $x_2 = y_2$ so $\sigma(1\ 3) = \sigma(2\ 4)$ and conversely if $\sigma(1\ 3) = \sigma(2\ 4)$ then $k \equiv 0 \bmod 2$. A similar argument holds for the other 4-cycles. Define

$$I := \{\rho \in \mathcal{R}(C) | \rho(d) = 0 \text{ for each 3-cycle } d \text{ and } \rho(c) \in \langle 2c\rangle \text{ for each 4-cycle } c\}.$$

We note that I is a nil ideal in $\mathcal{R}(C)$.

Now suppose $c = (1\ 2\ 3\ 4)$ and $\sigma(c) \in \{c, 3c\}$. Then $x_1 + y_1 \equiv 1 \equiv x_2 + y_2 \bmod 2$ and σ has the matrix representation $\left[\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right]$ on $\langle(1\ 3), (2\ 4)\rangle$.

If $x_1 = 1$ and $x_2 = 1$ then $y_1 = y_2 = 0$. Note $\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]$ and the second matrix in the sum represents a function in I restricted to $\langle(1\ 3),(2\ 4)\rangle$. Hence modulo I, each 4-cycle determines a copy of $\mathbb{Z}_2$. Thus we have $\mathcal{R}(C)/I \cong (\mathbb{Z}_3)^4 \oplus (\mathbb{Z}_2)^3$. Since I is a nil ideal and $J(\mathcal{R}(C)/I) = \{0\}$ we have $I = J(\mathcal{R}(C))$. (See [1], Corollary 15.12.)

Since we make use of this result from [1] several times in the sequel we state it for reference.

Theorem 2.3 ([1], Corollary 15.12)**.** *Let I be an ideal of the ring R. If I is nil and if $J(R/I) = \{0\}$, then $I = J(R)$.*

We return to the general case and take $n \geq 5$. Let H be a maximal abelian subgroup of S_n. Then H is a direct sum of finite cyclic groups and each generator of these cyclic subgroups is of prime power order. We focus on cycles. However we should mention that the generators of H need not be cycles of prime power order, but can be sums of such cycles. For example in S_6, the subgroup, H, generated by the cycle $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$ is a maximal abelian subgroup and H has generators $\sigma_1 = (1\ 4) + (2\ 5) + (3\ 6)$ of order 2 and $\sigma_2 = (1\ 5\ 3) + (2\ 6\ 4)$. (Note $\sigma_1 + \sigma_2 = \sigma$.)

Theorem 2.4. *Let c be a cycle in S_n of order $|c|$, i.e., $|c|c = 0$ in S_n. Let $\sigma \in \mathcal{R}(C)$. Then $\sigma(c) \in \langle c\rangle$ unless $|c| = 2^m$, $m \geq 2$ and $n = |c| + 2$.*

Proof. If $|c| = n$ or $|c| = n-1$ then $\langle c\rangle$ is the unique maximal abelian subgroup containing c so by definition, $\sigma(c) \in \langle c\rangle$. If $n - |c| \geq 3$ then one can find suitable partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of those elements in $M = \{1, 2, \ldots, n\}$ not in c to determine maximal abelian subgroups H_1 and H_2 such that $H_1 \cap H_2 = \langle c\rangle$. Hence $\sigma(c) \in \langle c\rangle$.

It remains to consider $n - |c| = 2$. If $|c|$ is odd, let t be the 2-cycle determined by the elements in M not in c. From this we get that $\langle c, t\rangle$ is a maximal abelian subgroup and $\sigma(c) = xc + yt$. But then $0 = \sigma(|c|c) = |c|\sigma(c) = |c|yt$, so $y = 0$ and $\sigma(c) \in \langle c\rangle$. Next suppose that $|c| = 2^m\ell, \ell$ odd, $\ell \geq 3$ and $m \geq 1$. Again let t be the 2-cycle associated with c and so, as above, $\sigma(c) = xc + yt$ for each $\sigma \in \mathcal{R}(C)$. We note that ℓc is the sum of ℓ disjoint 2^m-cycles, say $\ell c = b_1 + b_2 + \cdots + b_\ell$. Using an appropriate partition, $\langle b_1, b_2, \ldots, b_\ell\rangle$ is a subgroup of a maximal abelian subgroup and also one finds $\sigma(b_i) \in \langle b_i\rangle$ for $\sigma \in \mathcal{R}(C)$. We take $\sigma(b_i) = k_i b_i$. Thus $\sigma(\ell c) = \ell\sigma(c) = \ell xc + \ell yt = xb_1 + xb_2 + \cdots + xb_\ell + \ell yt$. But also $\sigma(\ell c) = \sigma(b_1 + \cdots + b_\ell) = \bigoplus_{i=1}^{\ell} \sigma(b_i) = k_1 b_1 + \cdots + k_\ell b_\ell$. From this we get $y = 0$ and $\sigma(c) \in \langle c\rangle$.

Now let $n = 2^m + 2$ and let c be a cycle in S_n. If $|c|$ is odd then $n = |c| + 2k + 1$. For $\sigma \in \mathcal{R}(C)$, $\sigma(c) = sxc + y_1t_1 + \cdots + y_kt_k$ where the t_k are 2-cycles. Then $0 = |c|\sigma(c) = y_1|c|t_1 + \cdots + y_k|c|t_k$ which implies $y_i = 0$, $i = 1, 2, \ldots, k$ so $\sigma(c) \in \langle c \rangle$. If $|c|$ is even and $|c| < 2^m$ then $n = |c| + 2h$ and $n = |c| + (2h-1) + 1$. From suitable partitions we get two maximal abelian subgroups whose intersection is $\langle c \rangle$. Again we obtain $\sigma(c) \in \langle c \rangle$ for $\sigma \in \mathcal{R}(C)$. When $|c| = 2^m$ we get a unique 2-cycle, t_c associated with c and $\langle c, t_c \rangle$ is a maximal abelian subgroup so $\sigma(c) = xc + yt_c$, $\sigma \in \mathcal{R}(C)$. □

Let b be an element in S_n of prime power order, say $|b| = p^{m_1}$ where, if $n = 2^{m_1} + 2$, $|b| \neq 2^{m_1}$. If b is a cycle, then from the above theorem $\sigma(b) \in \langle b \rangle$, $\sigma \in \mathcal{R}(C)$, say $\sigma(b) = kb$. Now $k = qp + r$, $0 \leq r < p$ so $\sigma(b) = rb + qpb$, $r \in \mathbb{Z}_p$. If b is not a cycle then we first take b as the sum of disjoint cycles of order p^{m_1}, $b = b_1 + \cdots + b_t$. Then there is a cycle c of order tp^{m_1} such that $tc = b$. We know $\sigma(c) = kc$ so $\sigma(b) = \sigma(tc) = t\sigma(c) = tkc = kb$ and again we get $\sigma(b) = sb + \hat{q}pb$, $s \in \mathbb{Z}_p$. Note also that $\sigma(b_i) = k_i b_i$ so $\sigma(b) = \bigoplus_{i=1}^{t} k_i b_i$. This implies that $k \equiv k_i, \bmod p$, for each i.

For the general case we take b to be the sum of elements of order p^{m_i}, $m_1 \geq \cdots \geq m_t$. Let b_i be the sum of the summands of order p^{m_i}. We have just shown that $\sigma(b_i) = r_i b_i + q_i p b_i$. Using a suitable partition, $\langle b_1, \ldots, b_t \rangle$ is a subgroup of a maximal abelian subgroup so $\sigma(b) = \sigma(b_1) + \cdots + \sigma(b_t) = r_1 b_1 + r_2 b_2 + \cdots + r_t b_t + p\hat{b}$ where $r_i \in \mathbb{Z}_p$ and $\hat{b}$ is an element of prime power order. We want to show $r_i = r_j$ in $\mathbb{Z}_p$. Note that each $p^{m_i - 1} b_i$ is a sum of p-cycles, $b_{i1} + \cdots + b_{iN_i}$. Using these p-cycles we can form a cycle c of order $(N_1 + \cdots + N_t)p$ and we know $\sigma(c) = rc$. Then $\sigma((N_1 + \cdots + N_t)c) = r(N_1 + \cdots + N_t)c$ and from this we find $r_i \equiv r \equiv r_j \bmod p$.

We summarize the above.

Lemma 2.5. *If b is an element in S_n of prime power order p^m where $|b| \neq 2^m$ if $n = 2^m + 2$, then for $\sigma \in \mathcal{R}(C)$, $\sigma(b) = r_\sigma b + p\hat{b}$ where $\hat{b}$ is an element of prime power order and $r_\sigma \in \mathbb{Z}_p$.*

We now turn to one of our main results.

Theorem 2.6. *Let $C = \{A_1, \ldots, A_N\}$ be the cover of S_n by maximal abelian subgroups and let $P := \{p_i | p_i$ is a prime integer, $p_i \leq n\}$. Then $\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \bigoplus_{p_i \in P} (\mathbb{Z}_{p_i})^{n_i}$, $n_i \geq 1$.*

Proof. From abelian group theory each A_i decomposes into its primary components and each endomorphism of A_i decomposes into endomorphisms of these primary components. From Section 1 we have $\mathcal{R}(C) \cong$

Im ψ where $\psi(\sigma) = (\sigma_1, \ldots, \sigma_N)$, $\sigma \in \mathcal{R}(C)$. From the decomposition into primary components we get $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{i\ell_i})$. The primary components decompose further into cyclic groups in which each generator is an element of prime power order.

We first take $n \neq 2^m + 2$, $m \geq 2$. Define

$$I := \{\sigma \in \mathcal{R}(C) | \sigma(b) \in \langle p_i \hat{b} \rangle \text{ for any element } b \text{ of prime power order } p_i \in P, p_i^{n_i}, \text{ and } \hat{b} \text{ has order a power of } p_i\}.$$

One verifies that I is an ideal of $\mathcal{R}(C)$, moreover a nil ideal.

As we noted above we only have to consider elements, b, of prime power order and so from Lemma 2.5, for $\sigma \in \mathcal{R}(C)$, $\sigma(b) = r_\sigma b + p\hat{b}$, $r_\sigma \in \mathbb{Z}_p$. Thus we obtain an embedding $\mathcal{R}(C) \hookrightarrow \left(\bigoplus_{p_i \in P} \mathbb{Z}_{p_i} b \right) \oplus I, b$ an element of order a power of p_i. This leads to an embedding of $\mathcal{R}(C)/I$ into $\bigoplus_{p_i \in P} (\mathbb{Z}_{p_i})^{m_i}$ and thus we have $\mathcal{R}(C)/I \cong \bigoplus_{p_i \in P} (\mathbb{Z}_{p_i})^{n_i}$, $n_i \geq 1$.

Now take $n = 2^m + 2$, $m \geq 2$. We modify the definition of I. The difference here is when c is a cycle of order 2^m. Then there is a unique 2-cycle, t_c, associated with c and $\sigma(c) = x_c c + y_c t_c$. Define $I := \{\sigma \in \mathcal{R}(C) | \sigma(c) \in \langle 2c, t_c \rangle$ if c is a cycle of order 2^m and $\sigma(c) \in \langle p\hat{c} \rangle$ if c is any element of prime power order, not 2^m and $\hat{c}$ is an element of order a power of $2\}$.

Again one finds that I is a nil ideal. For example if $\sigma \in I$ and $|c| = 2^m$ then $\sigma(c) = k \cdot 2c + yt_c$ and $\sigma^{2^{m-1}}(c) = 0$. Now as in the previous case, for $\sigma \in \mathcal{R}(C)$, $\sigma(c) = x_\sigma c + y_\sigma t_c$ and $x_\sigma = q \cdot 2 + r$, so $\sigma(c) = rc + q2c + y_c t_c$ so $\mathcal{R}(C)/I \cong \bigoplus_{p_i \in P} (\mathbb{Z}_{p_i})^{n_i}$, $n_i \geq 1$.

From Theorem 2.3, $I = J(\mathcal{R}(C))$. □

The above result is not very precise. One would like to specify the exponents n_i for a given n. We now turn to this specification. As we have seen above, each element b of prime power order p^m gives rise to a copy of $\mathbb{Z}_p$ in the decomposition of $\mathcal{R}(C)/J(\mathcal{R}(C))$. We wish to find how many distinct copies of $\mathbb{Z}_p$ appear in this decomposition. We know, for $\sigma \in \mathcal{R}(C), \sigma(b) = kb$ modulo $J(\mathcal{R}(C))$. Further, $p^{m-1}b$ is a sum of p-cycles $p^{m-1}b = b_1 + \cdots + b_\ell$ and $\sigma(b_i) = k_i b_i$, $i = 1, 2, \ldots, t$. Just as we did in the discussion prior to Lemma 2.5 we find that $k_i \equiv k \bmod p$. Thus we can restrict to cycles of prime order, i.e., p-cycles. So when c_1 and c_2 are p-cycles and $\sigma \in \mathcal{R}(C)$ we have $\sigma(c_1) = k_1 c_1$ and $\sigma(c_2) = k_2 c_2$. We want to determine when $k_1 \equiv k_2 \bmod p$, that is when the same copy of $\mathbb{Z}_p$ is associated with any element of prime power p^m which contains either c_1 or c_2 as one of its disjoint summands.

If $k_1 \equiv k_2 \bmod p$ we say c_1 and c_2 are *p-equivalent* and write $c_1 \sim_p c_2$. In fact we note that $hc_1 \sim_p c_1$ for any nonzero element hc_1 in $\langle c_1 \rangle$ so $\sim_p$ is an equivalence relation on the subgroups of order p in S_n. We denote the number of equivalence classes by n_p. Thus the number of summands of $\mathbb{Z}_p$ in $\mathcal{R}(C)/J(\mathcal{R}(C))$ is n_p.

Lemma 2.7. *Disjoint p-cycles in S_n are p-equivalent.*

Proof. Let c_1 and c_2 be disjoint p-cycles in S_n so we must have $n \geq 2p$. Let $c_1 = (x_1, \ldots, x_p)$ and $c_2 = (y_1, \ldots, y_p)$. Form $c_3 = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, a cycle of order $2p$. If $n = 2p$ or $n = 2p + 1$ there is a unique maximal abelian subgroup containing c_3 and for $\sigma \in \mathcal{R}(C)$, $\sigma(c_3) = k_3c_3$. We also have $\sigma(c_1) = k_1c_1$ and $\sigma(c_2) = k_2c_2$. Therefore $\sigma(2c_3) = k_3(2c_3) = k_3(c_1 + c_2)$. But c_1 and c_2 are in a maximal abelian subgroup so $\sigma(2c_3) = \sigma(c_1 + c_2) = k_1c_1 + k_2c_2$ and we see $k_1 \equiv k_3 \equiv k_2 \bmod p$.

Next suppose $n = 2p + 2$. Let t denote the unique 2-cycle on the elements of M not in c_3. Then $\langle c_3, t \rangle$ is a maximal abelian subgroup and $\sigma(c_3) = xc_3 + yt$ so $\sigma(2c_3) = x2c_3$ and the result follows as above. If $n = 2p + n_1$, $n_1 \geq 3$ we get $\sigma(c_3) \in \langle c_3 \rangle$ and obtain $c_1 \sim_p c_2$. □

Lemma 2.8. *For $n \geq 5$ all 2-cycles are 2-equivalent.*

Proof. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be 2-cycles and $\sigma \in \mathcal{R}(C)$. Then $\sigma(a) = k_1a$ and $\sigma(b) = k_2b$. If a and b are disjoint, the result follows from the previous lemma. Otherwise we suppose $a_1 = b_1$. Since $n \geq 5$, there exist elements a_3, b_3 in M different from a_1, a_2, b_2. Thus $c = (a_3, b_3)$ is disjoint from a and b. Hence $a \sim_p c \sim_p b$ as desired. □

We note that, from the above lemma, when $n \geq 5$ only one copy of $\mathbb{Z}_2$ appears in the decomposition of $\mathcal{R}(C)/J(\mathcal{R}(C))$. We now take p to be an odd prime.

Theorem 2.9. *Let p be an odd prime and let x and y be p-cycles in S_n on $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_p\}$ respectively, where $X \subseteq M$, $Y \subseteq M$. Let $n_{12} = |X \cap Y|$. If $n \geq 2p + \min\{n_{12}, p - n_{12}\}$ then $x \sim_p y$.*

Proof. Without loss of generality we let $\{x_1, \ldots, x_{12}\} = \{y_1, \ldots, y_{12}\}$ so we have $n_{12} + 2(p - n_{12}) = 2p - n_{12}$ elements listed in $X \cup Y$. Note that $p - n_{12} \neq n_{12}$ since p is an odd prime.

Case (i). $p - n_{12} < n_{12}$.

We have $n \geq 2p + (p - n_{12})$ so we have at least $2p + (p - n_{12}) - (2p - n_{12}) = p$ elements from $M = \{1, 2, \ldots, n\}$ not yet listed in x and y. We use these p elements to obtain a p-cycle, w, disjoint from x and y. Thus $x \sim_p w \sim_p y$.

Case (ii). $p - n_{12} > n_{12}$.

In this case $n \geq 2p + n_{12}$ and so there are at least $2p + n_{12} - (2p - n_{12}) = 2n_{12}$ elements from M not yet listed. Note in this case $2n_{12} < p$. Let $w_1, w_2, \dots, w_{n_{12}}$ and $v_1, v_2, \dots, v_{n_{12}}$ be $2n_{12}$ elements not listed in x and y. Let $\overline{X} = \{w_1, w_2, \dots, w_{n_{12}}, x_{n_{12}+1}, \dots, x_p\}$ and $\overline{Y} = \{v_1, v_2, \dots, v_{n_{12}}, y_{n_{12}+1}, \dots, y_p\}$ and let $\bar{x}$ be a p-cycle from the elements of $\overline{X}$, $\bar{y}$ a p-cycle from the elements of $\overline{Y}$. Then $y \sim_p \bar{x} \sim_p \bar{y} \sim_p x$ giving the result. □

Corollary 2.10. *If $n \geq 2p + 1$ then all p-cycles in S_n are p-equivalent, i.e., $n_p = 1$.*

Proof. Suppose $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_p)$ are arbitrary p-cycles in S_n with $X = \{x_1, \dots, x_p\}$ and $Y = \{y_1, \dots, y_p\}$. If $X = Y$ then $n_{12} - p = 0$ while if $X \cap Y = \emptyset$ then $n_{12} = 0$. Thus by the above theorem, $x \sim_p y$. We take $|X \cap Y| \geq 1$. Let $y_i \in Y - (X \cap Y)$ and $x_j \in X - (X \cap Y)$. Replace x_j in x by y_i to obtain x'. From the above theorem, $x \sim_p x'$ since the intersection number $n_{12} = p - 1$ and by hypothesis, $n \geq 2p + \{p - 1, p - (p - 1)\}$. Continuing by replacing one element at a time we get $x \sim_p y$. □

We classify the primes in $P = \{p | p \text{ is a prime}, p \leq n\}$ into three subsets. Define $P_1 = \{p \in P |\ 2p+1 \leq n\}$, $P_2 = \{p \in P | 2p = n < 2p+1\}$ and $P_3 = \{p \in P | p \leq n < 2p\}$. As we have just seen, for primes $p \in P_1$, all p-cycles are p-equivalent, so $n_p = 1$ for $p \in P_1$.

To investigate the primes in P_3 we first indicate how many distinct subgroups of order p are in S_n. We choose p of the n elements in M and recall that each choice determines $(p-1)!$ p-cycles. But each subgroup of order p contains $p - 1$ of these cycles, so we have $\binom{n}{p}(p-2)!$ distinct subgroups of order p in S_n.

Suppose now $p \in P_2$ and $x = (x_1, \dots, x_p)$ is a p-cycle. As noted above there are $(p-2)!$ subgroups using $\{x_1, \dots, x_p\}$ and $(p-2)!$ for the $n - p = p$ other elements in M. Since these sets are disjoint we have $2(p-2)!$ subgroups in a class so in this case $n_p = \frac{\binom{n}{p}(p-2)!}{2(p-2)!} = \frac{1}{2}\binom{n}{p}$.

We summarize this section in the following result.

Theorem 2.11. *Let C be the cover of S_n, by maximal abelian subgroups and let P_1, P_2, P_3 be the sets of prime numbers defined above. Then*

$\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \bigoplus_{p\in P} (\mathbb{Z}_p)^{n_p}$ *where* $P = P_1 \cup P_2 \cup P_3$ *and*

$$n_p = \begin{cases} 1, & p \in P_1 \\ \frac{1}{2}\binom{n}{p}, & p \in P_2 \\ \binom{n}{p}(p-2)!, & p \in P_3. \end{cases}$$

We close this section with some examples for small n.

Example 2.12. C is the cover of S_n by maximal abelian subgroups.

(i) $n = 4; P_1 = \emptyset$, $P_2 = \{2\}$,$P_3 = \{3\}$, $n_2 = \frac{1}{2}\binom{4}{2} = 3$, $n_3 = \binom{4}{3}(3-2)! = 4$ so $\mathcal{R}(C)/J(\mathcal{R}(C)) \cong (\mathbb{Z}_3)^4 \oplus (\mathbb{Z}_2)^3$ as found in Example 2.2.

(ii) $n = 5$; $P_1 = \{2\}$, $P_2 = \emptyset$, $P_3 = \{3, 5\}$, $n_2 = 1$, $n_3 = \binom{5}{3}(3-2)! = 10$, $n_5 = \binom{5}{2}(5-5)! = 6$ so $\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_3)^{10} \oplus (\mathbb{Z}_5)^6$.

(iii) $n = 10$; $P_1 = \{2, 3\}$, $P_2 = \{5\}$, $P_3 = \{7\}$, $n_2 = n_3 = 1$, $n_5 = \frac{1}{2}\binom{10}{5}3!$, $n_7 = \binom{10}{7}5!$ so $\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (\mathbb{Z}_5)^{n_5} \oplus (\mathbb{Z}_7)^{n_7}$.

(iv) $n = 11$; $P_1 = \{2, 3, 5\}$, $P_2 = \emptyset$, $P_3 = \{7, 11\}$ and $\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus (\mathbb{Z}_7)^{n_7} \oplus (\mathbb{Z}_{11})^{n_{11}}$.

3. p-groups with a cyclic maximal subgroup

Let G be a finite p-group having a cyclic subgroup which is also a maximal subgroup. The structure of groups with this property is well-known.

Theorem 3.1 ([3, 5.3.4]). *A group of order p^n has a cyclic maximal subgroup if and only if it is one of the following types:*

(i) *a cyclic group of order p^n;*

(ii) *the direct product of a cyclic group of order p^{n-1} and one of order p, i.e., $\mathbb{Z}_{p^{n-1}} \oplus \mathbb{Z}_p$;*

(iii) *the dihedral group $D_{2^{n-1}} = \langle x, y | 2^{n-1}x = 2y = 0,\ y + x = (2^{n-1} - 1)x + y\rangle, n \geq 3$;*

(iv) *the group $M_n(p) := \langle x, y | p^{n-1}x = py = 0, -y + x + y = (1 + p^{n-2})x\rangle, n \geq 3$;*

(v) $SD_n := \langle x, y | 2^{n-1}x = 2y = 0, -y + x + y = (2^{n-2} - 1)x\rangle, n \geq 3$;

(vi) $GQ := \langle x, y | 2^{n-1}x = 0, 2y = 2^{n-2}x, -y + x + y = (2^{n-1} - 1)x\rangle, n \geq 3$.

We consider the nonabelian cases separately in the following subsections. The cyclic group of order p^n has no cover by maximal abelian subgroups. The abelian case, i.e. part (ii) will be handled in the next section.

3.1. Dihedral group D_n

We consider here the collection of all dihedral groups rather than just dihedral p-groups. So we let $D_n := \langle x, y | nx = 0 = 2y, y + x = (n-1)x + y\rangle$.

Case A.1. n odd.

The maximal abelian subgroups are the cyclic subgroups

$$C = \{\langle x\rangle, \langle y\rangle, \langle x + y\rangle, \langle 2x + y\rangle, \dots, \langle (n-1)x + y\rangle\}.$$

Note that C is a partition so we have $\mathcal{R}(C) \cong \mathbb{Z}_n \oplus (\mathbb{Z}_2)^n$. If $n = p_1^{\alpha_1} \dots p_t^{\alpha_t}, p_i$ an odd prime, then $J(\mathcal{R}(C)) \cong J(\mathbb{Z}_n) \oplus \{0\} = \left(\bigoplus_{i=1}^{t} p_i \mathbb{Z}_{p_i^{\alpha_i}}\right) \oplus \{0\}$ so $\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \left(\bigoplus_{i=1}^{t} \mathbb{Z}_{p_i}\right) \oplus (\mathbb{Z}_2)^n$.

Case A.2. n even.

Let $C = \{\langle x\rangle, \langle \frac{n}{2}x, y\rangle, \langle \frac{n}{2}x, x+y\rangle, \dots, \langle \frac{n}{2}x, (\frac{n}{2}-1)x+y\rangle\}$. Note that C is a cover of D_n and each cell is abelian since the center of D_n, $Z(D_n)$, is $\langle \frac{n}{2}x\rangle$. We show each cell is a maximal abelian subgroup. Since $|\langle x\rangle| = n$, $\langle x\rangle$ is a maximal subgroup. Suppose H is an abelian subgroup, $H \supseteq \langle \frac{n}{2}x, rx + y\rangle$. For $w \in H$, $w = hx + y$ and we have $hx + y + rx + y = rx + y + hx + y$ so $hx + (n-1)rx = rx + (n-1)hx$ or $2hx = 2rx$. Thus $2h - 2r = qn$ or $h = r + q \cdot \frac{n}{2}$. Hence $w = hx + y = q \cdot \frac{n}{2}x + rx + y$ which is in $\langle \frac{n}{2}x, rx + y\rangle$. Hence $H = \langle \frac{n}{2}x, rx + y\rangle$ giving the result.

For notational convenience we let $A := \langle x\rangle$ and $A_i := \langle \frac{n}{2}x, ix + y\rangle$, $i = 0, 1, \dots, \frac{n}{2} - 1$ and take $\sigma \in \mathcal{R}(C)$ where as we have shown above, C is the cover of D_n by maximal abelian subgroups. On A, $\sigma(x) = kx$. If we use the basis $\{\frac{n}{2}x, ix + b\}$ on A_i then σ has the representation $\begin{bmatrix} \bar{k} & b_{i1} \\ 0 & b_{i2} \end{bmatrix}$ on A_i where $\bar{k} \equiv k \bmod 2$. Thus $\sigma \mapsto \psi(\sigma) = \left(k, \begin{bmatrix} \bar{k} & b_{01} \\ 0 & b_{02} \end{bmatrix}, \dots, \begin{bmatrix} \bar{k} & b_{\frac{n}{2}-1\ 1} \\ 0 & b_{\frac{n}{2}-1\ 2} \end{bmatrix}\right)$. From this we see $|\mathcal{R}(C)| = n4^{\frac{n}{2}} = n \cdot 2^n$. Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ where the p_i are primes and we have $p_1 = 2$, $\alpha_1 \geq 1$. Define $I := \{\sigma \in \mathcal{R}(C) | \sigma(x) = (p_1 \dots p_t)x$ and $\sigma(ix + b) \in \langle \frac{n}{2}x\rangle\}$. Calculations show that I is an ideal. Moreover for $\sigma \in I$, $\sigma^2(ix + y) = \sigma(h \cdot \frac{n}{2}x) = 0$

while $\sigma^2(x) = p_1^2 p_2^2 \dots p_t^2 \psi$. Thus I is a nil ideal of $\mathcal{R}(C)$ and we find $\mathcal{R}(C)/I \cong \frac{\mathrm{Im}\,\psi}{\psi(I)} \cong \frac{\mathbb{Z}_n}{p_1 \dots p_t \mathbb{Z}_n} \oplus (\mathbb{Z}_2)^{\frac{n}{2}} \cong \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_t} \oplus (\mathbb{Z}_2)^{\frac{n}{2}}$. Again, applying Theorem 2.3 we see that $I = J(\mathcal{R}(C))$.

Theorem 3.2. *Let D_n be the dihedral group of order $2n$ and let C be the cover of D_n by maximal abelian subgroups. If $n = 2^{\alpha_0} p_1^{\alpha_1} \dots p_t^{\alpha_t}, p_i$ an odd prime, then*

$$\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \begin{cases} \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_t} \oplus (\mathbb{Z}_2)^n & \text{if } \alpha_0 = 0, \\ \mathbb{Z}_{p_1} \oplus \dots \oplus \mathbb{Z}_{p_t} \oplus (\mathbb{Z}_2)^{\frac{n}{2}+1} & \text{if } \alpha_0 > 0. \end{cases}$$

3.2. The group

$$M_n(p) := \langle x, y | p^{n-1}x = py = 0; -y + x + y = (1 + p^{n-2})x \rangle$$

The group $M_n(p)$ has p^n elements and its center $Z(M_n(p)) = \langle px \rangle$. One finds that

$$C := \{\langle x \rangle, \langle x + y \rangle, \dots, \langle x + (p-1)y \rangle, \langle y, px \rangle\}$$

is the cover by maximal abelian subgroups. Let $A_i := \langle x + iy \rangle$, $i = 0, 1, \dots, p-1$ and $A := \langle y, px \rangle$. For $\sigma \in \mathcal{R}(C)$, let $\sigma(x) = kx$ and $\sigma(x + iy) = k_i(x + iy)$. Since $\langle px \rangle$ is contained in each of the cells of C, there exist h_i such that $h_i(x + iy) = px$. Thus $\sigma(px) = h_i \sigma(x + iy) = h_i k_i (x + iy) = k_i px$. But also $\sigma(px) = p\sigma(x) = kpx$. Thus we find $k \equiv k_i$, $i = 0, 1, 2, \dots, p-1$. On the cell A, with respect to the bases $\{y, px\}, \sigma$ has representation $\begin{bmatrix} y_1 & 0 \\ y_2 & k \end{bmatrix} = \begin{bmatrix} y_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ y_2 & k \end{bmatrix}$. If we let $I := \{\sigma \in \mathcal{R}(C) | \sigma(w) \in pM_n(p)$ for each w in $M_n(p)\}$ then I is a nil ideal with $\mathcal{R}(C)/I \cong \mathbb{Z}_p \oplus \mathbb{Z}_p = (\mathbb{Z}_p)^2$. Applying Theorem 2.3 gives $I = J(\mathcal{R}(C))$.

3.3. Semidihedral group

$$SD_n := \langle x, y | 2^{n-1}x = 0 = 2y; -y + x + y = (2^{n-2} - 1)x \rangle$$

Since $2y = 0$, from $-y+x+y = (2^{n-2}-1)x$ we get $y+x = (2^{n-2}-1)x+y$. Using this we see if a is odd, $\langle ax + y \rangle = \{0, ax + y, 2^{n-2}x, (2^{n-2} + a)x + y\}$ while if a is even, $2(ax + y) = 0$ and $\langle ax + y, 2^{n-2}x \rangle = \{0, ax + y, 2^{n-2}x, (2^{n-2}x + a)x + y\}$. Since the center $Z(SD_n) = \{0, 2^{n-2}x\}$ we find that the cover by maximal abelian subgroups is

$$C = \{\langle x \rangle, \langle x+y \rangle \langle 2x+y, 2^{n-2}x \rangle, \langle 3x+y \rangle, \dots, \langle (2^{n-2}-1)x+y \rangle, \langle 2^{n-2}x, y \rangle\}.$$

Let $A := \langle x\rangle$ and $A_i := \begin{cases} \langle ix+y\rangle & \text{if } i \text{ is odd} \\ \langle ix+y, 2^{n-2}x\rangle, & \text{if } i \text{ is even.} \end{cases}$

For $\sigma \in \mathcal{R}(C), \sigma(x) = kx$ and $\sigma(ix+y) = k_i(ix+y)$ if i is odd. But then $\sigma(2^{n-2}x) = k2^{n-2}x$ and $2\sigma(ix+y) = \sigma(2^{n-2}x) = k_i 2^{n-2}x$ which gives $k \equiv k_i \bmod 2$ when i is odd. For i even, using the basis $\{2^{n-2}x, ix+y\}, \sigma$ has the representation $\left[\begin{smallmatrix} k & b_{i1} \\ 0 & b_{i2}\end{smallmatrix}\right]$ on A_i. If we define $I := \{\sigma \in \mathcal{R}(C) | \sigma(x) \in \langle 2x\rangle$ and $\sigma(ix+y) \in \langle 2^{n-2}x\rangle$ for i even$\}$ then calculations show that I is a nil ideal of $\mathcal{R}(C)$ and $\mathcal{R}(C)/I \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^{2^{n-3}}$ where the second summand arises from the 2^{n-3} subgroups containing $ix+y, i$ even. Hence from Theorem 2.3, $I = J(\mathcal{R}(C))$ and $\mathcal{R}(C)/J(\mathcal{R}(C)) \cong (\mathbb{Z}_2)^{2^{n-3}+1}$.

3.4. Generalized quaternion groups

$$GQ := \langle x, y | 2^{n-1}x = 0, 2y = 2^{n-2}x, -y+x+y = (2^{n-1}-1)x\rangle$$

Since $(2^{n-1}-1)x = -x$ we find $y+x = -x+y = (2^{n-1}-1)x+y$. Using this we find the cover by maximal abelian subgroups is

$$C = \{\langle x\rangle, \langle x+y\rangle, \langle 2x+y\rangle, \dots, \langle 2^{n-2}x+y\rangle = \langle y\rangle\}.$$

For $\sigma \in \mathcal{R}(C), \sigma(x) = kx$ and $\sigma(ix+y) = k_i(ix+y), i = 1, 2, \dots, 2^{n-2}$. Since $2(ix+y) = 2^{n-2}x$ we find $\sigma(2^{n-2}x) = 2\sigma(ix+y) = k_i 2^{n-2}$ and $\sigma(2^{n-2}x) = k2^{n-2}x$ so $k \equiv k_i \bmod 2, i = 1, 2, \dots, 2^{n-2}$. Let $I := \{\sigma \in \mathcal{R}(C) | \sigma(x) \in \langle 2x\rangle\}$. (Note $\sigma(x) \in \langle 2x\rangle$ implies $\sigma(w) \in \langle 2x\rangle$ for all $w \in GQ$.) Again I is a nil ideal and $\mathcal{R}(C)/I \cong \mathbb{Z}_2$. Thus $I = J(\mathcal{R}(C))$ (using Theorem 2.3) and we see $\mathcal{R}(C)$ is a local ring.

4. Finite abelian p-groups

As in the above section we let p be an arbitrary but fixed prime integer and let A be a finite abelian p-group. Thus we have $A \cong \bigoplus_{i=1}^{t} \mathbb{Z}_{p^{n_i}}$, so without loss of generality, we take $A = \bigoplus_{i=1}^{t} \mathbb{Z}_{p^{n_i}}$ with the natural basis $\{e_1, e_2, \dots, e_t\}$. As usual C is the cover by maximal abelian subgroups, which in this case, is the cover by maximal subgroups. As is well known the intersection of all maximal subgroups of A is $pA = \langle pe_1, \dots, pe_t\rangle$.

Case (i). $t = 2, A = \mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^m}, n \geq m$.

First we consider $n \geq m \geq 2$. Let $C = \{\langle e_1, pe_2\rangle, \langle e_1+e_2, pe_2\rangle, \dots, \langle e_1+(p-1)e_2, pe_2\rangle, \langle pe_1, e_2\rangle\}$ and let $w = ae_1 + be_2$ be arbitrary in A. If $p|a$

then $w \in \langle pe_1, e_2\rangle$ or if $p|b$ then $w \in \langle e_1, pe_2\rangle$. Otherwise we have a is invertible modp^n and $a^{-1}w = e_1 + a^{-1}be_2$ and $a^{-1}b \not\equiv 0 \bmod p$ so $a^{-1}b = qp + r$, $0 < r < p$. Thus $a^{-1}w = e_1 + re_2 + qpe_2 \in \langle e_1 + re_2, pe_2\rangle$. Thus we see C is a cover and since the order of each cell is p^{n+m-1}, each cell is a maximal subgroup, i.e. C is the cover by maximal abelian subgroups. Let $A_i := \langle e_1 + ie_2, pe_2\rangle$, $i = 0, 1, \dots, p-1$ and let $A_p := \langle pe_1, e_2\rangle$. Let $\sigma \in \mathcal{R}(C)$. Then on A_i, $i = 0, 1, \dots, p-1, \sigma$ has representation $\begin{bmatrix} k_{i1} & h_{i1} \\ k_{i2} & h_{i2} \end{bmatrix}$ using the generating set $\{e_1 + ie_2, pe_2\}$ and on A_p, using $\{pe_1, e_2\}, \sigma$ has representation $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$. We then have $\sigma(e_1 + ie_2) = k_{i1}(e_1 + ie_2) + k_{i2}pe_2$ so $\sigma(pe_1 + ipe_2) = k_{i1}pe_1 + k_{i1}ipe_2 + k_{i2}p^2e_2$. But $\sigma(pe_1 + ipe_2) = pae_1 + be_2 + ip(cpe_1 + de_2)$. Hence $pa + icp^2 \equiv k_{i1}p \bmod p^m$ or $k_{i1} \equiv a \bmod p$.

Also, we get $b \equiv 0 \bmod p$. For, $\sigma(pe_1) = ape_1 + be_2$ and $\sigma(pe_1) = pk_{01}e_1 + pk_{02}pe_2$ so $b \equiv k_{02}p^2 \bmod p^n$ giving the result. Further, $\sigma(pe_2) = h_{i1}(e_1 + ie_2) + h_{i2}pe_2$, $i = 0, 1, 2, \dots, p-1$ and also from $\sigma(e_2) = cpe_1 + de_2$ one gets $\sigma(pe_2) = cp^2e_1 + pde_2$. Hence $(ih_{i1} + h_{i2})p \equiv pd \bmod p^n$ and $h_{i1} \equiv cp^2 \bmod p^m$. From this $h_{i1} \equiv 0 \bmod p^2$ which in turn gives $h_{i2} \equiv d \bmod p$. Therefore on A_i, $i = 0, 1, 2, \dots, p-1, \sigma$ has representation $\begin{bmatrix} k_{i1} & h_{i1} \\ k_{i2} & h_{i2} \end{bmatrix} = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] + \begin{bmatrix} \hat{a} & h_{i1} \\ k_{i2} & \hat{h}_{i2} \end{bmatrix}$ where the second summand maps A_i into pA. Also, on A_p, σ has representation $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & c \\ \hat{b} & 0 \end{smallmatrix}\right]$ where again the second summand map A_p into pA.

Define $I := \{\sigma \in \mathcal{R}(C) | \sigma(w) \in pA \text{ for all } w \in A\}$ and note I is a nil ideal. Moreover $\mathcal{R}(C)/I \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ so from Theorem 2.3, $I = J\mathcal{R}(C)$.

If $m = n = 1$ then $A = \mathbb{Z}_p + \mathbb{Z}_p$. The maximal abelian subgroups are the cyclic groups $\langle e_1 + ie_2\rangle$, $i = 0, 1, 2, \dots, p-1$ and $\langle e_2\rangle$. Thus we have a partition and $\mathcal{R}(C) \cong (\mathbb{Z}_p)^{p+1}$ with $J(\mathcal{R}(C)) = \{0\}$.

Case (ii). $A = \bigoplus_{i=1}^{t} \mathbb{Z}_{p^{n_i}}, n_1 \geq n_2 \geq \dots \geq n_t$ and $t \geq 3$.

We remark first that since $t \geq 3$, any two elements of A are contained in a maximal subgroup, so $\mathcal{R}(C) \subsetneq \text{End}(A)$.

Lemma 4.1. *For any element $w \in A$, let $I(w)$ denote the intersection of all cells in C containing w. Then $I(e_i) = \langle e_i\rangle + pA$ and $I(e_i + e_j) = (e_i + e_j) + pA$, $1 \leq i, j \leq t$, $i \neq j$.*

Proof. To illustrate the proof we let $i = 1$ and $j = 2$. First $\langle e_1, pe_2, e_3, \dots, e_t\rangle, \dots, \langle e_1, e_2, \dots, e_{t-1}, pe_t\rangle$ are maximal subgroups of A containing e_1. Hence $I(e_1) \subseteq \langle e_1, pe_2, \dots, pe_t\rangle \subseteq \langle e_1\rangle + pA$. On the other hand, the intersection of all maximal subgroups is contained in $I(e_1)$ which means $pA \subseteq I(e_1)$. But $\langle e_1\rangle \subseteq I(e_1)$ giving $\langle e_1\rangle + pA \subseteq I(e_1)$ and hence equality. Moreover, $\langle e_1 + e_2, pe_2, e_3, \dots, e_t\rangle$,

$\langle e_1, e_2, pe_3, e_4, \ldots, e_t\rangle, \ldots, \langle e_1, e_2, \ldots, e_{t-1}, pe_t\rangle$ are maximal subgroups containing $e_1 + e_2$ and we get $I(e_1 + e_2) = \langle e_1 + e_2\rangle + pA$. □

Now for $\sigma \in \mathcal{R}(C), \sigma(e_i) = a_i e_i + pw_i$, $w_i \in A$ and $\sigma(e_i + e_j) = a_{ij}(e_i + e_j) + pw_{ij}$, $w_{ij} \in A$. Since $\sigma(e_i + e_j) = \sigma(e_i) + \sigma(e_j)$ we get $a_i \equiv a_{ij} \equiv a_j \bmod p$ so for each i, $1 \le i \le t$, $a_i = r + q_i p$. From this we then get $\sigma(e_i) = re_i + b_{1i}e_1 + \cdots + b_{ti}e_t$ where $p|b_{ji}$. Using the natural basis, σ has matrix representation

$$\begin{bmatrix} r + b_{11} & b_{12} & \ldots & b_{1t} \\ b_{21} & r + b_{22} & & \\ \vdots & b_{32} & & \\ & \vdots & & \\ b_{t1} & b_{t2} & & r + b_{tt} \end{bmatrix} = \begin{bmatrix} r & & \\ & \ddots & \bigcirc \\ \bigcirc & & \ddots \\ & & & r \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1t} \\ b_{21} & & \\ \vdots & & \vdots \\ b_{t1} & \ldots & b_{tt} \end{bmatrix}$$

where $p|b_{ij}$ and $r \in \mathbb{Z}_p$. If we let $I = \{\sigma \in \mathcal{R}(C) | \sigma(w) \in pA \text{ for } w \in A\}$ then I is a nil ideal, $\mathcal{R}(C)/I \cong \mathbb{Z}_p$ and $I = J(\mathcal{R}(C))$ by Theorem 2.3.

We summarize the results of this section.

Theorem 4.2. *Let A be a finite p-group, $A = \bigoplus_{i=1}^{t} \mathbb{Z}_{p^{n_i}}$, $n_1 \ge n_2 \ge \cdots \ge n_t$ and let C be the cover of A by maximal subgroups. Then*

$$\mathcal{R}(C)/J(\mathcal{R}(C)) \cong \begin{cases} \mathbb{Z}_p, & \text{if } t \ge 3; \\ \mathbb{Z}_p + \mathbb{Z}_p, & \text{if } t = 2, n_1 \ge 2; \\ (\mathbb{Z}_p)^{p+1} & \text{if } t = 2, n_1 = 1 = n_2. \end{cases}$$

References

[1] Anderson, F.W. and Fuller, K.R., *Rings and Categories of Modules*, 2nd Ed., Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[2] Cannon, G. Alan, Maxson, C.J., and Neuerburg, Kent M., "Rings and Covered Groups," J. of Algebra, 320 (2008), 1586–1598.

[3] Robinson, D.J.S., *A Course in the Theory of Groups*, 2nd Ed., Springer-Verlag, Berlin-Heidelberg-New York, 1996.

[4] Winkler, Reinhard, "On Maximal Abelian Groups of Maps," J. Austral. Math. Soc. (Series A), 55 (1993), 414–420.

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