Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 1. **(2009).** pp. 1 – 13

Weak comultiplication modules over a pullback of commutative local Dedekind domains

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Communicated by D. Simson

Abstract. The goal point of recent attempts to classify indecomposable modules over non-artinian rings has been pullback rings. The purpose of this paper is to outline a new approach to the classification of indecomposable weak comultiplication modules with finite-dimensional top over certain kinds of pullback rings.

1. Introduction

Let $R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$ be the pullback of two semiperfect rings R_1, R_2 for which the factor rings $R_1/J(R_1)$ and $R_2/J(R_2)$ are isomorphic, say to $\bar{R}$ (V. V. Kirichenko named this ring the diad of the semiperfect rings R_1 and R_2 with common factor ring $\bar{R}$ [12]). In [12], Kirichenko has given a description of the representations of the diad of generalized uniserial algebras over algebraically closed field. It turns out that this problem is equivalent to the classification of pairs of mutually annihilating nilpotent operators acting in a space with gradation, and, in particular, the problem solved in [22] in connection with the study of irreducible representations of the Lorentz group is precisely the problem of describing representations of the diad of generalized uniserial algebras. His method was to reduce the problem to a matrix problem and then solve the matrix problem. The possibility of this reduction is based on the fact that one nilpotent operator acting in a graded space can be reduced to Jordan normal form. The problem of constructing all the finite metabelian

2000 Mathematics Subject Classification: *13C05, 13C13, 16D70.*

Key words and phrases: *Pullback, Separated modules and representations, Non-separated modules, Weak comultiplication modules, Dedekind domains, Pure-injective modules, Prüfer modules.*

groups (that is, groups with an abelian commutator subgroup) is fundamentally settled the Schreier theory of group extensions. Szekeres in [26], a more extensive class of metabelian groups were characterized by numerical invariants. In 1948 Brauer with his student K. A. Fowler, began the investigation of CA-groups of even order i.e. groups of even order in which the centralizer of every non-identity element is abelian. Around the same time, Wall began similar research at the suggestion of Graham Higman (there was some antecedents in the work of Szekeres [26]).

Modules over pullback rings has been studied by several authors (see for example, [3], [22], [16], [29]). In the present paper we introduce a new class of R-modules, called weak comultiplication modules, the dual notion of weak multiplication modules, (see Definition 2.1), and we study it in details from the classification problem point of view (see [1], [24, Chapter 1] and [25, Chapter 19]). We are mainly interested in case either R is a Dedekind domain or R is a pullback of two local Dedekind domains. First, we give a complete description of the weak comultiplication modules over a Dedekind domain. Let R be a pullback of two local Dedekind domains over a common factor field. Next, the main purpose of this paper is to give a complete description of the indecomposable weak comultiplication R-modules with finite-dimensional top over $R/\mathrm{rad}(R)$ (for any module M we define its top as $M/\mathrm{rad}(R)M$). The classification is divided into two stages: the description of all indecomposable separated weak comultiplication R-modules and then, using this list of separated weak comultiplication modules we show that non-separated indecomposable weak comultiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable weak comultiplication R-modules. Then we use the classification of separated indecomposable weak comultiplication modules from Section 3, together with results of Levy [16], [17] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable weak comultiplication modules M with finite-dimensional top (see Theorem 4.8). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable weak comultiplication modules (where infinite length weak comultiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated weak comultiplication modules. There are a connection between the weak multiplication modules, comultiplication modules and the weak comultiplication modules. In fact, every indecomposable comultiplication R-module M is weak comultiplication and every indecomposable weak comultiplication R-module M with finite-dimensional top is weak multiplication R-module, so they are pure-injectivr when $M \neq R$ (see [10, 11, 5]).

For the sake of completeness, we state some definitions and notations used throughout. In this paper all rings are commutative with identity and all modules unitary. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [15]).

Definition 1.1. *An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$).*

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [15, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also S is separated if and only if $P_1S \cap P_2S = 0$ [15, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi = (S \xrightarrow{f} S' \to M)$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [15, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [15, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [15, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [15, Theorem 2.6].

If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be prime submodule if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r \in (N : M)$, so $(N : M) = P$ is a prime ideal of R, and N is said to be P-prime submodule. The set of all prime submodules in an R-module M is denoted $\mathrm{Spec}(M)$.

Definition 1.2. *(a) An R-module M is defined to be a weak multiplication module if $\mathrm{Spec}(M) = \emptyset$ or for every prime submodule N of M,*

$N = IM$, *for some ideal* I *of* R *(note that we can take* $I = (N : M)$*).*

(b) An R*-module* M *is defined to be a multiplication module if for each submodule* N *of* M, $N = IM$, *for some ideal* I *of* R. *In this case we can take* $I = (N : M)$.

(c) An R*-module* M *is defined to be a comultiplication module if for each submodule* N *of* M, $N = (0 :_M I$, *for some ideal* I *of* R. *In this case we can take* $I = \mathrm{Ann}(N)$ *[2].*

(d) We say that an R*-module* M *is prime if the zero submodule of* M *is a prime submodule of* M *(so if* N *is a prime* R*-submodule of* M, *then* M/N *is a prime* R*-module).*

(e) A submodule N *of an* R*-module* M *is called pure submodule if any finite system of equations over* N *which is solvable in* M *is also solvable in* N. *So if* N *is pure in* M, *then* $IN = N \cap IM$ *for each ideal* I *of* R.

(f) A submodule N *of an* R*-module* M *is called relatively divisible (or an RD-submodule) in* M *if* $rN = N \cap rM$ *for all* $r \in R$. *An important property of modules* N, M *over a Dedekind domains is that* N *is pure in* M *if and only if* N *is an RD-submodule of* M *(see [28] and [23] for more details).*

(g) A module M *is pure-injective if it has the injective property relative to all pure exact sequences [28, 23]. In particular, by [14] and [28], an* R*-module is pure-injective if and only if it is algebraically compact.*

2. Weak comultiplication modules over a Dedekind domain

The aim of this section is to classify weak comultiplication modules over a Dedekind domain. We begin the key definition of this paper.

Definition 2.1. *Let* R *be a commutative ring. An* R*-module* M *is defined to be a weak comultiplication module if* $\mathrm{Spec}(M) = \emptyset$ *or for every prime submodule* N *of* M, $N = (0 :_M I)$, *for some ideal* I *of* R.

One can easily show that if M is a weak comultiplication module, then $N = (0 :_M \mathrm{Ann}(N))$ for every prime submodule N of M. We note that every comultiplication module is weak comultiplication module.

Proposition 2.2. *If* R *is a domain (not a field) and* M *is a weak comultiplication* R*-module with torsion submodule* $T(M) \neq M$, *then* $\mathrm{Spec}(M) = \{T(M)\}$.

Proof. Since $T(M)$ is a prime submodule of M, it suffices to show that if N is a non-zero prime submodule of M, then $N = T(M)$. By assumption, $N = (0 :_M I)$ for some non-zero ideal I of R, so $N \subseteq T(M)$. For the reverse inclusion, assume that $x \in T(M)$. Then $rx = 0 \in N$ for some $0 \neq r \in R$; hence $x \in N$ since $(N : M) \subseteq (T(M) : M) = 0$, and so we have equality.

Note, if we assume the additional condition that 0 is a prime submodule of M, then since $0 \subseteq T(M)$, $(0 : M) \subseteq (T(M) : M) = 0$ which implies $0 \in \operatorname{Spec}(M)$ and so $T(M) = 0$. □

Proposition 2.3. *Let M be a weak comultiplication module over a commutative ring R. Then the following hold:*

(i) If N is a pure submodule of M, then M/N is a weak comultiplication R-module.

(ii) Every direct summand of M is a weak comultiplication submodule. In particular, if L is a direct summand of M, then M/L is a weak comultiplication R-module.

Proof. (i) Let L/N be a prime submodule of M/N. Then by [21, Lemma 4.1], L is a prime submodule of M, so $L = (0 :_M I)$ for some ideal I of R; we show that $L/N = (0 :_{M/N} I)$. Let $m + N \in (0 :_{M/N} I)$. Then $Im \subseteq N$; thus $Im = Im \cap N \subseteq IM \cap N = IN = IL \cap N = 0$ (since N pure in M implies N is pure in L). Therefore, $m \in L$, and so $(0 :_{M/N} I) \subseteq L/N$. The proof of the other inclusion is clear, and so we have equality. (ii) follows from (i) (since direct summands are pure). □

Proposition 2.4. *Let M be a module over a Dedekind domain R. Then M is a weak comultiplication if and only if the R_P-module M_P is a weak comultiplication for every prime ideal P of R.*

Proof. Assume that M is a weak comultiplication R-module and let G be a prime submodule of M_P, where P is a prime ideal of R. According to [18, Proposition 1], there exists a prime submodule N of M such that $G = N_P$, so $N = (0 :_M J)$ for some ideal J of R. Therefore, $G = N_P = (0 :_M J)_P = (0 :_{M_P} J_P)$ by [27, Exercise 9.13]. The proof of the other implication is like that in [11, Proposition 2.3]. □

Proposition 2.5. *Let M be a module over a Dedekind domain R. Then M is an indecomposable weak comultiplication R-module if and only if M_P is indecomposable weak comultiplication as an R_P-module for every prime ideal P of R.*

Proof. The proof is straightforward by Proposition 2.4. □

Reduction to the local case. Let R be a Dedekind domain. Our aim here is to classify weak comultiplication R-modules. By Proposition 2.5, it suffices to consider the case where R is a local Dedekind domain (i.e. a discrete valuation domain) with a unique maximal ideal $P = Rp$.

Remark 2.6. *Assume that R is a local Dedekind domain with maximal ideal $P = Rp$ and let $M = R$ (as a R-module). For a prime submodule PM of M we have $(0 :_M \mathrm{Ann}(PM)) = R$. Therefore, M is not a weak comultiplication R-module, but it is a weak multiplication R-module.*

Theorem 2.7. *Let R be a discrete valuation domain with a unique maximal ideal $P = Rp$. Then the following is a complete list, without repetitions, of the indecomposable weak comultiplication modules:*

(1) R/P^n, $n \geq 1$, the indecomposable torsion modules;

(2) $R_{P^\infty} = E(R/P)$, the P-Prüfer module;

(3) $Q(R)$, the field of fractions of R.

Proof. First we note that each of the preceding modules is indecomposable (by [5, Proposition 1.3]) and weak comultiplication. Since the only prime submodule of $Q(R)$ is 0, by [19, Theorem 1], and $0 = (0 :_{Q(R)} R)$, we must have that $Q(R)$ is weak comultiplication. Moreover, as $E(R/P)$ is a torsion divisible R-module, we get $\mathrm{Spec}(E(R/P)) = \emptyset$ by [20, Lemma 1.3]; hence it is weak comultiplication. In the case of R/P^n this follows because module R/P^n is comultiplication.

Now let M be an indecomposable weak comultiplication and choose any non-zero element $a \in M$. Let $h(a) = \sup\{n : a \in P^n M\}$ (so $h(a)$ is a nonnegative integer or ∞). Also let $(0 : a) = \{r \in R : ra = 0\}$: thus $(0 : a)$ is an ideal of the form P^m or 0. Because $(0 : a) = P^{m+1}$ implies that $p^m a \neq 0$ and $p.p^m a = 0$, we can choose a so that $(0 : a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $(0 : a)$.

Case 1. If $\mathrm{Spec}(M) = \emptyset$, then by an argument like that in [9, Proposition 3.3 Case 1], we get $M \cong E(R/P)$. So we may assume that $\mathrm{Spec}(M) \neq \emptyset$.

Case 2. If $h(a) = n$, then $(0 : a) = P$. Suppose not. Then $(0 : a) = 0$. Say $a = p^n b$. Then $rb = 0$ implies $ra = 0$ and so $r = 0$. Thus $Rb \cong R$ and we also have that Rb is pure in M (see [9, Theorem 2.12 Case 1]). As M is a torsion-free R-module by [13, Theorem 10], we must have Rb is a prime submodule of M (see [18, Result 2]), so $R \cong Rb = (0 :_M 0) = M$, which is a contradiction by Remark 2.6. So we may assume that $h(a) = n$, $(0 : a) = P$. Say $a = p^n b$. Then we have $Rb \cong R/P^{n+1}$. Furthermore Rb is pure in M. Hence, since Rb is a pure submodule of bonded order of M, we obtain Rb is a direct summand of M by [13, Theorem 5]; thus $M = Rb \cong R/P^{n+1}$.

Case 3. $h(a) = \infty$, $(0 : a) = P$. By an argument like that in [11, Theorem 2.5 Case 2], we get $M \cong E(R/P)$, as needed.

Case 4. $h(a) = \infty$, $(0 : a) = 0$. By an argument like that in [9, Theorem 2.12 Case3], we obtain $M \cong Q(R)$. □

Theorem 2.8. *Let M be a weak comultiplication module over a discrete valuation domain with maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K \oplus L$, where N is a direct sum of copies of the modules as described in (1), K is a direct sum of copies of the module as described in (2) and L is a direct sum of copies of the module as described in (3) of Theorem 2.7. In particular, every weak comultiplication R-module is pure-injective (also it is weak multiplication).*

Proof. Apply Proposition 2.3 (ii), Theorem 2.7, [5, Proposition 1.3] and [9, Theorem 3.5]. □

3. The separated case

Throughout this section we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \qquad (1)$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field.

In particular, R is a commutative noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$).

In this section we determine the indecomposable weak comultiplication separated R-modules where R is the pullback of two local Dedekind domains (we do not need the a priori assumption of finite-dimensional top for this classification).

Theorem 3.1. *Let R be a pullback ring as described in (1), and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module. Then S is a weak comultiplication R-module if and only if each S_i is a weak comultiplication R_i-module, $i = 1, 2$.*

Proof. Note that by [10, Theorem 2.7], $\mathrm{Spec}(S) = \emptyset$ if and only if $\mathrm{Spec}(S_i) = \emptyset$ for $i = 1, 2$. So we may assume that $\mathrm{Spec}(S) \neq \emptyset$. Assume that S is a separated weak comultiplication R-module and let $0 \neq L$ be a non-zero prime submodule of S_1. Then there exists a separated submodule $T = (T/P_2S = T_1 \xrightarrow{g_1} \bar{T} = T/PS \xleftarrow{g_2} T_2 = T/P_1S)$ of S, where g_i is the restriction of f_i over T_i, $i = 1, 2$ such that $L = T_1$. We split the proof into three cases for $(T : S)$.

Case 1. $(T : S) = P$. Since T_1 is a prime submodule of S_1, we must have that $T/(0 \oplus P_2)S$ is a prime R-submodule of $S/(0 \oplus P_2)S$; hence T is a prime R-submodule of S. By assumption, $T = (0 :_S P_1^n \oplus P_2^m)$ for some integers m, n; we show that $T_1 = (0 :_{S_1} P_1^n)$. Let $s_1 \in (0 :_{S_1} P_1^n)$. Then $P_1^n s_1 = 0$, so $(P_1^n \oplus P_2^m)(s_1, 0) = 0$; hence $(s_1, 0) \in T$. Therefore, $(0 :_{S_1} P_1^n) \subseteq T_1$. Now suppose that $x \in T_1$. Then there is an element $y \in T_2$ such that $g_1(x) = g_2(y)$, so $(x, y) \in T$; hence $P_1^n x = 0$, and so we have equality. Similarly, S_2 is a weak comultiplication R_2-module. Clearly, in this case, for each $i = 1, 2$, $S_i \neq 0$.

Case 2. $(T : S) = P_1 \oplus 0$. Since S is a weak comultiplication R-module, we have $T = (0 :_S P_1^n \oplus 0)$ for some positive integer n. By [10, Proposition 2.6], $S_1 = 0$ and T_2 is a 0-prime submodule of S_2. Clearly, $T_2 = (0 :_{S_2} R)$, and the proof is complete. The third case of $(T : S) = 0 \oplus P_2$ is similar.

Conversely, assume that S_1, S_2 are weak comultiplication R_i-modules and let T be a non-zero prime submodule of S. If $(T : S) = P$, then for each i, $S_i \neq 0$ and there exist positive integers n, m such that $T_1 = (0 :_{S_1} P_1^n)$, $T_2 = (0 :_{S_2} P_2^m)$, and so $T = (0 :_S P_1^n \oplus P_2^m)$. If $(T : S) = P_1 \oplus 0$, then $S_1 = 0$ $T_2 = 0$, and so $T = (0 :_S R)$. The case of $(T : S) = 0 \oplus P_2$ is similar (see [10, Proposition 2.6]). Therefore S is a weak comultiplication R-module in every case. □

Lemma 3.2. *Let R be a pullback ring as described in (1). The following separated R-modules are indecomposable and weak comultiplication:*

(1) $S = (E(R_1/P_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow E(R_2/P_2)$ where $E(R_i/P_i)$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;

(2) $S = (Q(R_1) \to 0 \leftarrow 0)$ where $Q(R_1)$ is the field of fractions of R_1;

(3) $(0 \to 0 \leftarrow Q(R_2)$ where $Q(R_2)$ is the field of fractions of R_2;

and, for all positive integers n, m,

(4) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$.

Proof. By [5, Lemma 2.8], these modules are indecomposable. Weak comultiplicativity follows from Theorem 2.7 and Theorem 3.1 □

We refer to modules of type (1) in Lemma 3.2 as P_1-Prüfer and P_2-Prüfer respectively.

Proposition 3.3. *Let R be a pullback ring as described in (1), and let S be a separated weak comultiplication R-module. Then S is of the form $S = M \oplus N \oplus K$, where M is a direct sum of copies of the modules as described in (1), N is a direct sum of copies of the modules described in 2) - 3) and K is a direct sum of copies of the modules described in (4) of the Lemma 3.2. In particular, every separated weak comultiplication R-module is pure-injective.*

Proof. Let T denote an indecomposable summand of S. Then we can write $T = (T_1 \to \bar{T} \leftarrow T_2)$, and T is a weak comultiplication R-module by Proposition 2.3. We split the proof into three cases.

Case 1. If $\mathrm{Spec}(T) = \emptyset$, then $\mathrm{Spec}(T_i) = \emptyset$ by [10, Theorem 2.7], so $T_i = P_iT_i$ for each $i = 1, 2$ by Theorem 2.7; hence $T = PT = P_1T_1 \oplus P_2T_2 = T_1 \oplus T_2$. Therefore, $T = T_1$ or T_2 and so T is of type (1) in the list of Lemma 3.2 by Theorem 2.7.

Case 2. If T has a $(P_1 \oplus 0)$-prime R-submodule U, then U/P_1T is a 0-prime R_2-submodule of the weak comultiplication T_2 and $T_1 = 0$ (so $\bar{T} = 0$) by [10, Proposition 2.6] and Theorem 3.1; hence T is of type (2) in the list of Lemma 3.2. Similarly, if T has a $(0 \oplus P_2)$-prime R-submodule, then T is of type (3) in the list of Lemma 3.2.

Case 3. If T has a P-prime R-submodule $N = (N_1 \to \bar{N} \leftarrow N_2)$, then $PT \subseteq N \neq T$, so $PT \neq T$ (that is, $\bar{T} \neq 0$). Then by [10, Proposition 2.6] and Theorem 3.1, we must have $P_1T_1 = N_1 \neq T_1$ and $P_2T_2 = N_2 \neq T_2$; hence for each $i = 1, 2$, T_i is torsion and it is not divisible R_i-module (see Theorem 2.7). By an argument like that in [11, Proposition 3.3], we get T is a type (4) in the list of Lemma 3.2 by Theorem 2.7. □

Theorem 3.4. *Let R be a pullback ring as described in (1), and let S be an indecomposable separated weak comultiplication R-module. Then S is isomorphic to one of the modules listed in Lemma 3.2.*

Proof. Apply Proposition 3.3 and Lemma 3.2. □

4. The non-separated case

We continue to use the notation already established, so R is a pullback ring as in (1).

Proposition 4.1. *Let R be a pullback ring as described in (1). Then $E(R/P)$ is a non-separated weak comultiplication R-module.*

Proof. By [10, Theorem 3.2], $\mathrm{Spec}(E(R/P)) = \emptyset$. Therefore, $E(R/P)$ is a weak comultiplication R-module. □

Proposition 4.2. *Let R be a pullback ring as described in (1) and let M be any weak comultiplication R-module. If M has either a $P_1 \oplus 0$-prime submodule or a $0 \oplus P_2$-prime submodule, then M is separated.*

Proof. The proof is like that in [10, Proposition 3.4]. □

Theorem 4.3. *Let R be a pullback ring as described in (1) and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated* *representation of M. Then S is weak comultiplication if and only if M is weak comultiplication.*

Proof. By [10, Proposition 3.6], we may assume that $\mathrm{Spec}(S) \neq \emptyset$. Suppose that M is a weak comultiplication R-module and let T be a non-zero prime submodule of S. Then by [10, Lemma 3.5], $K \subseteq T$, and so T/K is a prime submodule of S/K. By an argument like that in [11, Theorem 4.4], we get S is weak comultiplication. Conversely, assume that S is a weak comultiplication R-module and let N be a non-separated prime submodule of M. Then $\varphi^{-1}(N) = U$ is a prime submodule of S (see [7, Lemma 3.1]), so $U = (0 :_S P_1^n \oplus P_2^m)$ for some integers m, n. By [7, Lemma 3.1], $U/K \cong N$ is a prime submodule of $S/K \cong M$, so an inspection will show that $N = U/K = (0 :_{S/K} P_1^n \oplus P_2^m)$, as required. □

Proposition 4.4. *Let R be a pullback ring as described in (1) and let M be an indecomposable weak comultiplication non-separated R-module with finite-dimensional top over $\bar{R}$. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is pure-injective.*

Proof. By [5, Proposition 2.6 (i)], $S/PS \cong M/PM$, so S is finite-dimensional top. Now the assertion follows from Theorem 4.3 and Proposition 3.3. □

Let R be a pullback ring as described in (1) and let M be an indecomposable weak comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$. Consider the separated representation $0 \to K \to S \to M \to 0$. By Proposition 4.4, S is pure-injective. So in the proofs of [5, Lemma 3.1, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [5, Proposition 2.6 (ii)]) we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable weak comultiplication non-separated R-module": because the main key in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following results:

Corollary 4.5. *Let R be a pullback ring as described in (1) and let M be an indecomposable weak comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then the quotient fields $Q(R_1)$ and $Q(R_2)$ of R_1 and R_2 do not occur among the direct summand of S.*

Corollary 4.6. *Let R be the pullback ring as described in (1) and let M be an indecomposable weak comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a* *separated representation of M. Then S is a direct sum of finitely many indecomposable weak multiplication modules.*

Corollary 4.7. *Let R be the pullback ring as described in (1) and let M be an indecomposable weak comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then at most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Recall that every indecomposable R-module of finite length is weak comultiplication (see Lemma 3.2 and Proposition 4.3). So by Corollary 4.7, the infinite length non-separated indecomposable weak comultiplication modules are obtained in just the same way as the deleted cycle type indecomposable are except that at least one of the two "end" modules must be a separated indecomposable weak comultiplication of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one cannot have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules: we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer module and a P_2-Prüfer module. If the P_1-Prüfer and the P_2-Prüfer are direct summands of S then we will describe these modules as **doubly infinite**. Those where S has just one infinite length summand we will call **singly infinite** (the reader is referred to [5], [10] and [11] for more details). It remains to show that the modules obtained by these amalgamations are indeed, indecomposable weak comultiplication modules.

Theorem 4.8. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the indecomposable non-separated weak comultiplication modules with finite-dimensional top are the following:*

(i) the indecomposable modules of finite length (apart from R/P which is separated),

(ii) the doubly infinite weak comultiplication modules as described above,

(iii) the singly infinite weak comultiplication modules as described above, apart from the two Prüfer modules (1) in Lemma 3.2.

Proof. Let M be an indecomposable non-separated weak comultiplication R-module with finite-dimensional top and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.

(i) Clearly, M is a weak comultiplication R-module. The indecomposability follows from [17, 1.9].

(ii) and (iii) (involving one or two Prüfer modules) M is weak comultiplication (see Proposition 3.3 and Proposition 4.1). Finally, the indecomposability follows from [5, Theorem 3.5]. □

Corollary 4.9. *Let R be the pullback ring as described in Theorem 4.9. Then every indecomposable weak comultiplication R-module with finite-dimensional top is pure-injective.*

Proof. Apply [5, Theorem 3.5] and Theorem 4.8. □

Acknowledgement. The authors wish to express their sincere thanks to Professor Daniel Simson for his valuable suggestions. The authors also would like to thank the referee for several useful suggestions on the first draft of the manuscript.

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Received by the editors: 22.03.2009
and in final form 01.05.2009.