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Gorenstein Latin squares

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Abstract. We introduce the notion of a Gorenstein Latin square and consider loops and quasigroups related to them. We study some properties of normalized Gorenstein Latin squares and describe all of them with order $n \leq 8$.

1. Preliminaries

The notion of a Latin square was introduced by L. Euler at the end of the XVIII century (see [5] for details). Nowadays Latin squares have their applications in such modern branches of mathematics as compiler testing in statistics and cryptography. The most popular example of an application of Latin squares is the game SUDOKU, well known through over the world as the Japanese game [10].

Definition 1.1. *A* **Latin square** $\mathcal{L}_n$ **of order** n *is a square* $n \times n$ *matrix, such that its rows and columns are permutations of some set* $S = \{s_1, \ldots, s_n\}$.

In what follows we shall take $S = \{0, 1, \ldots, n-1\}$. So $\mathcal{L}_n = (\alpha_{ij})$, where $\alpha_{ij} \in \{0, 1, \ldots, n-1\}$. We say that a Latin square of order n is *normalized* if its first row is $(0, 1, \ldots, n-1)$ and the first column is $(0, 1, \ldots, n-1)^T$, where T (as exponent index) means transposition. The normalized Latin squares are also called "reduced Latin squares" or "Latin squares of standard form".

The numbers of the Latin squares of small orders are as follows:

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There exist a unique normalized Latin square for each $n = 2, 3$:

$$\mathcal{L}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}_3 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

For $n = 4$ the normalized Latin squares are:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}.$$

There exist 56 normalized Latin squares for $n = 5$ (L. Euler (1782), A. Cayley (1890)), 9408 normalized Latin squares for $n = 6$ (M. Frolov (1890), G. Ferry (1900)) and 16 942 080 normalized Latin squares for $n = 7$ [15]. For $n = 8, 9, 10$ see [17], [3], [12], and for further information consult [12], [13].

Next we introduce the notion of a Gorenstein Latin square which appears in the study of Gorenstein rings [8].

Definition 1.2. *A Latin square $\mathcal{L}_n = (\alpha_{ij})$ is called Gorenstein if its main diagonal consists of zeros and there exists a permutation $\sigma : i \to \sigma(i)$ $i = 1, \ldots, n$, such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for all $i, k = 1, \ldots, n$.*

Example 1.1. *The Cayley table of the Klein four-group (2)×(2) can be written in the following form:*

$$K(4) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Example 1.2. *The Cayley table of the elementary Abelian 2-group (2)×(2)×(2) is as follows:*

$$K(8) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}.$$

The Latin square $K(4)$ is Gorenstein with $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$, whereas $K(8)$ is Gorenstein with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Recall that a **quasigroup** is a non-empty set Q with a binary operation $*$ such that the equations $a * x = b$ and $y * a = b$ have unique solutions x and y in Q. Obviously, every Latin square is the Cayley table of a finite quasigroup. In particular, the Cayley table of a finite group is a Latin square. See [1], [2], [4], [14] for more information on the theory of quasigroups.

A quasigroup Q is called a **loop** if it has an **identity element** $e \in Q$ ($e * x = x * e = x$ for every $x \in Q$). Evidently every normalized Latin square is a Cayley table of some loop.

2. Gorenstein loops

Definition 2.1. *A finite quasigroup Q defined on the set $\{0, 1, \ldots, n-1\}$ is called* **Gorenstein** *if its Cayley table $C(Q) = (\alpha_{ij})$ is a Gorenstein Latin square.*

The permutation corresponding to Q (see Definition 1.2) is denoted by $\sigma = \sigma(Q)$.

Lemma 2.1. *Let $C = (\alpha_{ij})$ be a Gorenstein Latin square with permutation σ. Then $\alpha_{i\sigma(i)} = n - 1$ for all $i = 1, \ldots, n$.*

Proof. Summing over k the equations

$$\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)} \tag{1}$$

we obtain

$$\frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n\alpha_{i\sigma(i)}$$

□

Corollary 2.1. *σ has no cycles of length 1.*

Proof. If $i = \sigma(i)$ then $n - 1 = \alpha_{i\sigma(i)} = \alpha_{ii} = 0$. □

In this section we consider Gorenstein loops. We will denote the identity element of such a loop by 0, so its Cayley table is supposed to be normalized. We abbreviate the term “normalized Gorenstein Latin square” by NGLS.

Proposition 2.1. *If $C = (\alpha_{ij})$ is a NGLS then*
(i) $\sigma(i) = n - i + 1$ for all $i = 1, \dots, n$;
(ii) n is even.

Proof. By Lemma 2.1 the equation (1) becomes

$$\alpha_{ik} + \alpha_{k\sigma(i)} = n - 1. \tag{2}$$

Put $k = 1$ in it. Since C is normalized, $\alpha_{1i} = \alpha_{i1} = i - 1$. Hence $\sigma(i) = n - i + 1$.

Suppose $n = 2m + 1$. Then $\sigma(m + 1) = m + 1$ in contradiction with Corollary 2.1. □

Proposition 2.1 means that for all NGLS's σ is uniquely determined and equals to a product of transpositions:

$$\sigma(C) = (1, n)(2, n - 1)(3, n - 2) \dots (m, m + 1)$$

for $n = 2m$.

Corollary 2.2. *All entries of the secondary diagonal of a NGLS equal $n - 1$.*

Proof. $\alpha_{i(n-i+1)} = \alpha_{i\sigma(i)} = n - 1$ by Lemma 2.1. □

Corollary 2.3. *Every NGLS is centrally symmetric.*

Proof. From the equation (2) we have

$$\alpha_{ik} + \alpha_{k(n-i+1)} = n - 1, \tag{3}$$

$$\alpha_{k(n-i+1)} + \alpha_{(n-k+1)(n-i+1)} = n - 1$$

Subtracting we obtain $\alpha_{(n-k+1)(n-i+1)} = \alpha_{ik}$. □

The next assertion gives an abstract characterization of Gorenstein loops:

Theorem 2.1. *A finite loop $L(*)$ with an identity e is isomorphic to a Gorenstein loop if and only if it satisfies the following conditions:*
(i) $x * x = e$ for all $x \in L$
and there exists such an element $a \in L$, $a \neq e$, that
(ii) $a * x = x * a$ for all $x \in L$,
(iii) for all $x, y \in L$

$$(x * y) * a = y * (x * a). \tag{4}$$

Proof. Let $C = (\alpha_{ij})$ be a NGLS with permutation σ, $L(*)$ its loop with multiplication $i * j = \alpha_{(i+1)(j+1)}$, $i, j = 0, \dots, n-1$. The condition (i) is evident. Put $a = n - 1$. From the equation (3) we have

$$(n-1) * i = \alpha_{n(i+1)} = n - 1 - \alpha_{(i+1)1} = n - i - 1 = i * (n-1) \qquad (5)$$

and (ii) is true. Now from (3) and (5)

$$\begin{aligned}(i * j) * a &= n - 1 - i * j = n - 1 - \alpha_{(i+1)(j+1)} = \alpha_{(j+1)(n-i)} \\ &= j * (n - i - 1) = j * (i * a).\end{aligned}$$

Conversely, let $L(*)$ be a loop of order n satisfying (i)-(iii). Preliminarily we prove some properties of L.

Putting sequentially $x = a$ and $x = y$ in (4) we obtain

$$(x * a) * a = x \qquad (6)$$

$$x * (x * a) = a. \qquad (7)$$

By (6) the translation $x \to x * a$ is a product of cycles of length two; cycles of length one are absent since $a \neq e$. In particular, it follows that $|L| = n$ is even.

Choose a bijection $\omega : L \to S = \{0, \dots, n-1\}$ which satisfies the conditions: $\omega(x) + \omega(x*a) = n-1$ and $\omega(e) = 0$ (evidently such bijections exist). In what follows we will identify $\omega(x)$ with x and thus consider L to be defined on S. In particular, $e = 0$, $a = n-1$ and $x + x * a = n - 1$.

We can compare with L a normalized Latin square $C = (\alpha_{ij})_{1 \le i,j \le n}$ where $\alpha_{ij} = (i-1)*(j-1)$. Verify that C is a NGLS with the permutation $\sigma : i \to n - i + 1$ $(1 \le i \le n)$:

$$\begin{aligned}\alpha_{ik} + \alpha_{k\sigma(i)} &= (i-1) * (k-1) + (k-1) * (n-i) \\ &= (i-1) * (k-1) + (k-1) * [(i-1) * a] \\ &= (i-1) * (k-1) + [(i-1) * (k-1)] * a = n - 1\end{aligned}$$

□

We apply Theorem 2.1 to study Gorenstein loops of small order.

Let L is a Gorenstein loop. Obviously, if $|L| = 4$ then $L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

We will use following two remarks:

1) For every $x \in L$, $x \neq e, a$, the subset $\{e, a, x, x * a\} \subset L$ is a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2) If H is a subgroup of L, $p, p * q \in H$ then $q \in H$ (since $H = p * H$).

Corollary 2.4. *There are no Gorenstein loops of order 6.*

Proof. Let $|L| = 6$, $L = \{e, a, x, x * a, y, y * a\}$. Then $G = \{e, a, x, x * a\}$ and $H = \{e, a, y, y * a\}$ are subgroups and $L = G \cup H$. So $x * y$ belongs to either G or H. If, e. g., $x * y \in G$ then $y \in G$. A contradiction. $\square$

Corollary 2.5. *Every Gorenstein loop of order 8 is commutative.*

Proof. Let $|L| = 8$, $L = \{e, a, x, x*a, y, y*a, z, z*a\}$ and $x*y \neq y*x$. The subsets $G = \{e, a, x, x * a\}$, $H = \{e, a, y, y * a\}$ and $K = \{e, a, z, z * a\}$ are subgroups. So $x * y, y * x \in K$ (as in the proof of Corollary 2.4). Evidently, $x * y, y * x \neq e$ (else $x * x = x * y$) and $x * y, y * x \neq a$ (else $y * (x * a) = (x * y) * a = e$ and $x * a = y$). Therefore, e. g., $x * y = z$, $y * x = z * a$. Hence $y * x = (x * y) * a = y * (x * a)$ and $x = x * a$. A contradiction. $\square$

3. Calculations of NGLS

Here we consider NGLS for $n \leq 12$. By Corollary 2.4 we have to examine the following cases: $n = 8$, $n = 10$ and $n = 12$. First we give a full list of NGLS of the order 8.

Let $\mathcal{L}_8 = (\alpha_{ij})$ be a NGLS of order 8. By Proposition 2.1 $\sigma(\mathcal{L}_8) = (18)(27)(36)(45)$, by Corollary 2.2 $\alpha_{i(9-i)} = 7$. Therefore $\mathcal{L}_8$ has the following preliminary form:

$$\mathcal{L}_8 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & x & y & z & t & 7 & 6 \\ 2 & 7-t & 0 & p & s & 7 & 7-x & 5 \\ 3 & 7-z & 7-s & 0 & 7 & 7-p & 7-y & 4 \\ 4 & 7-y & 7-p & 7 & 0 & 7-s & 7-z & 3 \\ 5 & 7-x & 7 & s & p & 0 & 7-t & 2 \\ 6 & 7 & t & z & y & x & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}.$$

Since the matrix $\mathcal{L}_8$ is symmetrical (see Corollary 2.5), then we immediately have that the equalities $x = 7 - t$, $y = 7 - z$, $s = 7 - p$ hold, and hence our matrix can be rewritten in the form

$$\mathcal{L}_8 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & x & y & 7-y & 7-x & 7 & 6 \\ 2 & x & 0 & p & 7-p & 7 & 7-x & 5 \\ 3 & y & p & 0 & 7 & 7-p & 7-y & 4 \\ 4 & 7-y & 7-p & 7 & 0 & p & y & 3 \\ 5 & 7-x & 7 & 7-p & p & 0 & x & 2 \\ 6 & 7 & 7-x & 7-y & y & x & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}.$$

From the first line we obviously obtain that $x \notin \{1, 2, 5, 6, 7\}$ and $y \notin \{1, 3, 4, 6, 7\}$, which means that $x \in \{3, 4\}$ and $y \in \{2, 5\}$. Also from the second line we have $p \notin \{2, 3, 4, 5, 7\}$, which means that $p \in \{1, 6\}$.

Since the sets of possible values for x, y and p do not intersect, then the inclusions $x \in \{3, 4\}$, $y \in \{2, 5\}$ and $p \in \{1, 6\}$, together with the form of $\mathcal{L}_8$ in terms of these three variables, form a sufficient condition for $\mathcal{L}_8$ to be a NGLS.

Note that from this description the number 8 of all NGLS of order 8 comes immediately. Here is the list of them:

$$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&3&2&5&4&7&6\\ 2&3&0&1&6&7&4&5\\ 3&2&1&0&7&6&5&4\\ 4&5&6&7&0&1&2&3\\ 5&4&7&6&1&0&3&2\\ 6&7&4&5&2&3&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&3&5&2&4&7&6\\ 2&3&0&1&6&7&4&5\\ 3&5&1&0&7&6&2&4\\ 4&2&6&7&0&1&5&3\\ 5&4&7&6&1&0&3&2\\ 6&7&4&2&5&3&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&3&2&5&4&7&6\\ 2&3&0&6&1&7&4&5\\ 3&2&6&0&7&1&5&4\\ 4&5&1&7&0&6&2&3\\ 5&4&7&1&6&0&3&2\\ 6&7&4&5&2&3&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&3&5&2&4&7&6\\ 2&3&0&6&1&7&4&5\\ 3&5&6&0&7&1&2&4\\ 4&2&1&7&0&6&5&3\\ 5&4&7&1&6&0&3&2\\ 6&7&4&2&5&3&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&4&2&5&3&7&6\\ 2&4&0&1&6&7&3&5\\ 3&2&1&0&7&6&5&4\\ 4&5&6&7&0&1&2&3\\ 5&3&7&6&1&0&4&2\\ 6&7&3&5&2&4&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&4&5&2&3&7&6\\ 2&4&0&1&6&7&3&5\\ 3&5&1&0&7&6&2&4\\ 4&2&6&7&0&1&5&3\\ 5&3&7&6&1&0&4&2\\ 6&7&3&2&5&4&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&4&2&5&3&7&6\\ 2&4&0&6&1&7&3&5\\ 3&2&6&0&7&1&5&4\\ 4&5&1&7&0&6&2&3\\ 5&3&7&1&6&0&4&2\\ 6&7&3&5&2&4&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&3&4&5&6&7\\ 1&0&4&5&2&3&7&6\\ 2&4&0&6&1&7&3&5\\ 3&5&6&0&7&1&2&4\\ 4&2&1&7&0&6&5&3\\ 5&3&7&1&6&0&4&2\\ 6&7&3&2&5&4&0&1\\ 7&6&5&4&3&2&1&0 \end{pmatrix}.$$

So we proved the next:

Theorem 3.1. *There are 8 normalized Gorenstein Latin squares of order 8. All of them are doubly symmetric.*

Now let $n = 10$ and $\mathcal{L}_{10}$ be a NGLS of order 10. By Proposition 2.1 $\sigma(\mathcal{L}_{10}) = (1, 10)(2, 9)(3, 8)(4, 7)(5, 6)$. Therefore $\mathcal{L}_{10}$ has the following preliminary form:

$$\begin{pmatrix} 0&1&2&3&4&5&6&7&8&9\\ 1&0&X&Y&Z&9-G&9-D&9-A&9&8\\ 2&A&0&S&T&9-H&9-B&9&9-X&7\\ 3&D&B&0&W&9-C&9&9-S&9-Y&6\\ 4&G&H&C&0&9&9-W&9-T&9-Z&5\\ 5&9-Z&9-T&9-W&9&0&C&H&G&4\\ 6&9-Y&9-S&9&9-C&W&0&B&D&3\\ 7&9-X&9&9-B&9-H&T&S&0&A&2\\ 8&9&9-A&9-D&9-G&Z&Y&X&0&1\\ 9&8&7&6&5&4&3&2&1&0 \end{pmatrix}. \tag{8}$$

Our computations show that for $n = 10$ there exist 1024 normalized Gorenstein Latin squares.

Theorem 3.2. *There are no* 10×10 *normalized Gorenstein Latin squares which are doubly symmetric.*

Proof. The formula (8) gives the following form for the doubly symmetric NGLS $\mathcal{L}_{10}$ of the order 10:

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\
1 & 0 & X & Y & Z & 9-Z & 9-Y & 9-X & 9 & 8\\
2 & X & 0 & S & T & 9-T & 9-S & 9 & 9-X & 7\\
3 & Y & S & 0 & W & 9-W & 9 & 9-S & 9-Y & 6\\
4 & Z & T & W & 0 & 9 & 9-W & 9-T & 9-Z & 5\\
5 & 9-Z & 9-T & 9-W & 9 & 0 & W & T & Z & 4\\
6 & 9-Y & 9-S & 9 & 9-W & W & 0 & S & Y & 3\\
7 & 9-X & 9 & 9-S & 9-T & T & S & 0 & X & 2\\
8 & 9 & 9-X & 9-Y & 9-Z & Z & Y & X & 0 & 1\\
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{pmatrix}.$$

Note that the numbers $\{X, 9-X, Y, 9-Y\}$ from the second column of $\mathcal{L}_{10}$ are pairwise different. Obviously, $S \notin \{X, 9-X, Y, 9-Y\}$. Let $S \in \{Z, 9-Z\}$. Since $Z \notin \{0, 1, 4, 5, 8, 9\}$, then $9-Z \notin \{0, 1, 4, 5, 8, 9\}$. So, the condition $S \in \{Z, 9-Z\}$ gives that $S \notin \{0, 1, 4, 5, 8, 9\}$. Looking at the position $(3, 4)$ ($S = \alpha_{34}$) we obtain that $S \notin \{2, 3, 6, 7\}$. Therefore, $S \notin \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, a contradiction which gives us $S \notin \{Z, 9-Z\}$.

Consider the second row of $\mathcal{L}_{10}$. We see that all elements $X, 9-X, Y, 9-Y, Z, 9-Z$ are pairwise different. Also from the third row and the fourth column we obtain that $S \notin \{X, 9-X, Y, 9-Y\}$. This together with the condition $S \notin \{Z, 9-Z\}$ gives us that all elements of the list $\{X, 9-X, Y, 9-Y, Z, 9-Z, S, 9-S\}$ are pairwise different. Note that each of these 8 numbers is neither 0 nor 9, whence $\{X, 9-X, Y, 9-Y, Z, 9-Z, S, 9-S\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Now take $\alpha_{35} = T$. Since $T \notin \{0, 9\}$, then $T \in \{X, 9-X, Y, 9-Y, Z, 9-Z, S, 9-S\}$. It obviously follows, looking at the third row and the fifth column, that $T \notin \{X, 9-X, S, 9-S, Z, 9-Z\}$. Then $\{T, 9-T\} = \{Y, 9-Y\}$. Since $Y \notin \{0, 1, 3, 6, 8, 9\}$, $9-Y \notin \{0, 1, 3, 6, 8, 9\}$ and $T \notin \{2, 4, 5, 7\}$, it follows that $T \notin \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. So, we can make the final conclusion that there is no doubly symmetric NGLS of order 10. $\square$

Note that a Gorenstein loop of order 10 can be non-commutative:

Example 3.1.

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\
1 & 0 & 3 & 2 & 6 & 7 & 4 & 5 & 9 & 8\\
2 & 4 & 0 & 1 & 3 & 8 & 5 & 9 & 6 & 7\\
3 & 5 & 4 & 0 & 1 & 2 & 9 & 8 & 7 & 6\\
4 & 2 & 1 & 7 & 0 & 9 & 8 & 6 & 3 & 5\\
5 & 3 & 6 & 8 & 9 & 0 & 7 & 1 & 2 & 4\\
6 & 7 & 8 & 9 & 2 & 1 & 0 & 4 & 5 & 3\\
7 & 6 & 9 & 5 & 8 & 3 & 1 & 0 & 4 & 2\\
8 & 9 & 5 & 4 & 7 & 6 & 2 & 3 & 0 & 1\\
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{pmatrix}.$$

Our computations give us the number of normalized Gorenstein Latin squares for $n = 12$. This number is 448512.

The following normalized Gorenstein Latin square is an example of a non-symmetric Latin square of order 12:

Example 3.2.

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 0 & 3 & 2 & 5 & 4 & 8 & 9 & 6 & 7 & 11 & 10 \\
2 & 4 & 0 & 5 & 3 & 1 & 10 & 6 & 7 & 11 & 8 & 9 \\
3 & 5 & 4 & 0 & 1 & 7 & 2 & 10 & 11 & 6 & 9 & 8 \\
4 & 2 & 5 & 1 & 0 & 3 & 9 & 11 & 10 & 8 & 6 & 7 \\
5 & 3 & 1 & 9 & 2 & 0 & 11 & 8 & 4 & 10 & 7 & 6 \\
6 & 7 & 10 & 4 & 8 & 11 & 0 & 2 & 9 & 1 & 3 & 5 \\
7 & 6 & 8 & 10 & 11 & 9 & 3 & 0 & 1 & 5 & 2 & 4 \\
8 & 9 & 6 & 11 & 10 & 2 & 7 & 1 & 0 & 4 & 5 & 3 \\
9 & 8 & 11 & 7 & 6 & 10 & 1 & 3 & 5 & 0 & 4 & 2 \\
10 & 11 & 7 & 6 & 9 & 8 & 4 & 5 & 2 & 3 & 0 & 1 \\
11 & 10 & 9 & 8 & 7 & 6 & 5 & 6 & 3 & 2 & 1 & 0
\end{pmatrix}.$$

According to Corollary 2.5 all normalized Gorenstein Latin squares of order 8 are symmetrical. Earlier we had the conjecture that all NGLS whose order is the power of two are symmetrical. However we give the following non-symmetric Latin square of order 16:

Example 3.3.

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 & 14 & 15 & 12 & 13 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 & 15 & 14 & 13 & 12 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 12 & 14 & 13 & 15 & 8 & 9 & 10 & 11 \\
5 & 4 & 7 & 6 & 2 & 0 & 3 & 1 & 13 & 12 & 15 & 14 & 9 & 8 & 11 & 10 \\
6 & 7 & 4 & 5 & 1 & 3 & 0 & 2 & 14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 15 & 13 & 14 & 12 & 11 & 10 & 9 & 8 \\
8 & 9 & 10 & 11 & 12 & 14 & 13 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 & 2 & 0 & 3 & 1 & 5 & 4 & 7 & 6 \\
10 & 11 & 8 & 9 & 14 & 15 & 12 & 13 & 1 & 3 & 0 & 2 & 6 & 7 & 4 & 5 \\
11 & 10 & 9 & 8 & 15 & 13 & 14 & 12 & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
12 & 13 & 14 & 15 & 8 & 9 & 10 & 11 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
13 & 12 & 15 & 14 & 9 & 8 & 11 & 10 & 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 & 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\
15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{pmatrix}.$$

4. Gorenstein quasigroups

Unfortunately, we know no general properties of Gorenstein quasigroups except Lemma 2.1 and Corollary 2.1. Here we consider two classes of Gorenstein quasigroups.

Let $P(*)$ and $Q(\circ)$ be quasigroups. Recall that an **isotopy** is a triple (λ, μ, ν) of bijections from P to Q such that $\lambda(x) \circ \mu(y) = \nu(x * y)$ for all $x, y \in P$. We will write $Q = P^{(\lambda,\mu,\nu)}$ and regard λ and μ as permutations of rows and columns of the corresponding Latin square.

Generally speaking, the Gorenstein property is not preserved under isotopy. However we have:

Proposition 4.1. *Let P be a Gorenstein quasigroup and ε the identical permutation. Then $Q = P^{(\lambda,\lambda,\varepsilon)}$ is also a Gorenstein quasigroup for any permutation λ.*

Proof. Let $C = (\alpha_{ij})$ and $D = (\beta_{ij})$ are the Latin squares of P and Q respectively and σ is the permutation for C. Set

$$\tau(i) = \lambda^{-1}[\sigma(\lambda(i-1)+1)-1]+1$$

and prove that D is a Gorenstein Latin square with the permutation τ.

Indeed,

$$\begin{aligned}\beta_{ik} + \beta_{k\tau(i)} &= \lambda(i-1) * \lambda(k-1) + \lambda(k-1) * (\sigma(\lambda(i-1)+1)-1) \\ &= \alpha_{\lambda(i-1)+1,\lambda(k-1)+1} + \alpha_{\lambda(k-1)+1,\sigma(\lambda(i-1)+1)} = n-1\end{aligned}$$

The equality $\beta_{ii} = 0$ is easily verified. $\square$

Proposition 4.1 means that Q is obtained from P by simultaneous permutation of rows and columns. So we can obtain Gorenstein quasigroups from a given Gorenstein loop P.

Entropic quasigroups give other examples.

Definition 4.1. *A quasigroup Q is called* **entropic** *or* **medial** *if it satisfies the identity $(xu)(vy) = (xv)(uy)$ for all $x, y, u, v \in Q$ (see [4, 14]).*

By the well-known theorem of Toyoda [16] every entropic quasigroup $Q(*)$ can be obtained from an abelian group $Q(\oplus)$ in the following way:

$$x * y = \varphi x \oplus \psi y \oplus c \tag{9}$$

where φ and ψ are commuting automorphisms of $Q(\oplus)$ and $c \in Q$ is some fixed element.

We shall consider a partial case only: the entropic quasigroup $Q(*)$ is given on the set $S = \{0, \dots, n-1\}$, the group $Q(\oplus)$ is cyclic and $x \oplus y \equiv x + y (\bmod n)$. We call $Q(*)$ a **cyclic quasigroup**.

Suppose that $Q(*)$ satisfies the condition: $x * x = 0$ for all $x \in Q(*)$. Putting $x = y = 0$ in (9) we get $c = 0$. Again from (9) we have: $0 = x * x = \varphi x \oplus \psi x$ for all x, and the equation (9) transforms into $x * y = \varphi(x \ominus y)$.

The automorphism φ has the form $\varphi(x) \equiv rx (\bmod n)$ for such $r < n$ that g.c.d.$(r, n) = 1$. Finally we have

$$x * y = r(x - y)(\bmod n). \tag{10}$$

Proposition 4.2. *The cyclic quasigroup Q with operation (10) is a Gorenstein quasigroup with a cyclic permutation.*

Proof. Let $C = (\alpha_{ij})$ is the Latin square of $Q(*)$. Find the permutation $\sigma = \sigma(C)$ from the equation (2) which has the form

$$r(i - k)(\bmod n) + r(k - \sigma(i))(\bmod n) = n - 1 \tag{11}$$

in our case.

Since g.c.d.$(r, n) = 1$, there is such $s < n$ that $rs \equiv 1 (\bmod n)$. Set $k \equiv i + s (\bmod n)$:

$$(-1)(\bmod n) + r(i + s - \sigma(i))(\bmod n) = n - 1,$$

whence

$$\sigma(i) \equiv i + s(\bmod n). \tag{12}$$

Since g.c.d.$(s, n) = 1$, this permutation is cyclic.

Conversely, if a permutation σ satisfies (12), one can easily see that equation (11) holds. $\square$

Corollary 4.1. *There are Gorenstein quasigroups of arbitrary orders.*

Example 4.1. *The matrix*

$$\begin{pmatrix} 0 & n-1 & n-2 & \dots & 2 & 1 \\ 1 & 0 & n-1 & \dots & 3 & 2 \\ 2 & 1 & 0 & \dots & 4 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n-2 & n-3 & n-4 & \dots & 0 & n-1 \\ n-1 & n-2 & n-3 & \dots & 1 & 0 \end{pmatrix}$$

is the Cayley table of an entropic Gorenstein quasigroup with permutation $\sigma = (12 \dots n)$.

5. Final remarks: some related concepts

In conclusion we discuss some mathematical objects related to Gorenstein Latin squares.

5.1. Exponent matrices

Denote by $M_n(\mathbb{Z})$ the ring of square $n \times n$-matrices over the integers $\mathbb{Z}$. Let $\mathcal{E} \in M_n(\mathbb{Z})$. We shall call a matrix $\mathcal{E} = (\alpha_{ij})$ an **exponent matrix** if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \ldots, n$ and $\alpha_{ii} = 0$ for $i = 1, \ldots, n$.

An exponent matrix $\mathcal{E}$ is called **reduced** if $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$, i.e. $\mathcal{E}$ has no symmetric zeros.

Definition 5.1. *A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called* **Gorenstein** *if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.*

The permutation σ is denoted by $\sigma(\mathcal{E})$. Obviously, $\sigma(\mathcal{E})$ has no cycles of length 1.

Gorenstein matrices are closely related to prime semiperfect semidistributive rings A with non-zero Jacobson radical and $inj.dim_A A_A = 1$ [11].

Example 5.1. *The following exponent matrix*

$$T_{n,\alpha} = \begin{pmatrix} 0 & 0 & \ldots & \ldots & \ldots & 0 \\ \alpha & 0 & 0 & \ldots & \ldots & 0 \\ \alpha & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha & & \ddots & \alpha & 0 & 0 \\ \alpha & \ldots & \ldots & \alpha & \alpha & 0 \end{pmatrix}$$

is Gorenstein with $\sigma(T_{n,\alpha}) = (12 \ldots n)$.

There exist Gorenstein loops whose Cayley tables are not exponent matrices. The first example of such a loop was given by B.V. Novikov ([8], example 14.7.3):

Example 5.2. *The matrix*

$$C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 0 & 5 & 2 & 3 & 4 & 7 & 8 & 9 & 6 & 11 & 10 \\ 2 & 5 & 0 & 4 & 1 & 3 & 8 & 10 & 7 & 11 & 6 & 9 \\ 3 & 2 & 4 & 0 & 5 & 1 & 10 & 6 & 11 & 7 & 9 & 8 \\ 4 & 3 & 1 & 5 & 0 & 2 & 9 & 11 & 6 & 10 & 8 & 7 \\ 5 & 4 & 3 & 1 & 2 & 0 & 11 & 9 & 10 & 8 & 7 & 6 \\ 6 & 7 & 8 & 10 & 9 & 11 & 0 & 2 & 1 & 3 & 4 & 5 \\ 7 & 8 & 10 & 6 & 11 & 9 & 2 & 0 & 5 & 1 & 3 & 4 \\ 8 & 9 & 7 & 11 & 6 & 10 & 1 & 5 & 0 & 4 & 2 & 3 \\ 9 & 6 & 11 & 7 & 10 & 8 & 3 & 1 & 4 & 0 & 5 & 2 \\ 10 & 11 & 6 & 9 & 8 & 7 & 4 & 3 & 2 & 5 & 0 & 1 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

is the Cayley table of a Gorenstein loop. We see that

$$\alpha_{17} + \alpha_{79} = 7 < \alpha_{19} = 8.$$

Therefore the matrix C is not exponent. Observe that C is doubly symmetric, i.e., symmetric with respect to the main and the secondary diagonals.

Note that a reduced exponent matrix turns out to be a distance matrix of some finite metric space if and only if it is symmetric. In particular, $K(4)$ and $K(8)$ from Examples 1.1 and 1.2 are distance matrices of metric spaces with 4 and 8 elements, respectively.

5.2. Cayley tables of elementary Abelian 2-Groups and finite metric spaces

In this subsection we cite results which were obtained in [9], §7.6, and [6].

Introduce the following notation:

$$\Gamma_0 = (0), \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

$$U_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in M_n(\mathbb{Z}), \qquad X_{k-1} = 2^{k-1} U_{2^{k-1}};$$

$$\Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \Gamma_{k-1} + X_{k-1} \\ \Gamma_{k-1} + X_{k-1} & \Gamma_{k-1} \end{pmatrix} \text{ for } k = 1, 2, \ldots .$$

The matrix $\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Cayley table of the cyclic group G_1 of order 2 and is a Gorenstein matrix with permutation $\sigma(\Gamma_1) = (12)$.

Proposition 5.1. ([9], §7.6) Γ_k *is an exponent matrix for any positive integer* k.

Proposition 5.2. ([9], §7.6). Γ_k *is the Cayley table of the elementary Abelian group* G_k *of order* 2^k.

Proposition 5.3. ([9], §7.6). *The matrix* Γ_k *is Gorenstein with permutation*

$$\sigma(\Gamma_k) = \begin{pmatrix} 1 & 2 & 3 & \ldots & 2^k - 1 & 2^k \\ 2^k & 2^k - 1 & 2^k - 2 & \ldots & 2 & 1 \end{pmatrix}.$$

Theorem 5.1. ([6] and [9], §7.6) *Suppose that a Latin square* $\mathcal{L}_n$ *with first row and first column* $(0\,1\,\ldots\,n-1)$ *is an exponent matrix. Then* $n = 2^k$ *and* $\mathcal{L}_n = \Gamma_k$ *is the Cayley table of the direct product of* k *copies of the cyclic group of order 2.*

Conversely, the Cayley table Γ_k *of the elementary Abelian group*

$$G_k = \mathbb{Z}/(2) \times \ldots \times \mathbb{Z}/(2) = (2) \times \ldots \times (2)$$

(k factors) of order 2^k *is a Latin square and a Gorenstein symmetric matrix with the first row* $(0, 1, \ldots, 2^k - 1)$ *and permutation*

$$\sigma(\Gamma_k) = \begin{pmatrix} 1 & 2 & 3 & \ldots & 2^k - 1 & 2^k \\ 2^k & 2^k - 1 & 2^k - 2 & \ldots & 2 & 1 \end{pmatrix}.$$

Now we consider the case when a Latin square $\mathcal{L}_n$ with first row and first column $(0\,1\,\ldots\,n-1)$ is a distance matrix $D = D(M)$ of a finite metric space $M = \{m_1, \ldots, m_n\}$. Obviously, if $\mathcal{L}_n = D(M)$ then $\mathcal{L}_n$ is an exponent matrix. So we obtain the following theorem:

Theorem 5.2. *Suppose that a normalized Latin square* $\mathcal{L}_n$ *is a distance matrix* $D = D(M)$ *of a finite metric space* $M = \{m_1, \ldots, m_n\}$*. Then* $n = 2^k$ *and* $\mathcal{L}_n = \Gamma_k$ *is the Cayley table of the direct product of* k *copies of the cyclic group of order 2.*

Conversely, the Cayley table Γ_k *of the elementary Abelian group*

$$G_k = \mathbb{Z}/(2) \times \ldots \times \mathbb{Z}/(2) = (2) \times \ldots \times (2)$$

(k factors) of order 2^k *is a Latin square and the distance matrix* $D = D(M)$ *of a finite metric space with* 2^k *elements.*

5.3. d-Matrices

The notion of a d-matrix was introduced in [9, §7.5].

Definition 5.2. *Let $A \in M_n(\mathbb{R})$ and $A \geq 0$ (i.e., if $A = (a_{ij})$ then $a_{ij} \geq 0$). We say that A is a d-**matrix** for some $d > 0$, if $\sum\limits_{j=1}^{n} a_{ij} = d$ and $\sum\limits_{i=1}^{n} a_{ij} = d$ for all i, j.*

Obviously, every Latin square $\mathcal{L}_n$, defined on the set $\{0, 1, \dots, n-1\}$, is a d-matrix with $d = \frac{(n-1)n}{2}$. Consider the matrix $T_n = \frac{2}{(n-1)n}\mathcal{L}_n$. It is clear that T_n is a doubly stochastic matrix and T_n^2 is positive. Therefore, T_n is a primitive matrix and T_n is the transition matrix of a regular ergodic homogenous Markov chain.

Denote by e_{ij} the $(n \times n)$-matrix with 1 in the (i, j) position and zeroes elsewhere. These n^2 matrices e_{ij} are called the matrix units and form a basis of $M_n(\mathbb{R})$ over $\mathbb{R}$. Let $\sigma \in S_n$ be a permutation of $\{1, \dots, n\}$. The matrix $P_\sigma = \sum\limits_{i=1}^{n} e_{i\sigma(i)}$ is called a permutation matrix of σ.

Let $B = (b_{ij}) \in M_n(\mathbb{R})$ be a non-negative matrix. A **normal set of elements of** B is a set of n elements $b_{1j_1}, \dots, b_{nj_n}$ of B, where

$$\begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$$

is an element of the symmetric group S_n of degree n.

Lemma 5.1. ([9, p.343]) *Let $B = (b_{ij}) \in M_n(\mathbb{R})$ be a non-negative matrix and*

$$\sum_{i=1}^{n} b_{ij} = \sum_{j=1}^{n} b_{ij} = \omega.$$

Then there exists a normal set $b_{1i_1}, \dots, b_{ni_n}$ of strictly positive elements of B.

Let $\mathcal{L}_n = (\alpha_{ij})$ be a Latin square, defined on $\{0, 1, \dots, n-1\}$. With every $k \in \{1, \dots, n-1\}$ we associate the permutation $\sigma = \begin{pmatrix} 1 & \dots & n \\ t_1 & \dots & t_n \end{pmatrix}$ such that $\alpha_{it_i} = k$. The permutation σ_k exists by the definition of a Latin square. Therefore, $\mathcal{L}_n = \sum\limits_{k=1}^{n-1} kP_{\sigma(k)}$, where $P_{\sigma(k)}$ is a permutation matrix of $\sigma(k)$.

Example 5.3. *The Latin square*

$$\mathcal{L}_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

is the following sum of permutation matrices:

$$\mathcal{L}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 2 \times \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + 3 \times \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the vector $(1, \ldots, 1)^T$ is the eigenvector of $\mathcal{L}_n$ with the eigenvalue $\frac{n(n-1)}{2}$.

By Perron-Frobenius theorem (see, for example, [7]) every another eigenvector $\vec{v}$ of $\mathcal{L}_n$ with eigenvalue $\frac{n(n-1)}{2}$ is $\vec{v} = (\alpha, \alpha, \ldots, \alpha)$, where $\alpha \neq 0$ and if $\lambda \neq \frac{n(n-1)}{2}$ is an eigenvalue of $\mathcal{L}_n$, then $|\,\lambda\,| < \frac{n(n-1)}{2}$.

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