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Construction of free $\mathfrak{g}$-dimonoids

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Abstract. In this paper, the concept of a $\mathfrak{g}$-dimonoid is introduced and the construction of a free $\mathfrak{g}$-dimonoid is described. (A $\mathfrak{g}$-dimonoid is a duplex satisfying two additional identities.)

Introduction

The concepts of a dimonoid and a dialgebra were introduced by Loday [1]. Dimonoids are a tool to study Leibniz algebras [1]. A dimonoid is a set with two binary associative operations satisfying the additional identities. A dialgebra is a linear analogue of the dimonoid. One of the first results about dimonoids is the description of the free dimonoid generated by the given set. Using properties of the free dimonoid, the free dialgebras were described and the cohomologies of dialgebras were studied in [1]. In [2], using the concept of a dimonoid, the concept of a unileteral diring was introduced and the basic properties of dirings were studied. In [4, 5] free dimonoids and free commutative dimonoids were described. In [6] the concept of a duplex (which generalizes the concept of a dimonoid) and the construction of a free duplex were introduced. A duplex is a set equipped with two associative operations. In [7, 8] the concept of a Boolean bisemigroup (which generalizes the concept of a Boolean algebra) was introduced and a Stone-type representation theorem was proved (cf. [9–13]).

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The concept of a 0-dialgebra was introduced in [14]. The 0-dialgebra under the field F is a vector space under F with two binary operations, $\dashv$ and $\vdash$, such that the following two identities are satisfied:

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \quad z \dashv (x \vdash y) = z \dashv (x \dashv y).$$

In this paper the concept of a $\mathfrak{g}$-dimonoid (a generalized dimonoid) is introduced and the construction of a free $\mathfrak{g}$-dimonoid is described. (A $\mathfrak{g}$-dimonoid is a duplex satisfying two additional identities.)

1. Auxiliary results

Definition 1. An algebra $(\mathcal{A}; \dashv, \vdash)$ is called a $\mathfrak{g}$-dimonoid if it satisfies the following identities:

$$\begin{array}{ll} (A1) & (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (A2) & (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (A3) & (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (A4) & (x \vdash y) \vdash z = x \vdash (y \vdash z). \end{array}$$

An $\mathfrak{g}$-dimonoid $(\mathcal{A}; \dashv, \vdash)$ is called a dimonoid, if it satisfies the following additional identity:

$$(x \vdash y) \dashv z = x \vdash (y \dashv z).$$

Let us give an example of a $\mathfrak{g}$-dimonoid, which is not a dimonoid ([3]). Let X be an arbitrary nonempty set, $|X| > 1$ and let X^* be the set of all finite nonempty words in the alphabet X. Denote the first (respectively, the last) letter of a word $\omega \in X^*$ by $\omega^{(0)}$ (respectively, by $\omega^{(1)}$). Define the following operations $\dashv, \vdash$ on X^* by

$$\omega \dashv u = \omega^{(0)}, \quad \omega \vdash u = u^{(1)}$$

for all $\omega, u \in X^*$. It is easy to check that the binary algebra $(X^*, \dashv, \vdash)$ is a $\mathfrak{g}$-dimonoid, but is not a dimonoid.

Definition 2. A map $f : \mathcal{A}_1 \to \mathcal{A}_2$ between $\mathfrak{g}$-dimonoids $\mathcal{A}_1$ and $\mathcal{A}_2$ is called a homomorphism if $f(x \dashv y) = f(x) \dashv f(y)$ and $f(x \vdash y) = f(x) \vdash f(y)$ for all $x, y \in \mathcal{A}_1$. A bijective homomorphism between $\mathfrak{g}$-dimonoids is called an isomorphism.

A $\mathfrak{g}$-dimonoid F is called a free $\mathfrak{g}$-dimonoid if there exists a subset $X \subseteq F$ such that F is generated by X and for any $\mathfrak{g}$-dimonoid D and for any map $f : X \to D$ there exists a unique homomorphism of $\mathfrak{g}$-dimonoids $g : F \to D$ such that $g(x) = f(x)$ for all $x \in X$. If this holds, then we say that F is a free $\mathfrak{g}$-dimonoid with the system of free generators X ([15,16]).

We recall the concept of a term of a $\mathfrak{g}$-dimonoid $\mathcal{A}$ by the following: any element $x \in \mathcal{A}$ is a term of $\mathcal{A}$; if t_1, t_2 are terms of $\mathcal{A}$, then $t_1 \dashv t_2$ and $t_1 \vdash t_2$ also are the terms of $\mathcal{A}$; and there are no other terms. By $t(x_1, \ldots, x_n)$ we mean a term with the elements $x_1, \ldots, x_n$ each of which meets once and in the mentioned order.

Lemma 1. *Let $(\mathcal{A}; \dashv, \vdash)$ be a $\mathfrak{g}$-dimonoid. Then for any term $t = t(x_1, \ldots, x_n)$, $x \in \mathcal{A}$ the following equalities hold:*

(i) $x \dashv t = x \dashv x_1 \dashv \ldots \dashv x_n$,

(ii) $t \vdash x = x_1 \vdash \ldots \vdash x_n \vdash x$.

Proof. Prove (i) by induction on n. If $n = 1, 2$ then the statement is obvious. Let it be true for $n < k$, where $k > 2$. For $t = t_1 * t_2$, where $* \in \{\dashv, \vdash\}$, $t_1 = t_1(x_1, \ldots, x_{k_1}), t_2 = t_2(x_{k_1+1}, \ldots, x_k)$, $0 < k_1 < k$, let us consider the following cases. If $* = \dashv$, then

$$\begin{aligned} x \dashv t = x \dashv (t_1 \dashv t_2) &\overset{(A1)}{=} (x \dashv t_1) \dashv t_2 \\ &= (x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv t_2 \\ &= (x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv x_{k_1+1} \dashv \ldots \dashv x_k \\ &= x \dashv x_1 \dashv \ldots \dashv x_k. \end{aligned}$$

If $* = \vdash$, then

$$\begin{aligned} x \dashv t = x \dashv (t_1 \vdash t_2) &\overset{(A2)}{=} (x \dashv t_1) \dashv t_2 \\ &= (x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv t_2 \\ &= (x \dashv x_1 \dashv \ldots \dashv x_{k_1}) \dashv x_{k_1+1} \dashv \ldots \dashv x_k \\ &= x \dashv x_1 \dashv \ldots \dashv x_k. \end{aligned}$$

Hence, (i) holds for $n = k$. (ii) is proved analogously. □

Let e be an arbitrary symbol; introduce the following sets:

$$I^1 = \{e\}, \quad I^n = \{0, 1\}^{n-1} =$$

$$= \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1}) : \varepsilon_k \in \{0,1\}, k = \overline{1, n-1}\},\ n > 1,$$
$$I = \bigcup_{n \geqslant 1} I^n.$$

If $l = 0$ then the sequence $\varepsilon_1, \dots, \varepsilon_l$ without brackets we consider empty, and the sequence $(\varepsilon_1, \dots, \varepsilon_l)$ with brackets we consider e. For example, if $n = 1$, then the sequence $(\varepsilon_1, \dots, \varepsilon_{n-1}, \overbrace{1, 1, \dots, 1}^{m})$ is $(\overbrace{1, 1, \dots, 1}^{m})$.

Definition 3. Let $(\mathcal{A}; \dashv, \vdash)$ be a $\mathfrak{g}$-dimonoid. For any $x_1, x_2, \dots, x_n \in \mathcal{A}$ and for any $\varepsilon \in I^n$ define the element

$$x_1 x_2 \dots x_n \varepsilon \in \mathcal{A} \tag{$\star$}$$

by induction on $n \geqslant 1$ in the following way:

1. $x_1 e = x_1$,
2. $x_1 x_2 \dots x_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}, 0) = x_1 \vdash x_2 \dots x_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2})$,
   $x_1 \dots x_{n-1} x_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}, 1) = x_1 \dots x_{n-1}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}) \dashv x_n$, if $n > 1$.

In particular, if $\varepsilon = (\overbrace{1, 1, \dots, 1}^{n-1})$, then $x_1 \dots x_n \varepsilon = x_1 \dashv \dots \dashv x_n$; if $\varepsilon = (\overbrace{0, 0, \dots, 0}^{n-1})$, then $x_1 \dots x_n \varepsilon = x_1 \vdash \dots \vdash x_n$; and if $n = 1$, then $x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) = x_1 e$ (according to above agreement).

Lemma 2. *In any $\mathfrak{g}$-dimonoid $(\mathcal{A}; \dashv, \vdash)$ the following identities hold:*

$$x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1})$$
$$= x_1 x_2 \dots x_n y_1 y_2 \dots y_m(\varepsilon_1, \dots, \varepsilon_{n-1}, \overbrace{1, 1, \dots, 1}^{m}),$$

$$x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \vdash y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1})$$
$$= x_1 x_2 \dots x_n y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1}, \overbrace{0, 0, \dots, 0}^{n}).$$

Proof.

$$x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv y_1 y_2 \dots y_m(\theta_1, \dots, \theta_{m-1})$$
$$\overset{(i)}{=} x_1 x_2 \dots x_n(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv y_1 \dashv y_2 \dashv \dots \dashv y_m$$
$$= x_1 x_2 \dots x_n y_1 y_2 \dots y_m(\varepsilon_1, \dots, \varepsilon_{n-1}, \overbrace{1, 1, \dots, 1}^{m}),$$

$$
\begin{aligned}
x_1x_2\ldots x_n(\varepsilon_1,\ldots,\varepsilon_{n-1}) &\vdash y_1y_2\ldots y_m(\theta_1,\ldots,\theta_{m-1})\\
&\overset{(ii)}{=} x_1\vdash x_2\vdash\cdots\vdash x_n\vdash y_1y_2\ldots y_m(\theta_1,\ldots,\theta_{m-1})\\
&= x_1x_2\ldots x_ny_1y_2\ldots y_m(\theta_1,\ldots,\theta_{m-1},\overbrace{0,0,\ldots,0}^{n}).
\end{aligned}
$$
□

All terms of a given $\mathfrak{g}$-dimonoid can be described by the elements of the form ($\star$); namely, we can show that any term of a given $\mathfrak{g}$-dimonoid can be reduced to ($\star$).

Theorem 1. *Let $t=t(x_1,\ldots,x_n)$ be a term of a given $\mathfrak{g}$-dimonoid. Then there is such $\varepsilon\in I^n$ that*

$$t=x_1x_2\ldots x_n\varepsilon.$$

Proof. Prove the theorem by induction on n. For $n=1,2$, the statement is obvious. Let it be true for $n<k$, where $k>2$. Suppose $t=t_1(x_1,\ldots,x_{k_1})*t_2(x_{k_1+1},\ldots,x_k)$, where $*\in\{\dashv,\vdash\}$, $0<k_1<k$.

Since $k_1<k, k-k_1<k$, then for the terms $t_1(x_1,\ldots,x_{k_1})$ and $t_2(x_{k_1+1},\ldots,x_k)$ there are such $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_{k_1-1})$, $\theta=(\theta_1,\ldots,\theta_{k-k_1-1})$ that

$$
\begin{aligned}
t_1(x_1,\ldots,x_{k_1}) &= x_1x_2\ldots x_{k_1}\varepsilon,\\
t_2(x_{k_1+1},\ldots,x_k) &= x_{k_1+1}x_{k_1+2}\ldots x_k\theta.
\end{aligned}
$$

If $*=\dashv$, then

$$
\begin{aligned}
t &= t_1(x_1,\ldots,x_{k_1})\dashv t_2(x_{k_1+1},\ldots,x_k)\\
&\overset{(i)}{=} t_1(x_1,\ldots,x_{k_1})\dashv x_{k_1+1}\dashv\cdots\dashv x_k\\
&= x_1\ldots x_{k_1}(\varepsilon_1,\ldots,\varepsilon_{k_1-1})\dashv x_{k_1+1}\dashv\cdots\dashv x_k\\
&\overset{2.}{=} x_1\ldots x_k(\varepsilon_1,\ldots,\varepsilon_{k_1-1},\overbrace{1,1,\ldots,1}^{k-k_1}).
\end{aligned}
$$

If $*=\vdash$, then

$$
\begin{aligned}
t &= t_1(x_1,\ldots,x_{k_1})\vdash t_2(x_{k_1+1},\ldots,x_k)\\
&\overset{(ii)}{=} x_1\vdash\ldots x_{k_1}\vdash t_2(x_{k_1+1},\ldots,x_k)\\
&= x_1\vdash\cdots\vdash x_{k_1}\vdash x_{k_1+1}\ldots x_k(\theta_1,\ldots,\theta_{k-k_1-1})\\
&\overset{2.}{=} x_1\ldots x_k(\theta_1,\ldots,\theta_{k-k_1-1},\overbrace{0,0,\ldots,0}^{k_1}).
\end{aligned}
$$

Therefore the theorem is valid for $n=k$. □

By virtue of Theorem 1, any term of a given $\mathfrak{g}$-dimonoid can be reduced to the form $(\star)$ which we call the canonical form of a given term. For example, for the term $((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6)$ the canonical form is:

$$\begin{aligned}((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) &\dashv (x_5 \vdash x_6)\\ &= (x_1x_2(1) \vdash x_3x_4(1)) \dashv x_5x_6(0)\\ &= x_1x_2x_3x_4(1,0,0) \dashv x_5x_6(0)\\ &= x_1x_2x_3x_4x_5x_6(1,0,0,1,1).\end{aligned}$$

Define operations $\dashv$ and $\vdash$ on I in the following way:

$$(\varepsilon_1,\dots,\varepsilon_{n-1}) \dashv (\theta_1,\dots,\theta_{m-1}) = (\varepsilon_1,\dots,\varepsilon_{n-1},\overbrace{1,1,\dots,1}^{m}),$$

$$(\varepsilon_1,\dots,\varepsilon_{n-1}) \vdash (\theta_1,\dots,\theta_{m-1}) = (\theta_1,\dots,\theta_{m-1},\overbrace{0,0,\dots,0}^{n}).$$

Lemma 3. *The algebra $(I;\dashv,\vdash)$ is a $\mathfrak{g}$-dimonoid.*

Proof. The axioms $(A1),(A2),(A3),(A4)$ are checked directly. □

Note that $(I;\dashv,\vdash)$ is not a dimonoid.

Lemma 4. *In the algebra $(I;\dashv,\vdash)$ we have*

$$\underbrace{ee\dots e}_{n}\varepsilon = \varepsilon$$

for any $\varepsilon \in I^n$.

Proof. Prove by induction on n. If $n = 1,2$, then the statement is clear. Let it be true for $n = k, k > 1$ and let $\varepsilon \in I^{k+1}$.

If $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{k-1},0) = e \vdash \varepsilon'$, where $\varepsilon' = (\varepsilon_1,\dots,\varepsilon_{k-1})$, then

$$\underbrace{ee\dots e}_{k+1}\varepsilon = e \vdash \underbrace{ee\dots e}_{k}\varepsilon' = e \vdash \varepsilon' = \varepsilon.$$

If $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{k-1},1) = \varepsilon' \dashv e$, where $\varepsilon' = (\varepsilon_1,\dots,\varepsilon_{k-1})$, then

$$\underbrace{ee\dots e}_{k+1}\varepsilon = \underbrace{ee\dots e}_{k}\varepsilon' \dashv e = \varepsilon' \dashv e = \varepsilon.$$ □

From definitions of operations $\dashv,\vdash$ it follows:

Lemma 5. *If* $\alpha \in I^n, \theta \in I^m$, *then* $\alpha \dashv \theta, \alpha \vdash \theta \in I^{n+m}$.

Now we can prove the uniqueness of the canonical form for the $\mathfrak{g}$-dimonoid $(I; \dashv, \vdash)$.

Theorem 2. *The canonical form is unique for any term of the* $\mathfrak{g}$*-dimonoid* $(I; \dashv, \vdash)$.

Proof. Assume that for some term t there are two canonical forms:

$$x_1 x_2 \ldots x_n \varepsilon = y_1 y_2 \ldots y_m \theta \tag{$\star\star$}$$

for some $\varepsilon \in I^n$ and $\theta \in I^m$.

Replacing all variables by $e \in I$, we get:

$$\underbrace{ee \ldots e}_{n} \varepsilon = \underbrace{ee \ldots e}_{m} \theta,$$

whence and from Lemma 4 it follows $\varepsilon = \theta$. Hence, $(\star\star)$ has the following form:

$$x_1 x_2 \ldots x_n \varepsilon = y_1 y_2 \ldots y_n \varepsilon.$$

Let the variables x_k and y_k be different for some $1 \leqslant k \leqslant n$. In the last equality, replacing all variables except y_k by e and replacing the variable y_k by (1) we get:

$$\varepsilon = e \ldots e(1)e \ldots e\varepsilon,$$

which is contradiction, because $\varepsilon \in I^n$ and $e \ldots e(1)e \ldots e\varepsilon \in I^{n+1}$ due to Lemma 5. Therefore, the canonical forms $x_1 x_2 \ldots x_n \varepsilon$ and $y_1 y_2 \ldots y_m \theta$ graphically coincide, which proves the uniqueness of the canonical form of the term t. □

2. Free $\mathfrak{g}$-dimonoids

Let us turn to the construction of a free $\mathfrak{g}$-dimonoid. Let X be an arbitrary and nonempty set. Denote:

$$Y_n = X^n \times I^n, \qquad n \in \mathcal{N},$$

where $X^n = \underbrace{X \times X \times \ldots \times X}_{n} = \{(x_1, x_2, \ldots, x_n) : x_k \in X, k = \overline{1, n}\}$,

$$\mathcal{G}(X) = \bigcup_{n \geqslant 1} Y_n.$$

For convenience the elements of $\mathcal{G}(X)$ are denoted by $(x_1, x_2, \ldots, x_n)\varepsilon$ instead of $((x_1, x_2, \ldots, x_n), \varepsilon)$, where $\varepsilon \in I^n$; we consider the sets $X \times I^1$ and X being the same, that is, we identify the symbol $x \in X$ with the element $xe \in \mathcal{G}(X)$. Define operations $\dashv, \vdash$ on $\mathcal{G}(X)$ in the following way:

$$(x_1, x_2, \ldots, x_k)\varepsilon \dashv (x_{k+1}, x_{k+2}, \ldots, x_l)\theta = (x_1, x_2, \ldots, x_l)(\varepsilon \dashv \theta),$$

$$(x_1, x_2, \ldots, x_k)\varepsilon \vdash (x_{k+1}, x_{k+2}, \ldots, x_l)\theta = (x_1, x_2, \ldots, x_l)(\varepsilon \vdash \theta).$$

Theorem 3. *The binary algebra $(\mathcal{G}(X); \dashv, \vdash)$ is a free $\mathfrak{g}$-dimonoid with the system of free generators X.*

Proof. The fact that the algebra $\mathcal{G}(X)$ is a $\mathfrak{g}$-dimonoid follows from Lemma 3. From the definition of operations $\dashv, \vdash$ it follows that $\mathcal{G}(X)$ is generated by X. Namely, if $(x_1, \ldots, x_n)\varepsilon \in \mathcal{G}(X)$, where $\varepsilon = (\varepsilon', 1) = \varepsilon' \dashv e$, then

$$\begin{aligned}(x_1, \ldots, x_n)\varepsilon &= (x_1, \ldots, x_n)(\varepsilon' \dashv e) \\ &= (x_1, \ldots, x_{n-1})\varepsilon' \dashv x_n e = (x_1, \ldots, x_{n-1})\varepsilon' \dashv x_n.\end{aligned}$$

Analogously, if $\varepsilon = (\varepsilon', 0) = e \vdash \varepsilon'$, then

$$\begin{aligned}(x_1, \ldots, x_n)\varepsilon &= (x_1, \ldots, x_n)(e \vdash \varepsilon') \\ &= x_1 e \vdash (x_2, \ldots, x_n)\varepsilon' = x_1 \vdash (x_2, \ldots, x_n)\varepsilon'.\end{aligned}$$

Hence, using induction, we can prove that any element $(x_1, \ldots, x_n)\varepsilon \in \mathcal{G}(X)$ can be written as a word in the alphabet $x_1, \ldots, x_n$ and $\dashv, \vdash$.

Let us prove that it is a free $\mathfrak{g}$-dimonoid. Let $(\mathcal{D}; \dashv, \vdash)$ be an arbitrary $\mathfrak{g}$-dimonoid and $\varphi : X \to \mathcal{D}$ be an arbitrary map. Define the map $\psi_0 : (\mathcal{G}(X); \dashv, \vdash) \to (\mathcal{D}; \dashv, \vdash)$ in the following way:

$$\psi_0((x_1, x_2, \ldots, x_n)\varepsilon) = \varphi(x_1)\varphi(x_2) \ldots \varphi(x_n)\varepsilon.$$

The map ψ_0 matches with φ on X:

$$\psi_0(x) = \psi_0(xe) = \varphi(x)e = \varphi(x), \ x \in X.$$

Show that ψ_0 is a homomorphism. From Lemmas 2 and 3 it follows:

$$\begin{aligned}\psi_0((x_1,&x_2, \ldots, x_k)\varepsilon \dashv (x_{k+1}, x_{k+2}, \ldots, x_l)\theta) \\ &= \psi_0((x_1, x_2, \ldots, x_l)(\varepsilon \dashv \theta)) = \varphi(x_1)\varphi(x_2) \ldots \varphi(x_l)(\varepsilon \dashv \theta) \\ &= \varphi(x_1)\varphi(x_2) \ldots \varphi(x_k)\varepsilon \dashv \varphi(x_{k+1})\varphi(x_{k+2}) \ldots \varphi(x_l)\theta \\ &= \psi_0((x_1, x_2, \ldots, x_k)\varepsilon) \dashv \psi_0((x_{k+1}, x_{k+2}, \ldots, x_l)\theta),\end{aligned}$$

$$
\begin{aligned}
\psi_0((x_1,&x_2,\dots,x_k)\varepsilon \vdash (x_{k+1},x_{k+2},\dots,x_l)\theta)\\
&= \psi_0((x_1,x_2,\dots,x_l)(\varepsilon\vdash\theta)) = \varphi(x_1)\varphi(x_2)\dots\varphi(x_l)(\varepsilon\vdash\theta)\\
&= \varphi(x_1)\varphi(x_2)\dots\varphi(x_k)\varepsilon \vdash \varphi(x_{k+1})\varphi(x_{k+2})\dots\varphi(x_l)\theta\\
&= \psi_0((x_1,x_2,\dots,x_k)\varepsilon) \vdash \psi_0((x_{k+1},x_{k+2},\dots,x_l)\theta).
\end{aligned}
$$

Prove that if $\psi:(\mathcal{G}(X);\dashv,\vdash)\to(\mathcal{D};\dashv,\vdash)$ is a homomorphism coinciding with φ on X, then $\psi\equiv\psi_0$.

We have that the maps ψ and ψ_0 match on Y_1. Let they match on the sets $Y_1,\dots,Y_n$. Then

$$
\begin{aligned}
\psi((x_1,&x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1},0))\\
&= \psi(x_1 \vdash (x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1}))\\
&= \psi(x_1) \vdash \psi((x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1}))\\
&= \psi_0(x_1) \vdash \psi_0((x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1}))\\
&= \psi_0(x_1 \vdash (x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1}))\\
&= \psi_0((x_1,x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1},0)),\\
\psi((x_1,&x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1},1))\\
&= \psi((x_1,x_2,\dots,x_n)(\varepsilon_1,\dots,\varepsilon_{n-1}) \dashv x_{n+1})\\
&= \psi((x_1,x_2,\dots,x_n)(\varepsilon_1,\dots,\varepsilon_{n-1})) \dashv \psi(x_{n+1})\\
&= \psi_0((x_1,x_2,\dots,x_n)(\varepsilon_1,\dots,\varepsilon_{n-1})) \dashv \psi_0(x_{n+1})\\
&= \psi_0((x_1,x_2,\dots,x_n)(\varepsilon_1,\dots,\varepsilon_{n-1}) \dashv x_{n+1})\\
&= \psi_0((x_1,x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_{n-1},1)).
\end{aligned}
$$

Hence

$$
\psi((x_1,x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_n)) = \psi_0((x_1,x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_n))
$$

for any $(x_1,x_2,\dots,x_{n+1})(\varepsilon_1,\dots,\varepsilon_n)\in Y_{n+1}$. So, the maps ψ and ψ_0 coincide on Y_{n+1}. Therefore, $\psi\equiv\psi_0$. $\square$

Let us give another description of a free $\mathfrak{g}$-dimonoid. Let $F[X]$ be the free semigroup with the system of free generators X. For any word $\omega\in F[X]$ we denote the length of ω by $|\omega|$. Define operations $\dashv,\vdash$ on the set

$$
FG = \{(\omega,\varepsilon) : \omega\in F[X], \varepsilon\in I^{|\omega|}\}
$$

in the following way:

$$
(\omega_1,\varepsilon)\dashv(\omega_2,\theta) = (\omega_1\omega_2,\varepsilon\dashv\theta),
$$

$$(\omega_1, \varepsilon) \vdash (\omega_2, \theta) = (\omega_1\omega_2, \varepsilon \vdash \theta),$$

where $(\omega_1, \varepsilon), (\omega_2, \theta) \in FG$. It is easy to verify that the binary algebra $(FG; \dashv, \vdash)$ is a $\mathfrak{g}$-dimonoid, which we denote by $FG[X]$.

Theorem 4. *The $\mathfrak{g}$-dimonoids $(\mathcal{G}(X); \dashv, \vdash)$ and $FG[X]$ are isomorphic.*

Proof. Define the map $\sigma : \mathcal{G}(X) \to FG[X]$ in the following way:

$$\sigma : (x_1, x_2, \dots, x_k)\varepsilon \mapsto (x_1x_2 \dots x_k, \varepsilon),\ (x_1, x_2, \dots, x_k)\varepsilon \in \mathcal{G}(X).$$

From the definition it follows that σ is a bijection and a homomorphism. □

Hence, the binary algebra $FG[X]$ is also a free $\mathfrak{g}$-dimonoid with the system of free generators X.

Lemma 6. *The $\mathfrak{g}$-dimonoid $(I; \dashv, \vdash)$ is a free $\mathfrak{g}$-dimonoid which is isomorphic to the $\mathfrak{g}$-dimonoid $FG[X]$, where $|X| = 1$.*

Proof. Let $X = \{a\}$. Define the map

$$\tau : (I; \dashv, \vdash) \to FG[X]$$

in the following way: $\tau(\varepsilon) = (a^n, \varepsilon) \in FG[X]$ for all $\varepsilon \in I^n$, $n \geqslant 1$. From the definition it follows that the map τ is a bijection. Prove that it is a homomorphism. Indeed, by Lemma 5, $\varepsilon \dashv \theta, \varepsilon \vdash \theta \in I^{n+m}$ for any $\varepsilon \in I^n$, $\theta \in I^m$, hence

$$\tau(\varepsilon \dashv \theta) = (a^{n+m}, \varepsilon \dashv \theta) = (a^n, \varepsilon) \dashv (a^m, \theta) = \tau(\varepsilon) \dashv \tau(\theta),$$

$$\tau(\varepsilon \vdash \theta) = (a^{n+m}, \varepsilon \vdash \theta) = (a^n, \varepsilon) \vdash (a^m, \theta) = \tau(\varepsilon) \vdash \tau(\theta).$$

Therefore, τ is an isomorphism. □

Thus, the free $\mathfrak{g}$-dimonoid of rank 1 coincides with the $\mathfrak{g}$-dimonoid $(I; \dashv, \vdash)$ up to isomorphism.

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