Algebra and Discrete Mathematics
Volume **19** (2015). Number 1, pp. 77–86 RESEARCH ARTICLE


# On Kaluzhnin-Krasner's embedding of groups

**A. Yu. Olshanskii**[1]

Abstract. In this note, we consider a 'thrifty' version of Kaluzhnin-Krasner's embedding in wreath products and apply it to extensions by finite groups and to metabelian groups.

## 1. Introduction

This note goes back to the pioneer paper of L. Kaluzhnin and M. Krasner [6], where wreath products of groups were introduced and studied. Later many other group theorists applied wreath products to construct various counter-examples and to prove embedding theorems, and now wreath products are among the main tools of Group Theory. Here I pay attention to a feature of Kaluzhnin-Krasner's works, that probably has not been used in subsequent research papers. Herewith I consider only standard wreath products of abstract groups (i.e., in terms of [6], of permutation groups with regular actions).

Let $A$ and $B$ be groups and $F$ a group of all functions $f : B \to A$ with multiplication $(f_1 f_2)(x) = f_1(x)f_2(x)$ for $x \in B$. The group $B$ acts on $F$ from the right by shift automorphisms: $(f \circ b)(x) = f(xb^{-1})$ for all $f \in F$, $b, x \in B$, and the associated with this action semidirect product $B \ltimes F$ is called the (complete) *wreath product* of the groups $A$ and $B$, denoted by $A \operatorname{Wr} B$. Thus, every element of $A \operatorname{Wr} B$ has a unique presentation as $bf$ ($b \in B, f \in F$) and the multiplication rule follows from the conjugation formula

$$(b^{-1}fb)(x) = f(xb^{-1}) \tag{1}$$

in $A \operatorname{Wr} B$ for any $b, x \in B$ and $f \in F$.

[1]The author was supported in part by the NSF grants DMS-1161294 and by the RFBR grant 15-01-05823.
**2010 MSC:** 20E22, 20F16, 20E07.
**Key words and phrases:** wreath product, embedding of group, metabelian group.

Observe that any homomorphism $A \to \bar{A}$ induces the homomorphism $A \operatorname{Wr} B \to \bar{A} \operatorname{Wr} B$ by the rule $bf \mapsto b\bar{f}$, where $\bar{f} \in \bar{F}$ is obtained by replacing the values of $f$ by their images in $\bar{A}$.

Given an arbitrary group $G$ with a normal subgroup $A$, one has a canonical homomorphism $\pi$ of $G$ onto the factor group $G/A = B$. Let $b \mapsto b^s$ be any transversal $B \to G$, i.e. $\pi(b^s) = b$. Then the Kaluzhnin - Krasner monomorphism $\phi$ of the (abstract) group $G$ into $A \operatorname{Wr} B$ is given by the formula (see [5], [10])

$$\phi(g) = \pi(g) f_g, \ \ \textit{where} \ \ f_g(x) = (x\pi(g)^{-1})^s \ g \ (x^s)^{-1} \tag{2}$$

Applying $\pi$ to $(x\pi(g)^{-1})^s \ g \ (x^s)^{-1}$, one obtains 1, and so $f_g \in F$. To check that $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$, one just exploits the formulas (1,2). Finally, $\phi$ is injective since obviously we have $\ker\phi \leqslant A$, and by (2), $f_g(x) = x^s g(x^s)^{-1} \neq 1$ if $g \in A\backslash 1$.

The above-defined form of the Kaluzhnin - Krasner embedding $\phi$ is well known, and, up to conjugation, the image $\phi(G)$ does not depend on the transversal $s$. However, the original paper [6] suggested a stronger form of such an embedding. Namely, assume now that a subgroup $C$ is normal in the normal subgroup $A$ of $G$, but $C$ does not contain nontrivial normal subgroups of $G$. Then consider the product $\kappa$ of $\phi$ and the homomorphism $A \operatorname{Wr} B \to \bar{A} \operatorname{Wr} B$ induces by the canonical homomorphism $A \to \bar{A} = A/C$ with $a \mapsto \bar{a}$ for $a \in A$. Thus

$$\kappa(g) = \pi(g) \overline{f}_g, \ \ \textit{where} \ \ \overline{f}_g(x) = \overline{(x\pi(g)^{-1})^s \ g \ (x^s)^{-1}} \tag{3}$$

If $g \in \ker\kappa$, then $\pi(g) = 1$ and so $g \in A$. Then formula (3) shows that the values $\overline{x^s g(x^s)^{-1}}$ are trivial in $A/C$ for any $x \in B$, i.e. $x^s g(x^s)^{-1} \in C$. Taking $x = 1$, we have $g \in C$ since $C$ is normal in $A$. Thus, $\ker\kappa \leqslant C$, which implies $\ker\kappa = 1$ because $C$ contains no nontrivial normal in $G$ subgroups.

Therefore $\kappa$ is also an embedding of $G$, and the following is a slightly modified version of the old Kaluzhnin - Krasner statement:

**Proposition 1.1.** *Let $G \rhd A \rhd C$ be a subnormal series of a group $G$ with factors $\bar{A} = A/C$ and $B = G/A$, and let $C$ contain no non-trivial normal subgroups of $G$. Then there is an isomorphic embedding $\kappa$ of the group $G$ in the wreath product $\bar{A} \operatorname{Wr} B = B \ltimes \bar{F}$, the embedding $\kappa$ can be defined by the formula (3), so $\kappa(A) \leqslant \bar{F}$ and $\kappa(G)\bar{F} = \bar{A} \operatorname{Wr} B$.*

The group $\bar{A}$ can be much smaller than $A$, and making use of this below, we

- apply the embedding $\kappa$ to wreath products with finite active group $B$,
- embed finitely generated metabelian groups into $\bar{A}\,\mathrm{Wr}\,B$ with 'small' abelian $\bar{A}$ and $B$, and
- observe that $(\mathbb{Z}/p\mathbb{Z})\,\mathrm{Wr}\,\mathbb{Z}$ and $\mathbb{Z}\,\mathrm{Wr}\,\mathbb{Z}$ contain $2^{\aleph_0}$ non-isomorphic locally polycyclic subgroups.

## 2. Splittings of some group extensions

The first application gives a characterization of wreath products with finite active group. If the normal subgroup $A$ is abelian, then the statement of Theorem 2.1 follows from Proposition I.8.3 [7].

**Theorem 2.1.** *Assume that a normal subgroup $A$ of a group $G$ is a direct product $\times_{i=1}^{n} H_i$, where $\{H_i\}_{1\leqslant i\leqslant n}$ is the set of all conjugate to $H = H_1$ subgroups of $G$. Also assume that the normalizer $N_G(H)$ is equal to $A$. Then $A$ has a semidirect compliment $B$ in $G$ and $G = H\,\mathrm{Wr}\,B$.*

*Proof.* Note that $n$ is the index of the normalizer of $H$ in $G$, therefore the group $B = G/A$ has order $n$.

Define $C = C_1 = \prod_{i\neq 1} H_i$. The normal in $A$ subgroup $C$ is conjugate in $G$ to any $C_j = \prod_{i\neq j} H_i$, as it follows from the assumptions. Therefore $\cap_{j=1}^{n} C_j = 1$, and $C$ does not contain nontrivial normal in $G$ subgroups.

By Proposition 1.1, we have the embedding $\kappa$ of $G$ in the wreath product $(A/C)\,\mathrm{Wr}\,B$. If $g \in A$, then by formula (3), $\kappa(g) = \bar{f}_g$, where $\bar{f}_g(x) = \overline{x^s g (x^s)^{-1}}$. We may further assume that $x^s = 1$ for $x = 1$. Then for $g \in H$, every value $x^s g (x^s)^{-1}$ of $f_g$ belongs to some $H_i \leqslant C$ if $x \neq 1$ and $f_g(1) = g$. Therefore all the values of $\bar{f}_g$, except for one, are trivial, and $\kappa(H) = H(1)$, where $H(1)$ is the subgroup of functions $\bar{f}$ supported by $\{1\}$ only. Since $H(1) \simeq A/C \simeq H_1 = H$, the subgroup $H(1)$ can be identified with $H$.

The conjugacy class of the subgroup $H(1)$ in the wreath product $H\,\mathrm{Wr}\,B$ consists of $n$ subgroups $H(b) = b^{-1}H(1)b$, where $b \in B$. Therefore the set $\{H(b)\}_{b\in B}$ is the $\kappa$-image of the set $\{H_i\}_{i=1}^{n}$. Since $\bar{F} = \times_{b\in B} H(b)$, we have $\kappa(A) = \bar{F}$ and, by Proposition 1.1, $A\,\mathrm{Wr}\,B = \kappa(G)\bar{F} = \kappa(G)$. Hence $\kappa$ is an isomorphism, and the theorem is proved. □

The following example shows that one cannot remove the finiteness of the quotient $B = G/A$ from the assumption of Theorem 2.1.

**Example.** Let $G$ be a free metabelian group with two generators $a$ and $b$. The commutator subgroup $A = [G, G]$ is the normal closure of one commutator $[a,b]$. Since the subgroup $A$ is abelian, every subgroup conjugate to $H = \langle [a,b]\rangle$ is of the form $H_{ij} = c_{ij}^{-1}\langle [a,b]\rangle c_{ij}$, where $c_{ij} = a^i b^j$, $i,j \in \mathbb{Z}$. We explain below the known fact that $A$ is the direct product $\times_{i,j} H_{i,j}$. It follows that $N_G(H) = A$. However $G$ does not split over $A$ because by A. Shmelkin's [12] result (also see [10], Theorem 42.56) two independent modulo $A$ elements must generate free metabelian subgroup of $G$.

To check that the elements $d_{ij} = c_{ij}^{-1}\langle [a,b]\rangle c_{ij}$ are linearly independent over $\mathbb{Z}$, one can apply the homomorphism $\mu$ of $G$ (in fact, a version of Magnus's embedding [8,8]) into the metabelian group of upper triangle $2\times 2$ matrices over the group ring $\mathbb{Z}\langle x,y\rangle$ of a free abelian group of rank two given by the rule

$$\mu(a) = \mathrm{diag}(x,1), \quad \mu(b) = \mathrm{diag}(y,1) + E_{12},$$

where $E_{12}$ is the matrix with 1 at position $(1,2)$ and zeros everywhere else. The matrix multiplication shows that $\mu(d_{ij}) = I + x^{-i}y^{-j-1}(1-x^{-1})E_{12}$, where the elements $x^{-i}y^{-j-1}(1-x^{-1})$ of $\mathbb{Z}\langle x,y\rangle$ $(i,j\in\mathbb{Z})$ are linearly independent over $\mathbb{Z}$.

## 3. Embeddings of metabelian groups

There are finitely generated torsion free metabelian groups which are not embeddable in $W = \mathbb{Z}\,\mathrm{Wr}\,B$ with finitely generated abelian $B$. For example, the derived subgroup $[G,G]$ of the Baumslag - Solitar group $G = \langle a,b \mid b^{-1}ab = a^n\rangle$ is isomorphic to the additive group of rationals whose denominators divide some powers of $n$, denote it by $D_n$; but for $|n| \geqslant 2$, $[W,W]$ has no nontrivial elements divisible by all powers of $n$. (Moreover, it is easy to construct 2-generated metabelian groups $G$ with $[G,G]$ containing an infinite direct power of the group $D_n$ [9].) However the following is true.

**Theorem 3.1.** *Let $G$ be a finitely generated metabelian group with infinite abelianization $B = G/[G,G]$. Then for some $n = n(G) \geqslant 1$,*

(a) *$G$ embeds in the wreath product $(D_n \times \mathbb{Z}/n\mathbb{Z})\,\mathrm{Wr}\,B$.*

(b) *$G$ is isomorphic to a subgroup of $W = D_n\,\mathrm{Wr}\,B$ if the derived subgroup $[G,G]$ is torsion free;*

(c) *$G$ is isomorphic to a subgroup of $W = (\mathbb{Z}/n\mathbb{Z})\,\mathrm{Wr}\,B$, provided the derived subgroup $[G,G]$ is a torsion group.*

The proof is based on the following

**Lemma 3.2.** *Let $G$ be a finitely generated metabelian group and $A$ an abelian normal subgroup of $G$. Then there is a subgroup $C$ in $A$ containing no non-trivial normal in $G$ subgroups, such that for some $n \geqslant 1$,*

(a) *the factor group $A/C$ is isomorphic to a subgroup of a finite direct power of the group $D_n \times (\mathbb{Z}/n\mathbb{Z})$;*

(b) *$A/C$ is isomorphic to a subgroup of a finite direct power of $D_n$ if $A$ is torsion free;*

(c) *$A/C$ is isomorphic to a subgroup of a finite direct power of $\mathbb{Z}/n\mathbb{Z}$ if $A$ is a torsion group.*

*Proof.* **(b)** We denote by $R$ the maximal normal in $G$ subgroup of finite (torsion-free) rank contained in $A$. Since $G$ satisfies the maximum condition for normal subgroups [3], this 'radical' $R$ exists and contains all normal in $G$ subgroups of $A$ having finite rank. Moreover, the maximal torsion subgroup $T/R$ of $L = A/R$ is trivial since the subgroup $T$ is normal in $G$ and its rank is equal to the rank of $R$.

By Ph.Hall's theorem on normal subgroups in metabelian groups (see [4], lemmas 8 and 5.2), there is $n \geqslant 1$, and a basis $(a_1, a_2, \dots)$ of a free abelian subgroup $M \leqslant A$ such that $A/M$ is a torsion group having non-trivial $p$-subgroups for $p|n$ only, and so the group $A$ embeds in a direct product $D_n^1 \times D_n^2 \times \dots$ of the copies of $D_n$ (such that $a_i \mapsto D_n^i$).

We will use additive notation for $A$. If $R = A$, i.e. if the basis of $M$ is finite, the statement (b) holds for $C = 0$. So we assume further that $M$ has infinite rank.

Since $R$ has finite rank, it follows from the above embedding of $A$ into the countable direct power of $D_n$ that the group $A$ has a subgroup $K$ such that the intersection $R \cap K$ is trivial and the factor group $A/K$ is isomorphic to a subgroup of a finite direct power of $D_n$.

Let us enumerate non-trivial normal subgroups $N$ of $G$ of infinite rank, which are contained in $A$ and have torsion free factor groups $A/N$: $N_1, N_2, \dots$ (This set is countable since $G$ satisfies the maximum condition for normal subgroups.) Using this enumeration, we will transform the basis of $M$ as follows.

Let $a_{01} = a_1, a_{02} = a_2, \dots,$ and assume that the basis $(a_{i-1,1}, a_{i-1,2}, \dots)$ of $M$ is defined for some $i \geqslant 1$, and $a_{i-1,k} = a_k$ if $k$ is greater than $m(i-1)$, where $m(0) = 0$.

Since the subgroup $N_i$ has infinite rank, $N_i \cap M$ has a non-zero element $g_i = \sum_{j>m(i-1)} \lambda_j a_j$. Let $m(i)$ be the maximal subscript at non-zero coefficients $\lambda_j$ of this sum. We may assume that the greatest common divisor of the coefficients $\lambda_{m(i-1)+1}, \dots, \lambda_{m(i)}$ is 1 since the group $A/N_i$ is torsion free. Therefore there exists a basis $(a_{i,m(i-1)+1}, \dots, a_{i,m(i)})$ of the free abelian subgroup $\langle a_{m(i-1)+1}, \dots, a_{m(i)} \rangle$ with $a_{i,m(i-1)+1} = g_i$. The other elements of the $(i-1)$-th basis of $M$ are left unchanged, i.e. $a_{i,k} = a_{i-1,k}$ if $k \leqslant m(i-1)$ or $k > m(i)$.

Now we define $e_k = a_{i,k}$ if $m(i-1) < k \leqslant (m(i))$ for some $i$ and obtain a new basis of $(e_1, e_2, \dots)$ of $M$ because we see from the construction that $\langle e_1, \dots, e_{m(i)} \rangle = \langle a_1, \dots, a_{m(i)} \rangle$ for every $m(i) > 0$. Thus every subgroup $N_i$ contains an element $g_i = a_{i,m(i-1)+1} = e_{m(i-1)+1}$ from this new basis.

Since the group $A$ is torsion free, every element of $A$ has a unique finite presentation of the form $\sum_i \lambda_i e_i$ with rational coefficients (although not every such a rational combination belongs to $A$). Hence the subgroup $H$ of $A$ given by the equation $\sum_i \lambda_i = 0$ is well defined, and the factor group $A/H$ is torsion free and has rank 1. Note that $(M+H)/H \simeq M/(H \cap M)$ is infinite cyclic and $A/(M+H)$ is the factor group of the torsion group $A/M$. Therefore the group $A/H$ is isomorphic to a subgroup of $D_n$.

It follows from the definition of $H$ that for every $i$, the subgroup $H$ does not contain any non-zero multiple $mg_i$ of the element $g_i = e_{m(i-1)+1}$.

Finally we set $C = H \cap K$. Then $A/C$ is embeddable in a finite direct power of $D_n$ since both $A/H$ and $A/K$ are. Assume now that $C$ contains a nontrivial normal in $G$ subgroup $L$. Then $L$ has to have infinite rank since $C \cap R \leqslant K \cap R = 0$. Let $T/L$ be the torsion part of $A/L$. Then $T$ is a normal in $G$ subgroup with torsion free factor group $A/T$. Hence $T = N_i$ for some $i$. Therefore $L$ and $C$ must contain a non-zero multiple of the element $g_i \in N_i$. But $H$ does not contain such elements, and the statement (b) is proved by contradiction.

**(c)** The subgroup $A$ is a direct sum of its Sylow $p$-subgroups $A_p$. For every prime $p$, the elements $x$ of $A_p$ with $px = 0$ form a normal in $G$ subgroup $A(p)$. It follows from the maximum condition for normal subgroups of $G$ that there are only finitely many primes $p$ with nonzero $A(p)$. For the same reason, $A$ has a finite exponent $n$. Arguing as in the proof of (b), but replacing $A$ by $A(p)$ and taking the maximal set $(a_1, a_2, \dots)$ in $A(p)$ as there (but linearly independent over $\mathbb{Z}/p\mathbb{Z}$), we obtain a subgroup (and subspace) $C_p$ of finite codimension in $A(p)$, which does not contain any nonzero normal in $G$ subgroup.

Now consider a maximal in $A_p$ subgroup $E_p$ such that $E_p \cap A(p) = C_p$. Then every element $x + E_p$ of order $p$ from $A_p/E_p$ must belong to the canonical image $(A(p)+E_p)/E_p$ of the subgroup $A(p)$ in $A_p/E_p$ since otherwise $(\langle x+E_p\rangle/E_p)\cap((A(p)+E_p)/E_p)$ is trivial, and so $\langle x+E_p\rangle\cap A(p)\leqslant E_p$, contrary to the maximality of $E_p$. Since the subgroup $(A(p)+E_p)/E_p\simeq A(p)/C_p$ is finite, we have finitely many elements of order $p$ in the $p$-group $A_p/E_p$ of finite exponent dividing $n$. Hence $A_p/E_p$ is a finite $p$-group.

If $N$ is a non-trivial $p$-subgroup normal in $G$ and $N\leqslant E_p$, then $N\cap A(p)\leqslant E_p\cap A(p)=C_p$, where $N\cap A(p)$ is non-trivial and normal in $G$, contrary to the choice of $C_p$. Thus $E_p$ contains no such subgroups $N$.

Since $A_p$ is a direct summand of $A$, for every $E_p$, one can find a subgroup $F_p\leqslant A$ with $A_p\cap F_p=E_p$ and $A/F_p\simeq A_p/E_p$. If a normal in $G$ subgroup $N$ with nontrivial $p$-torsion were contained in $F_p$, then $A_p\cap N\leqslant A_p\cap F_p=E_p$, which would provide a contradiction.

Now the intersection $C=\cap_{p|n}F_p$ contains no nonzero subgroups of $A$, which are normal in $G$. Since every $A/F_p$ is a finite group of exponent dividing $n$, the group $A/C$ is embeddable in a finite direct power of the group $\mathbb{Z}/n\mathbb{Z}$, as desired.

**(a)** Let $T$ be the torsion subgroup of $A$. By the statement (b) applied to $G/T$, we have a subgroup $C'$ containing $T$ but containing no bigger normal in $G$ subgroups, and $A/C'$ is embeddable in a finite direct power of some $D_n$. Note that $C'$ contains no nontrivial torsion free subgroup $N$ normal in $G$ since $N+T>T$.

Since $T$ has a finite exponent $m$, it has a torsion free direct compliment $K$ in $A$ by Kulikov's theorem [11,14]. The intersection $S$ of the subgroups conjugated to $K$ in $G$ is torsion free, normal in $G$, and the exponent of $A/S$ is equal to the exponent of $A/K\simeq T$, i.e. it is $m$.

The statement (c) applied to $G/S$ provides us with a subgroup $C''$ containing $S$ but containing no bigger normal in $G$ subgroups, with $A/C''$ embeddable in a finite direct power of $\mathbb{Z}/m\mathbb{Z}$. Note that $C''$ contains no nontrivial torsion subgroup $N$ normal in $G$ since $N+S>S$.

It follows that $C=C'\cap C''$ contains no nontrivial normal in $G$ subgroups and $A/C$ is embeddable in a finite direct power of $D_n\times\mathbb{Z}/m\mathbb{Z}$. Since both $m$ and $n$ can be replaced by their common multiple, the lemma is proved. □

**Remark 3.3.** One cannot generalize Lemma 3.2 to the slightly larger class of central-by-metabelian groups. Indeed, every countable abelian group $A$ embeds as a central subgroup in some finitely generated central-by-metabelian group $G$ [3], and so every subgroup $C$ of $A$ becomes normal in $G$.

**Proof of Theorem 3.1. (b)** By Lemma 3.2 (b) and Proposition 1.1, for some $n, m \geqslant 1$, the group $G$ embeds in a wreath product $W = D \operatorname{Wr} B$, where $D$ is a direct product of $m$ copies of the group $D_n$. Since $B$ is infinite, it follows from the Fundamental Theorem of finitely generated abelian groups, that $B$ contains a subgroup of index $m$ isomorphic to $B$. In other words, $B$ is a subgroup of index $m$ in a group $B_0$ isomorphic to $B$. The group $W_0 = D_n \operatorname{Wr} B_0 = B_0 \ltimes F$ , where $F$ is the subgroup of functions $B_0 \to D_n$, contains the subgroup $W_1 = B \ltimes F$, and it remains to show that $W_1$ is isomorphic with $W$. This isomorphism is identical on $B$ and maps every function $f_0 : B_0 \to D_n$ to the function $f : B \to D$ given by the rule $f(b) = (f(t_1 b), \ldots, f(t_m b)) \in D$, where $\{t_1, \ldots, t_m\}$ is a transversal to the subgroup $B$ in $B_0$.

**(c,a)** One should argue as in the proof of (b) but with reference to the items (c) and (a) of Lemma 3.2 and with the group $D_n$ replaced by $\mathbb{Z}/n\mathbb{Z}$ and $D_n \times \mathbb{Z}/n\mathbb{Z}$, respectively. □

## 4. Subgroups of $(\mathbb{Z}/p\mathbb{Z})\ Wr\ \mathbb{Z}$

Let us fix a prime number $p$. For every prime $q \neq p$, there is a finite-dimensional faithful, irreducible presentation of $\mathbb{Z}/q\mathbb{Z}$ over the Galois field $\mathbb{F}_p$. Let $V_q$ be the corresponding representation module. Each of these representations lifts to a representation of an infinite cyclic group $\langle b \rangle$, so the direct sum $V = \oplus_{q \neq p} V_q$ is a $\mathbb{F}_p\langle b \rangle$-module. The action of $\langle b \rangle$ defines the semidirect product $G = \langle b \rangle \ltimes V$.

**Lemma 4.1.**

(1) *Every finitely generated subgroup of $G$ is finite-by-cyclic, i.e. $G$ is locally finite-by-cyclic.*

(2) *$G$ contains $2^{\aleph_0}$ non-isomorphic subgroups.*

(3) *There is a subgroup $C$ in $V$ such that $V/C$ has order $p$ and $C$ contains no nontrivial normal in $G$ subgroups.*

*Proof.* **(1)** It is easy to see that any finite subset of $G$ is contained in a subgroup $\langle b \rangle \ltimes U$, where $U$ is the direct sum of the subgroups $V_{q_i}$ over a finite subset of prime numbers $q_i$. Since $U$ is finite, the statement (1) follows.

**(2)** Denote by $H_S$ the subgroup $\langle b \rangle \ltimes V_S$, where $S$ is a set of prime numbers $q \neq p$ and $V_S = \oplus_{q \in S} V_q$. Since different $V_q$-s are irreducible and non-isomorphic $\mathbb{F}_p\langle b \rangle$-modules, every normal in $H_S$ finite subgroup of $V_S$

is $V_T = \oplus_{q\in T} V_q$ for a finite subset $T \subset S$. The centralizer of $V_T$ in $H_S$ has index $\prod_{q\in T} q$. It follows that the groups $H_{S_1}$ and $H_{S_2}$ are not isomorphic for $S_1 \neq S_2$, which implies the statement (2).

**(3)** Let form an $\mathbb{F}_p$-basis $(e_1, e_2, \dots)$ as the union of the bases of the subspaces $V_q$ and define the subspace $C$ by the equation $\sum_i x_i = 0$ in the coordinates. Then clearly $V/C$ has order $p$ and $C$ contains none of the summands $V_q$. Since every normal in $G$ subgroup of $V$ is a submodule of the direct sum of some non-isomorphic irreducible $\mathbb{F}_p\langle b\rangle$-modules $V_q$, it coincides with some $V_S$, and the statement is proved. □

The next theorem demonstrates that the structure of subgroups of wreath products of cyclic groups is quite rich.

**Theorem 4.2.** *The wreath product $\mathbb{Z}_p \,\mathrm{Wr}\, \mathbb{Z}$ contains $2^{\aleph_0}$ non-isomorphic subgroups $H$, where each $H$ is locally finite-by-cyclic.*

*Proof.* The group $G$ from Lemma 4.1 is embeddable in $\mathbb{Z}_p \,\mathrm{Wr}\, \mathbb{Z}$ by the property (3) of Lemma 4.1 and Proposition 1.1. Therefore Theorem 4.2 follows from the properties (1) and (2) of Lemma 4.1. □

**Remark 4.3.** Similar approach shows that the wreath product $\mathbb{Z} \,\mathrm{Wr}\, \mathbb{Z}$ contains $2^{\aleph_0}$ countable locally polycyclic, non-isomorphic subgroups. But now, instead of $V_q$, one should start with $\mathbb{Z}\langle b\rangle$-modules $V_i$, which are non-isomorphic, finite-dimensional and irreducible over $\mathbb{Q}$.

## References

[1] L.Fuchs, *Infinite Abelian Groups, volume II*, Academic Press, N-Y and London (1973).

[2] M. Hall Jr., *The theory of groups*, New York, MacMillan. Co., 1959, 13+434.

[3] P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc., (3) 4 (1954), 419–436.

[4] P. Hall, *On the finiteness of certain soluble groups*, Proc. London Math. Soc. (3) 9 (1959), 595–622.

[5] M.I. Kargapolov, Yu.I. Merzlyakov, *Fundamentals of group theory*, Nauka, Moscow (1982) (in Russian).

[6] M. Krasner, L. Kaloujnine, *Produit complete des groupes de permutations et le problème d'extension de groupes III*, Acta Sci. Math. Szeged 14 (1951), 69–82 (in French).

[7] Yu. Kuz'min, *Homological Group Theory*, Factorilal Publisher, Moscow (2006, in Russian).

[8] W.Magnus, *On a theorem of Marshall Hall*, Ann. Math., 40 (1939), 764-768.

[9] V.H. Mikaelian, A.Yu. Olshanskii, *On abelian subgroups of finitely generated metabelian groups*, Journal of Group Theory, 16 (2013), no.5, 695–705.

[10] H. Neumann, *Varieties of Groups*, Springer–Verlag, Berlin (1967).

[11] D. J. S. Robinson, *A Course in the Theory of Groups*, second edition, Springer-Verlag, New York, Berlin, Heidelberg (1996).

[12] A.L. Shmelkin, *Free polynilpotent groups*, Izvestiya Akad. Nauk SSSR, Ser. Mat. 28 (1964), 91-122 (in Russian).

## Contact information

**Alexander Yu. Olshanskii** Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA, and
Department of Mathematics and Mechanics, Moscow State University, Moscow 119991, Russia
*E-Mail(s)*: alexander.olshanskiy@vanderbilt.edu

Received by the editors: 03.02.2015
and in final form 25.02.2015.