Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **19** (2015). Number 1, pp. 8–18

Uncountably many 2-generated just-infinite branch pro-2 groups

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Abstract. The aim of this note is to prove that there are $2^{\aleph_0}$ non-isomorphic 2 generated just-infinite branch pro-2 groups.

1. Introduction

In 1937 B. Neumann [Neu37] proved that there are $2^{\aleph_0}$ non-isomorphic 2-generated abstract groups (see [dlH00, Chapter III]). In 1983 the second author suggested a construction of continuously many 2-generated torsion 2-groups M_ω of intermediate growth and showed that among these there are $2^{\aleph_0}$ distinct groups not only up to isomorphism but up to the weaker equivalence relation, quasi-isometry. One of the important features of the groups M_ω is the fact that they are branch just-infinite groups.

Recall that a group is just-infinite if it is infinite but every proper quotient is finite. Just-infinite groups lie on the border between finite and infinite groups. Every finitely generated infinite group can be mapped onto a just-infinite group. The class of branch groups was introduced by the second author in [Gri00]. These groups act on a spherically homogeneous rooted tree such that the rigid stabilizers of levels have finite index (see next section for definitions). The class of just-infinite groups naturally splits into three subclasses [Gri00]. One of these subclasses is the class of just-infinite branch groups. The other two classes are related to hereditarily just-infinite groups and simple groups.

Branch groups and just-infinite groups were also defined within the category of profinite groups where now a dichotomy holds: Every just-infinite profinite group either is of branch type or is related to hereditarily just-infinite groups [Wil00].

[1]The second author was supported by NSF grant DMS-1207699

A natural question is about the cardinality of the class of finitely generated profinite groups. More precisely, one can fix the number of generators to 2 and restrict the consideration to pro-p groups where p is a prime. Suprisingly, it seems that there are no results in literature showing that the cardianlity of this class is $2^{\aleph_0}$. This question was raised recently in private conservations of the second author with A. Lubotzky and D. Segal. As a result, D. Segal suggested an elegant construction of $2^{\aleph_0}$ 2-generator centre-by-metabelian pro-p groups [D. Segal, private communication]. A. Lubotzky asked an analoguous question for finitely presented profinite groups which was answered in [Sno15], where for each $m \geqslant 3$, $2^{\aleph_0}$ non-isomorphic finitely presented metabelian pro-p groups are constructed.

The main result of this paper is the following:

Main Theorem. *There exist $2^{\aleph_0}$ non-isomorphic 2 generated just-infinite branch pro-2 groups.*

Unlike the groups of [Sno15], our examples are not finitely presented (by arguments from [Ben12]), but have additional properties of being branch and just-infinite. It is not known if there are finitely presented branch groups. It would be interesting to know the cardinality of the class of finitely generated hereditarily just-infinite pro-p groups, that is, pro-p groups with all subgroups of finite index being just-infinite groups.

The proof of the main theorem is based on a number of previous results, in particular the results of L. Lavreniuk and V. Nekrashevych from [LN02].

Although we do not do this here, the same techniques and construction of p-groups of [Gri85] may be used to prove the same result for the class of pro-p groups, for any prime p.

2. Preliminaries

2.1. Profinite groups and profinite completions

We refer to [RZ10] for a detailed account on profinite groups. We recall some basic material related to profinite groups and completions.

Let C b a class of finite groups. An inverse limit of a (surjective) inverse system of groups in C is called a pro-C group. Equivalently, G is a pro-C group if it is compact, totally disconnected and for any open normal subgroup N, G/N lies in the class C. When the class C is the class of all finite groups (respectively, the class of all finite p-groups for a prime p), then we will get the class of profinite groups (respectively, pro-p groups).

Given a group G, let $N_C(G)$ be the collection of finite index normal subgroups N of G, for which G/N lies in the class C. The pro-C topology on G is the topology having the collection $\{N \in N_C(G)\}$ as a neighbourhood basis around the identity element in G. This is the coarsest topology on G making it a topological group for which the canonical maps $G \to G/N, N \in N_C(G)$ are continuous (here G/N has the discrete topology). Assume that $N_C(G)$ is filtered from below, that is for any $N_1, N_2 \in N_C(G)$ there is $N \in N_C(G)$ such that $N \leqslant N_1 \cap N_2$ (observe that this is true if C is the class of finite groups or the class of finite p-groups, for a prime p). In this case, $N_C(G)$ is a directed set with order $N_1 \preceq N_2$ if $N_2 \subset N_1$, and $\{G/N, P_{N,H}, N \preceq H \in N_C(G)\}$ is an inverse system where $P_{N,H} : G/H \to G/N$ are the conical homomorphisms. The inverse limit of this inverse system is called the *pro-C completion of G* and is denoted by $\widehat{G}_{N_C(G)}$. If C is the class of finite groups (resp. the class of finite p-groups) the profinite completion (resp. the pro-p completion) of G will be denoted by $\widehat{G}$ (resp. $\widehat{G}_p$). If G is residually C, that is $\bigcap_{n \in N_C(G)} N = 1$, then one has a continuous embedding $G \to \widehat{G}_{N_C(G)}$ with dense image.

Definition 1.

1) A group G is called just-infinite if every proper quotient of G is finite.
2) A profinite group G is called just-infinite if every proper quotient by a closed normal subgroup is finite.

Every finitely generated infinite group can be mapped onto a just-infinite group and every finitely generated pro-p group can be mapped onto a just-infinite pro-p group [Gri00]. By an important result N. Nikolov and D. Segal [NS07], in a finitely generated profinite group a subgroup is open if and only if has finite index. Therefore, if $\phi : G \to H$ is a homomorphism between profinite groups and G is finitely generated, then ϕ is continuous.

2.2. Groups of tree automorphisms and the congruence subgroup property

We refer to [Gri00, Nek05] for detailed accounts on groups acting on rooted trees. Let $X = \{0, 1, \ldots, d-1\}$ be a finite set with d elements and let X^* denote the set of finite words (sequences) over X. Elements of X^* are naturally in bijection with the vertices of a rooted regular d-ary tree in which the root is identified with the empty word and vertices at distance n from the root with words of length n. We will denote the length of an element $u \in X^*$ by $|u|$ and will say that u lies in level $|u|$ of the tree. We will not distinguish between X^* and the tree it describes.

The group $\mathrm{Aut}(X^*)$ consists of all graph automorphisms of X^*. Equivalently, $\mathrm{Aut}(X^*)$ consists of all bijections of X^* which fix the empty word and preserve prefixes, that is, if two words have common prefix of length n, so do their images. It follows that an automorphism must preserve the level of a vertex. We will say a subgroup $G \leqslant \mathrm{Aut}(X^*)$ is *level transitive* if it acts transitively on each level of the tree.

Given $g \in \mathrm{Aut}(X^*)$ and $u \in X^*$, the *section* of g at u is the automorphism g_u uniquely determined by

$$g(uv) = g(u)g_u(v) \text{ for all } v \in X^*$$

There is an isomorphism

$$\begin{aligned} \mathrm{Aut}(X^*) &\cong S_d \ltimes \mathrm{Aut}(X^*)^d \\ g &\mapsto (\pi_g; g_0, \dots, g_{d-1}) \end{aligned} \tag{1}$$

where S_d is the symmetric group on X and π_g is the permutation determined by the action of g onto X.

Let $X^{(n)}$ denote the set of words of length at most n in X^*. $X^{(n)}$ describes the finite rooted tree of vertices up to level n. For each n, we have a map $r_n : \mathrm{Aut}(X^{(n+1)}) \to \mathrm{Aut}(X^{(n)})$ given by restriction. It follows that $\mathrm{Aut}(X^*)$ is isomorphic to the inverse limit of the inverse system $\{\mathrm{Aut}(X^{(n)}), r_n\}$, and hence is a profinite group. Given distinct $g, h \in \mathrm{Aut}(X^*)$, define a metric (in fact an ultra metric) on $\mathrm{Aut}(X^*)$ as follows: $d(g,h) = 2^{-m(g,h)}$ where $m(g,h)$ is the maximum number n such that the actions of g and h on X^* agree up to level n. A straightforward argument shows that the topology generated by this metric is the same as the topology on $\mathrm{Aut}(X^*)$ as a profinite group. For $n \geqslant 1$, the n-th level stabilizer $St(n)$ is the subgroup fixing (point wise) every vertex on level n. Each level has finitely many vertices, from which it follows that for each n, $St(n)$ is of finite index and since $\bigcap_{n \geqslant 1} St(n) = 1$, $\mathrm{Aut}(X^*)$ (and hence any subgroup of it) is residually finite.

Definition 2. A subgroup $G \leqslant \mathrm{Aut}(X^*)$ is said to have the congruence subgroup property, if every finite index subgroup of G contains the subgroup $St_G(n)$ for some n.

If $G \leqslant \mathrm{Aut}(X^*)$ has the congruence subgroup property then $\widehat{G} \cong \overline{G}$ where the latter denotes the closure of G in $\mathrm{Aut}(X^*)$ (see [Gri00, Theorem 9] and [RZ10, Lemma 1.1.9]).

For $u \in X^*$ let us denote the subtree at u by uX^*. The *rigid stabilizer* of $G \leqslant \mathrm{Aut}(X^*)$ at u is the subgroup $\mathrm{Rist}_G(u)$ consisting of elements which act trivially outside of the sub-tree uX^*. The rigid stabilizer of level n is the subgroup $\mathrm{Rist}_G(n) = \langle \mathrm{Rist}_G(u) \mid |u| = n \rangle = \prod_{|u|=n} \mathrm{Rist}_G(u)$.

Definition 3. A level transitive subgroup $G \leqslant \mathrm{Aut}(X^*)$ is called a branch group (resp. weakly branch group), if for all n, the subgroup $\mathrm{Rist}_G(n)$ has finite index in G (resp. is nontrivial).

The class of branch groups plays an important role in the description of just-infinite groups and contains many groups with unsual properties. We refer to [Gri00] for a detailed account on branch groups. We also note that many branch profinite groups have a universal embedding property [GHZ00]. The following gives a criterion for a branch group to be just-infinite:

Theorem 1 ([Gri00, Theorem 4])**.** *A branch group G is just-infinite if and only if for each $u \in X^*$, $\mathrm{Rist}_G(u)$ has finite abelianization.*

2.3. Grigorchuk groups[2]

We recall a construction of groups from [Gri84].

Throughout this section we fix $X = \{0, 1\}$. Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$ be the set of all infinite sequences over $\{0, 1, 2\}$ and let $\sigma : \Omega \to \Omega$ be the shift given by $\sigma(\omega_1\omega_2 \ldots) = \omega_2\omega_3 \ldots$. For each $\omega \in \Omega$ we will define the automorphisms $b_\omega, c_\omega, d_\omega$ recursively as follows.

For $v \in X^*$

$$\begin{array}{llllll} b_\omega(0v) = & 0\beta(\omega_1)(v) & c_\omega(0v) = & 0\zeta(\omega_1)(v) & d_\omega(0v) = & 0\delta(\omega_1)(v) \\ b_\omega(1v) = & 1b_{\tau(\omega)}(v) & c_\omega(1v) = & 1c_{\sigma\omega}(v) & d_\omega(1v) = & 1d_{\sigma\omega}(v), \end{array}$$

where

$$\begin{array}{lll} \beta(0) = a & \beta(1) = a & \beta(2) = e \\ \zeta(0) = a & \zeta(1) = e & \zeta(2) = a \\ \delta(0) = e & \delta(1) = a & \delta(2) = a \end{array}$$

and e denotes the identity. Also define the following automorphism a:

$$a(0v) = 1v \text{ and } a(1v) = 0v$$

[2]The first author insists on this terminology which is standard.

Note that from the definition, the following relations are immediate:

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = e.$$

For $\omega = \omega_1\omega_2 \ldots \in \Omega$, let G_ω be the subgroup of $\mathrm{Aut}(X^*)$ generated by $a, b_\omega, c_\omega, d_\omega$. The isomorphism (1) restricts to an embedding $G_\omega \to S_2 \ltimes (G_{\sigma\omega} \times G_{\sigma\omega})$ for which

$$\begin{array}{rcll} a & \mapsto & (01) & (e,e) \\ b_\omega & \mapsto & & (\beta(\omega_1), b_{\sigma\omega}) \\ c_\omega & \mapsto & & (\zeta(\omega_1), c_{\sigma\omega}) \\ d_\omega & \mapsto & & (\delta(\omega_1), d_{\sigma\omega}) \end{array}$$

The groups $\{G_\omega, \omega \in \Omega\}$ have a plethora of interesting and unusual properties related to various notions such as growth and amenability. We will mention some properties which will be used in the proof of the main theorem.

Let $\Omega_\infty \subset \Omega$ be the subset consisting of sequences in which the symbols $0, 1, 2$ appear infinitely often.

Theorem 2. *For $\omega \in \Omega_\infty$, G_ω is a just-infinite 2-group which is branch and has the congruence subgroup property.*

Proof. The facts that G_ω is a 2-group and is just infinite are established in [Gri84]. The branch property is also implicitly proven and used in [Gri84] (this property was not defined at time of [Gri84]). Explicit proofs can be found in [Per02] or in [BGN15]. The congruence subgroup property is proven in [Per00]. □

It was shown already in [Gri84] that the family $\{G_\omega \mid \omega \in \Omega\}$ contains $2^{\aleph_0}$ non-isomorphic groups. The complete solution of isomorphism problem for the groups G_ω was given by V. Nekrashevych [Nek05] based on the results of [LN02].

Theorem 3 ([Nek05, Theorem 2.10.13])**.** *For $\omega, \eta \in \Omega_\infty$, G_ω is isomorphic to G_η if and only if ω is obtained from η by an application of a permutation $\pi \in Sym\{0, 1, 2\}$ coordinate wise.*

The ideas in the proof of this theorem will be used by us as well.

3. Proof of the main theorem

Let $L_\omega = \langle ab_\omega, d_\omega\rangle \leqslant G_\omega$. It follows from the relations $a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = e$ that L_ω is a normal subgroup of index 2 in G_ω.

Lemma 1. *Let $\omega \in \Omega, u \in X^*$ with $|u| = n$. Then, for all $g \in L_{\sigma^n\omega}$ there exists $h \in St_{L_\omega}(u)$ such that $h_u = g$.*

Proof. Induction on n.

Suppose $n = 1$, then if ω starts with 0 we have

$$\begin{array}{ccc} (ab_\omega)^2 & \mapsto & (b_{\sigma\omega}a, ab_{\sigma\omega}) \\ (b_\omega a)^2 & \mapsto & (ab_{\sigma\omega}, b_{\sigma\omega}a) \\ d_\omega & \mapsto & (1, d_{\sigma\omega}) \\ ad_\omega a & \mapsto & (d_{\sigma\omega}, 1) \end{array}$$

If ω starts with 1 we have

$$\begin{array}{ccc} (ab_\omega)^2 & \mapsto & (b_{\sigma\omega}a, ab_{\sigma\omega}) \\ (b_\omega a)^2 & \mapsto & (ab_{\sigma\omega}, b_{\sigma\omega}a) \\ d_\omega & \mapsto & (a, d_{\sigma\omega}) \\ ad_\omega a & \mapsto & (d_{\sigma\omega}, a) \end{array}$$

and if ω starts with 2 we have

$$\begin{array}{ccc} (ab_\omega)^2 & \mapsto & (b_{\sigma\omega}, b_{\sigma\omega}) \\ d_\omega & \mapsto & (a, d_{\sigma\omega}) \\ ad_\omega a & \mapsto & (d_{\sigma\omega}, a) \\ d_\omega(ab_\omega)^2 & \mapsto & (ab_{\sigma\omega}, c_{\sigma\omega}) \\ ad_\omega a(ab_\omega)^2 & \mapsto & (c_{\sigma\omega}, ab_{\sigma\omega}) \end{array}$$

which show that the claim is true for $n = 1$. Assume now $n > 1$ and let $u = vx$ for $x \in X$. By induction assumption, there is $k \in St_{L_{\sigma^{n-1}\omega}}(x)$ such that $k_x = g$ and again by induction assumption there is $h \in St_{L_\omega}(v)$ such that $h_v = k$. Then clearly $h \in St_{L_\omega}(u)$ and $h_u = h_{vx} = k_x = g$. $\square$

Lemma 2. *The action of L_ω is level transitive.*

Proof. Let $u, v \in X^*$ be of length n. By induction on n, if $n = 1$, then $ab_\omega(u) = v$. Suppose $n > 1$ and let $xu, yv \in X^*$ be of length n. If $x = y$, let $g \in L_{\sigma\omega}$ such that $g(u) = v$ (such g exists by the induction assumption). By the previous Lemma, there is $h \in St_{L_\omega}(x)$ such that $h_x = g$. Then $h(xu) = h(x)h_x(u) = xg(u) = xv$. If $x \neq y$, let $g \in L_{\sigma\omega}$ such that $g\left((ab_\omega)_x(u)\right) = v$. Again by the previous Lemma, there is $h \in St_{L_\omega}(y)$ such that $h_y = g$. Then $(hab_\omega)(xu) = h(y(ab_\omega)_x(u)) = yg((ab_\omega)_x(u)) = yv$. $\square$

Lemma 3. *If $\omega \in \Omega_\infty$ then L_ω is a branch group.*

Proof. Let $u \in X^*$ with $|u| = n$. Since L_ω has finite index in G_ω, $\mathrm{Rist}_{L_\omega}(u) = \mathrm{Rist}_{G_\omega}(u) \cap L_\omega$ has finite index in $\mathrm{Rist}_{G_\omega}(u)$. Therefore $\mathrm{Rist}_{L_\omega}(n) = \prod_{|u|=n} \mathrm{Rist}_{L_\omega}(u)$ has finite index in $\mathrm{Rist}_{G_\omega}(n) = \prod_{|u|=n} \mathrm{Rist}_{G_\omega}(u)$. Since G_ω is branch, $\mathrm{Rist}_{G_\omega}(n)$ has finite index in G_ω and hence $\mathrm{Rist}_{L_\omega}(n)$ has finite index in L_ω. □

Lemma 4. *If $\omega \in \Omega_\infty$, L_ω is just-infinite and has the congruence subgroup property.*

Proof. G_ω has the congruence subgroup property and hence L_ω (having finite index in G_ω) has this property. As mentioned in Theorem 1, a branch group G is just infinite if and only if each $\mathrm{Rist}_G(u)$ has finite abelianization and clearly this is satisfied in L_ω since it is a periodic (torsion) group. □

It follows that the profinite completions $\widehat{L_\omega}$ are isomorphic to the closures $\bar{L}_\omega$ in $\mathrm{Aut}(X^\star)$ and by [Gri00, Theorem 2 and Corollary on Page 150], the groups $\widehat{L_\omega}$ are just-infinite branch profinite (in fact pro-2) groups. We will show that given $\omega \in \Omega_\infty$, there is only one $\eta \in \Omega_\infty \setminus \{\omega\}$ such that $\widehat{L_\omega} \cong \widehat{L_\eta}$, by using rigidity results and ideas of [LN02, Nek05].

Definition 4. Let $G_1, G_2 \leqslant \mathrm{Aut}(X^*)$ be level transitive. An isomorphism $\phi : G_1 \to G_2$ is called saturated if there exists a sequence of subgroups $H_n \leqslant G_1$ such that:

1) $H_n \leqslant St_{G_1}(n)$ and $\phi(H_n) \leqslant St_{G_2}(n)$,
2) for all $v \in X^*$ with $|v| = n$, H_n and $\phi(H_n)$ act level transitively on the sub-tree vX^*.

Proposition 1 ([Nek05, Proposition 2.10.7])**.** *Let $G_1, G_2 \leqslant \mathrm{Aut}(X^*)$ be weakly branch groups and let $\phi : G_1 \to G_2$ be a saturated isomorphism. Then ϕ is induced by an automorphism of the tree X^*.*

For $\omega \in \Omega$ define inductively $L_{0,\omega} = L_\omega$ and $L_{n,\omega} = L^2_{n-1,\omega}$ for $n \geqslant 1$.

Lemma 5. *Let $\omega \in \Omega$ and $u \in X^*$ with $|u| = n$. Then for any $g \in L_{\sigma^n\omega}$ there exists $h \in St_{L_{n,\omega}}(u)$ such that $h_u = g$.*

Proof. Induction on n. Suppose $n = 1$ and ω starts with 0 (the other cases being similar). Let $r = (ab_\omega)^2$ and $s = (ab_\omega d_\omega)^2 = (ac_\omega)^2$. Note $r, s \in St_{L_{1,\omega}}(1)$. We have

$$\begin{array}{ccc} r & \mapsto & (b_{\sigma\omega}a, ab_{\sigma\omega}) \\ r^a & \mapsto & (ab_{\sigma\omega}, b_{\sigma\omega}a) \\ (rs^{-1})^{ab_\omega a} & \mapsto & (ad_{\sigma\omega}, d_{\sigma\omega}) \\ (rs^{-1})^{ab_\omega} & \mapsto & (d_{\sigma\omega}, ad_{\sigma\omega}) \end{array}$$

hence the claim is true for $n = 1$. Now suppose $n > 1$ and let $u = vx$ for $v \in X^*$ and $x \in X$. Let $g \in L_{\sigma^n\omega} = L_{\sigma\sigma^{n-1}\omega}$. By induction assumption, there exists $k \in St_{L_{1,\sigma^{n-1}\omega}}(x)$ such that $k_x = g$. Since $k \in L_{1,\sigma^{n-1}\omega}$, k is of the form

$$k = t_1^2 \cdots t_m^2 \text{ for some } t_i \in L_{\sigma^{n-1}\omega}$$

Again by the induction assumption, there exits $h_i \in St_{L_{n-1,\omega}}(v)$ such that $(h_i)_v = t_i$ for $i = 1, 2, \dots, m$. Since $h_i \in L_{n-1,\omega}$ we have $h_i^2 \in L_{n,\omega}$. Now

$$(h_1^2 \cdots h_m^2)_u = (h_1^2 \cdots h_m^2)_{vx} = ((h_1)_v^2 \cdots (h_m)_v^2)_x = (t_1^2 \cdots t_m^2)_x = k_x = g. \quad \square$$

Lemma 6. *For all $\omega \in \Omega$ and n,*

i) $L_{n,\omega} \leqslant St_{L_\omega}(n)$

ii) *For all $v \in X^*$ with $|v| = n$, $L_{n,\omega}$ acts level transitively on the sub-tree vX^*.*

Proof. The first assertion follows from a straightforward induction. For the second assertion, let $vu_1, vu_2 \in vX^*$ where $|u_1| = |u_2|$. Since $L_{\sigma^n\omega}$ acts level transitively, there is $g \in L_{\sigma^n\omega}$ such that $g(u_1) = u_2$. By Lemma 5 there exists $h \in St_{L_{n,\omega}}(v)$ such that $h_v = g$. Hence $h(vu_1) = vu_2$. $\square$

Theorem 4. *Let $\omega = \omega_1\omega_2\dots$ and $\eta = \eta_1\eta_2\dots$ be two sequences in Ω_∞. Then the groups $\widehat{L_\omega}$ and $\widehat{L_\eta}$ are isomorphic if and only if $\omega = \eta$ or $\omega_i = \pi(\eta_i)$ for all i, where $\pi = (12) \in Sym\{0, 1, 2\}$.*

Proof. If $\omega_i = \pi(\eta_i)$ then by the definition of the groups, $d_\omega = d_\eta$ and $b_\omega = c_\eta = b_\eta d_\eta$ and hence $ab_\omega = ab_\eta d_\eta \in L_\eta$. Therefore, in this case $L_\omega = L_\eta$ as subgroups of $\mathrm{Aut}(X^*)$.

Now suppose that $\phi : \widehat{L_\omega} \to \widehat{L_\eta}$ is an isomorphism. As mentioned above, we have $\widehat{L_\omega} \cong \overline{L}_\omega$ for any $\omega \in \Omega_\infty$ and each $\overline{L}_\omega$ is a branch group. Let $H_0 = \overline{L}_\omega$ and $H_n = \overline{H_{n-1}^2}$ for $n \geqslant 1$ and $K_0 = \overline{L}_\eta$ and $K_n = \overline{K_{n-1}^2}$

for $n \geqslant 1$. By Lemma 6 and continuity of ϕ we see that $\phi(H_n) = K_n$ and H_n satisfies the conditions of Definition 4 and hence ϕ is a saturated isomorphism. It follows from Proposition 1 that ϕ is induced by an automorphism and hence $\bar{L}_\omega$ and $\bar{L}_\eta$ are conjugate in $\mathrm{Aut}(X^*)$.

Given a vertex $v \in X^*$ and $g \in \mathrm{Aut}(X^*)$, we say that g is active at v if g_v acts non-trivially on the first the level. Consider the function $p : \mathrm{Aut}(X^*) \to \mathbb{Z}_2^{\mathbb{N}}$ where $p(g)_n$ is the number of active vertices for g on level n modulo 2. It is straightforward to check that p is a continuous group homomorphism (in fact one can check that this gives the abelianization). It follows that $p(\overline{L}_\omega) = p(\overline{L}_\eta)$. But by continuity we have $p(\overline{L}_\omega) = \overline{p(L_\omega)} = p(L_\omega)$ and hence $p(L_\omega) = p(L_\eta)$. It remains to observe that one can reconstruct the sequence ω from $p(L_\omega)$ up to the permutation π.

We have

$$p(ab_\omega) = (1, \beta(\omega_1), \beta(\omega_2), \dots)$$

$$p(d_\omega) = (0, \delta(\omega_1), \delta(\omega_2), \dots)$$

where $\beta(0) = \beta(1) = 1, \beta(2) = 0$ and $\delta(0) = 0, \delta(1) = \delta(2) = 1$. Note that non-trivial elements of $p(L_\omega)$ are $\{p(ab_\omega), p(d_\omega), p(ab_\omega) + p(d_\omega)\}$. Given such a set of three sequences, the only sequence whose first entry is 0 corresponds to $p(d_\omega)$ and one of the remaining ones corresponds to $p(ab_\omega)$. If $\delta(\omega_i) = 0$ then $\omega_i = 0$. If $\delta(\omega_i) = 1$ then depending on the choice of the sequence corresponding to $p(ab_\omega)$, we will have either $w_i = 1$ or $w_i = 2$. □

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Received by the editors: 19.03.2015
and in final form 19.03.2015.