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SURVEY ARTICLE

On various rank conditions in infinite groups

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To Professor V. V. Kirichenko on the occasion of his 65th birthday

Abstract. In the current survey the authors consider some of the main theorems concerning groups satisfying certain rank conditions. They present these theorems starting with recently established results. This order of exposition is different, indeed opposite to chronological, but it allows them to present the main development of the theory. They illustrate the connections between the different ranks emphasizing, in particular, the connection between the special rank and the Hirsch–Zaitsev rank.

1. Introduction

The concept of *rank* naturally evolved in group theory from the concept of the dimension of a vector space. The latter proved to be a very efficient instrument, thanks to which finite dimensional vector spaces form one of the most developed and perfectly shaped algebraic theories. The appearance of analogs of this notion in distinct branches of algebra is a very natural process. Thus, because the concept of a module is a generalization of the concept of a vector space, the analog of dimension, namely the notion of rank, was first introduced in module theory. The rank of a module A over a ring R formally has the same definition as the dimension of a module does, namely it is the size of any maximal R-independent system of elements of A. However, in contrast to vector spaces, this concept did not play a central role in module theory for several reasons. Firstly, the rank of a module does not always make sense in non-commutative rings since there exist non-commutative rings R that have finite maximal independent systems of elements of modules that have different numbers of elements. Secondly, this notion usually only makes sense for modules without R-torsion.

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Recall, that any abelian group is a module over the ring $\mathbb{Z}$ of integers. Initially, therefore, the notion of rank appeared in abelian group theory in the following two ways.

Let G be an abelian group. Observe, that the set of elements of finite order is easily seen to form a subgroup, which is usually called the *periodic part or torsion subgroup* of G, and which we denote by $Tor(G)$. In fact, this is a characteristic subgroup of G. The number of elements in a maximal independent subset consisting of elements of infinite order is called the 0*–rank* or *torsion–free rank* of a group G, and is denoted by $r_0(G)$. It is easy to see that $r_0(G) = r_0(G/Tor(G))$.

On the other hand, if we consider G as a $\mathbb{Z}$-module then $r_0(G)$ is exactly the $\mathbb{Z}$*–rank* of the $\mathbb{Z}$–module G and it is not hard to see that

$$r_0(G) = dim_Q(G \otimes_{\mathbb{Z}} \mathbb{Q}).$$

From this definition, an abelian group G has finite $0-$rank r if and only if $G/Tor(G)$ is isomorphic to a subgroup of the additive group

$$\underbrace{\mathbb{Q} \oplus ... \oplus \mathbb{Q}}_{r}.$$

Following A.I. Maltsev [MAI 1951], we say that an abelian group G is an *abelian* A_1*–group*, if $r_0(G)$ is finite.

If p is a prime, then the set of all elements of $p-$power order forms a characteristic subgroup of the abelian group G. We denote this subgroup by $Tor_p(G)$. This is the p*-component* of G and *the maximal* p-subgroup of G. We have $Tor(G) = \mathbf{Dr}_{p\in\Pi(G)} Tor_p(G)$. In the periodic case, this decomposition allows us to reduce many arguments to p-groups.

If P is an abelian p-group then the p*-rank* $r_p(P)$ of P is defined as follows. We let $\Omega_1(P) = \{a \in P \mid a^p = 1\}$, the *lower layer of* P. Then $\Omega_1(P)$ is an elementary abelian $p-$group. We can think of it as a vector space over the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and we define

$$r_p(P) = dim_{F_p} \Omega_1(P).$$

For an arbitrary abelian group G we set $r_p(G) = r_p(Tor_p(G))$. It is not hard to see that $r_p(G)$ is exactly the number of elements in a maximal independent subset consisting of elements of p-power order.

A *quasicyclic* p*-group* (sometimes called a *Prüfer* p*-group*) is an important example of an abelian group of $p-$rank equal to 1. In terms of generators and relations this group is

$$C_{p^\infty} = \left\langle a_n \mid a_1^p = 1, a_{n+1}^p = a_n, n \in \mathbb{N} \right\rangle.$$

This group can be thought of as the multiplicative group of complex p th roots of unity or, alternatively, as the set of elements of p-power order in the additive abelian group $\mathbb{Q}/\mathbb{Z}$, that is, $C_{p^\infty} = Tor_p(\mathbb{Q}/\mathbb{Z})$.

In general the Prüfer groups define the structure of an abelian p-group of finite p-rank. A group G is called a *Chernikov group* if G contains a normal subgroup D of finite index, which is itself a direct product of finitely many Prüfer p-groups. The subgroup D is a maximal divisible subgroup of P which we call *the divisible part* of P.

The following two results provide us with the description of abelian p-groups of finite p-rank.

1.1. Proposition. *Let P be an abelian $p-$group for some prime p. Then $r_p(P) = r$ is finite if and only if every elementary abelian $p-$section U/V of P is finite, $r_p(U/V) \le r$, and there is an elementary abelian section A/B of P such that $r_p(A/B) = r$.*

1.2. Proposition. *Let P be an abelian $p-$group for some prime p. Then $r_p(P)$ is finite if and only if P is a Chernikov group.*

Another key notion has been introduced by A.I. Maltsev. It is based on the following important property of dimension. If A is a vector space of finite dimension k and B is a subspace of A, then B is also finite dimensional of dimension at most k. Similarly, if G is an abelian group with k generators and B is a subgroup of G, then B is finitely generated and has at most k generators. However, there are non–abelian groups that do not possess this property. The next important concept of rank, generalizing these natural situations, appeared in connection with this.

Let G be a group. We say that G has *finite special rank* $r(G) = r$, if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property.

The notion of rank in this general form initially appeared in a paper of A. I. Maltsev [MAI1948]. In [BR1966] the special rank has also been called the *Prüfer rank*. The notion of finite special rank turns out to be one of the most important in infinite group theory. The investigations related to this notion significantly influenced infinite group theory and partially determined its future development. A large variety of papers containing interesting results have been dedicated to this topic. The current survey only reflects some of these results, which can naturally be split into two main parts: results reflecting the investigation of periodic (mainly locally finite) groups, and results concerned with groups having no normal periodic subgroups. The latter turn out to be connected with another important notion of rank which initially appeared in the class of polycyclic–by-finite groups.

We recall that a group G is said to be *polycyclic–by–finite* if it has a finite subnormal series whose factors are either finite or infinite cyclic.

The number of infinite cyclic factors in every such series is always the same, that is, it is an invariant of the group. K. A. Hirsch [HKA1938] first studied this invariant which was later named the *Hirsch number* of the polycyclic–by–finite group. For polycyclic–by–finite groups the Hirsch number plays approximately the same role as the dimension plays in vector spaces. Nowadays, the theory of polycyclic–by–finite groups is a very well developed algebraic theory, having interesting and mutually influential connections with ring and module theories (see, for example, [PDS1971, PDS1977, PDS1984, RJE1985, SD1983]).

This idea can be naturally extended to other classes of groups, for example to the class of soluble A_1-groups (in the sense of A.I. Maltsev [MAI1951]). These are the soluble groups containing a finite subnormal series, whose factors are abelian A_1-groups. Such groups have a finite subnormal series, whose factors are either periodic or infinite cyclic. However, no-one really seriously focused on this notion in the class of soluble A_1-groups until the paper of D. I. Zaitsev [ZDI1971A] appeared, initiating research on this topic in a certain subclass of soluble A_1-groups.

A group G is called *polyrational* if it has a finite subnormal series whose factors are torsion–free locally cyclic groups. Observe, that every torsion–free locally cyclic group is isomorphic to a subgroup of $\mathbb{Q}_+$, and for this reason a torsion–free locally cyclic group is also called *rational*. Every polyrational group possesses a finite subnormal series, whose factors either are periodic or infinite cyclic. D. I. Zaitsev [ZDI1971A] proved, that the number of infinite cyclic factors in every such series coincides with the special rank of the group. Later, in [ZDI1975], D. I. Zaitsev introduced the following class of groups G. A group G from this class contains a polycyclic–by–finite subgroup H with the property: for every finitely generated subgroup K of G such that $K \geq H$ the index $|K : H|$ is finite. Every group from this class also possesses a finite subnormal series, whose factors are either periodic or infinite cyclic.

Analogs of the Hirsch number turned out to be very useful in the investigation of groups with complemented systems of subgroups, groups with factorization, and in module theory as well (see, for example, [ZDI1980A, ZDI1980B, ZDI1981, ZKT1985, KTZ1991]). It also found natural applications in the study of groups satisfying the weak minimal and weak maximal conditions for normal subgroups (see [KLA1979, KLA1984, KLA1985, KLA1990A, KLA1990B]). Finally, in [DKP2007] the following generalization of this notion was considered.

A group G is said to have *finite Hirsch–Zaitsev rank* $r_{hz}(G) = r$ if G has an ascending series, whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factor is exactly r. Otherwise, we will say that G *has infinite* Hirsch–Zaitsev rank.

It is not hard to see that $r_{hz}(G)$ is an invariant of G.

In the current survey we want to consider some of the main theorems concerning groups satisfying certain rank conditions. We present these theorems starting with recently established results. This order of exposition is different, indeed opposite to chronological, but it allows us to present the main development of the theory. We would like to illustrate the connections between the different ranks emphasizing, in particular, the connection, from our point of view, between the special rank and the Hirsch–Zaitsev rank.

2. Groups of finite Hirsch–Zaitsev rank

One of the first results on groups of finite Hirsch–Zaitsev rank is the following key theorem due to A. I. Maltsev.

2.1. Theorem [MAI1951, Theorem 5]. *Let G be a torsion–free locally nilpotent group. Suppose that every abelian subgroup of G has a finite $0-$rank. Then G is a nilpotent group of finite Hirsch–Zaitsev rank. Moreover, if A is a maximal normal abelian subgroup in G and $r_0(A) = k$, then $r_{hz}(G) \leq \frac{k(k+1)}{2}$ and $ncl(G) \leq 2k$.*

Here we let $ncl(G)$ denote the *nilpotency class of G*.

2.2. Corollary. *Let G be a torsion–free locally nilpotent group. If G has finite Hirsch–Zaitsev rank, then G is nilpotent. In particular, G is polyrational.*

As we will see later, the polyrational groups play an important role in the class of groups of finite Hirsch–Zaitsev rank.

If G is an arbitrary group, we let $Tor(G)$ denote the largest normal periodic subgroup of G.

A group G is *generalized radical* if G has an ascending series, whose factors are locally nilpotent or locally finite. Hence, a generalized radical group G either contains an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $Lnr(G)$ of G is nontrivial. In the second case, it is not hard to see, that G contains a nontrivial normal locally finite subgroup. Clearly, in every group G, the subgroup generated by all normal locally finite subgroups is the largest normal locally finite subgroup (*the locally finite radical*). Thus, every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors.

We will consider the class of locally generalized radical groups. This class is quite large: it includes all locally radical (in particular, locally soluble) groups, all locally finite groups and all locally (soluble–by–finite) groups. On the other hand, all periodic locally generalized radical groups are locally finite.

Generalized radical groups are naturally connected to the study of groups with finite Hirsch-Zaitsev rank as the following result shows.

2.3. Proposition [DKP2007]. *Let G be a group. Then the following assertions are equivalent:*

(i) G has an ascending series whose factors are either infinite cyclic or locally finite and the number of infinite cyclic factors is exactly r;

(ii) G is a generalized radical group of finite Hirsch–Zaitsev rank r;

(iii) G is a locally generalized radical group of finite Hirsch–Zaitsev rank r.

The following fundamental result describes the structure of locally generalized radical groups of finite Hirsch–Zaitsev rank. We let $scl(G)$ denote the derived length of the soluble group G.

2.4. Theorem [DKP2007]. *Let G be a locally generalized radical group of finite Hirsch–Zaitsev rank. Then G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that T is locally finite, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, G/K is finite and S/K is soluble. Moreover, if $r_{hz}(G) = r$, then there are functions $f_1, f_2 : \mathbb{N} \longrightarrow \mathbb{N}$ such that $|G/K| \leq f_1(r)$ and $scl(S/T) \leq f_2(r)$.*

As a special case of this result we mention the following result which appears in the paper [FdeGK1995].

2.5. Corollary. *Let G be a locally (soluble–by–finite) group. If G has finite Hirsch–Zaitsev rank r, then G has the normal subgroups $T \leq L \leq K \leq S \leq G$ such that T is locally finite, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, G/K is finite and S/K is soluble. Moreover, $|G/K| \leq f_1(r)$ and $scl(S/T) \leq f_2(r)$.*

We also make the following observation.

2.6. Corollary. *Let G be a locally generalized radical group of finite Hirsch–Zaitsev rank. If G is not periodic, then $G/Tor(G)$ contains a normal polyrational subgroup of finite index.*

The following theorem on local properties of groups of finite Hirsch–Zaitsev rank allows us to obtain local versions of many results.

2.7. Theorem [DKP2007]. *Let G be a locally generalized radical group. Let r be a positive integer such that every finitely generated subgroup of G has finite Hirsch–Zaitsev rank at most r. Then G has Hirsch–Zaitsev rank at most r. In particular, G is a generalized radical group.*

We can obtain also some useful generalizations of this local theorem. For example, Theorem 2.8 is almost an immediate consequence of Theorems 2.7 and 2.4.

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is called *an almost $\mathfrak{X}$ –group* if G contains a normal subgroup $H \in \mathfrak{X}$ such that the index $|G : H|$ is finite. The class of all almost $\mathfrak{X}$ – group is denoted by $\mathfrak{X}\mathfrak{F}$.

In particular, a group is *almost locally soluble* if it has a locally soluble subgroup of finite index.

2.8. Theorem. *Let G be a group and suppose that G satisfies the following conditions:*

(i) for every finitely generated subgroup L of G the factor – group $L/Tor(L)$ is a generalized radical group;

(ii) there is a positive integer r such that $r_{hz}(L) \leq r$ for every finitely generated subgroup L.

Then $G/Tor(G)$ contains a normal soluble subgroup $D/Tor(G)$ of finite index. Moreover, G has finite Hirsch–Zaitsev rank r and there is a function $f_3 : \mathbb{N} \longrightarrow \mathbb{N}$ such that $|G/D| \leq f_3(r)$.

3. Groups of finite section p –rank

Let p be a prime. Propositions 1.1 and 1.2 allow us to extend the concept of the p-rank to an arbitrary group as follows. We say that a group G has *finite section p –rank $r_p(G) = r$* if every elementary abelian p-section U/V of G is finite of order at most p^r and *there is* an elementary abelian p-section A/B of G such that $|A/B| = p^r$. We use here the same notation $r_p(G)$ that we used in Section 1. This is not a reason for any misunderstanding since farther we will deal with the finite section $p-$rank only.

For locally finite p-groups we have the same picture that we observed for abelian p-groups.

3.1. Theorem. *Let P be a locally finite p–group for some prime p. Then P has finite section $p-$rank if and only if P is a Chernikov group.*

By analogy, we say that a group G has f*inite section $0-$rank $r_0(G) = r$*, if for every torsion–free abelian section U/V of G, $r_0(U/V) \leq r$ and *there is* an abelian torsion–free section A/B such that $r_0(A/B) = r$. For soluble groups these concepts were introduced by A. I. Maltsev [MAI1951] and D. J. S. Robinson [RD1968, 6.1]. We would like to underline the following relationship between the section p-rank and the section 0-rank.

3.2. Proposition. *Let G be a group and p be a prime. If G has finite section $p-$rank, then G has finite section $0-$rank. Moreover, $r_0(G) \leq r_p(G)$.*

Using Theorems 2.4 and 2.7 the following description of locally generalized radical groups of finite section 0-rank can be obtained.

3.3. Theorem [DKP2007]. *Let G be a locally generalized radical group of finite section $0-$rank r_0. Then $r_{hz}(G)$ is finite, moreover $r_{hz}(G) \leq \frac{r_0(r_0+3)}{2}$. In particular, G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that T is locally finite, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, G/K is finite and S/K is soluble.*

Moreover, there are functions f_4 and $f_5 : \mathbb{N} \longrightarrow \mathbb{N}$ such that $|G/K| \leq f_4(r_0)$ and $scl(S/T) \leq f_5(r_0)$.

We will start our discussion of groups of finite section p-rank, $p > 0$, with locally finite groups. By Theorem 1.16, every $p-$subgroup (and hence, maximal p-subgroup) of G is Chernikov so we are dealing with locally finite groups whose maximal $p-$subgroups are Chernikov, for the single prime p. We remark at once that in general the maximal p-subgroups of such groups need not be isomorphic. Locally finite groups with Chernikov maximal $p-$subgroups can have poorly behaved maximal $p-$subgroups. However, such groups always have well-behaved maximal p-subgroups in the following sense.

Let G be a locally finite group, and p be a prime. A maximal $p-$subgroup P of G is called a *Wehrfritz $p-$subgroup* [DMR1994, Definition 2.5.2] if P contains an isomorphic copy of every p-subgroup of G.

3.4. Theorem. *Let G be a locally finite group whose maximal $p-$subgroups are Chernikov for some prime p. Then G has Wehrfritz $p-$subgroups and every finite $p-$subgroup lies in at least one of them. Furthermore*

If P is a $p-$subgroup of G and Q is a Wehrfritz $p-$subgroup of G, then the following assertions are equivalent.

(i) P is a Wehrfritz $p-$subgroup of G.

(ii) P includes an isomorphic copy of every finite $p-$subgroup of G.

(iii) $P \cong Q$.

3.5. Corollary. *Let G be a locally finite group, and let p be a prime. Then G has finite section $p-$rank if and only if all maximal $p-$subgroups of G are Chernikov. In particular, G has finite section $p-$rank if and only if every elementary abelian $p-$section of G is finite.*

Using Theorem 3.3 and Proposition 3.2, we obtain

3.6. Theorem. *Let G be a locally generalized radical group of finite section $p-$rank r_p for some prime p. Then $r_{hz}(G)$ is finite. Moreover, $r_{hz}(G) \leq \frac{r_p(r_p+3)}{2}$. In particular, G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that T is a locally finite group, whose maximal p-subgroup are Chernikov, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, G/K is finite, and S/K is soluble.*

Moreover, $|G/K| \leq f_4(r_p)$ and $scl(S/T) \leq f_5(r_p)$.

The next result is very important for the description of locally soluble groups of finite section $p-$rank.

3.7. Theorem [KMI1961]. *Suppose G is a locally finite group of finite section $p-$rank for some prime p. Then $G/O_{p',p}(G)$ is finite if and only if every simple section of G containing elements of order p is finite.*

This result has a number of useful consequences which we now list.

3.8. Corollary. *Suppose G is a periodic locally soluble group of finite section $p-$rank for some prime p. Then $G/O_{p'}(G)$ is a Chernikov group.*

3.9. Corollary [CSN1960]. *Suppose G is a periodic locally soluble group. If the maximal $p-$subgroups of G are finite for a prime p, then $G/O_{p'}(G)$ is finite.*

3.10. Corollary [CSN1960]. *Suppose G is a periodic locally soluble group. If the maximal $p-$subgroups of G are finite for all prime p, then G is residually finite.*

By Kargapolov's theorem, for locally radical groups we can obtain the following significant specification of Theorem 3.6.

3.11. Corollary. *Let G be a locally radical group of finite section $p-$rank r_p for some prime p. Then G has normal subgroups $Q \leq T \leq L \leq K \leq G$ such that Q is a periodic locally soluble $p'-$subgroup, T/Q is a soluble Chernikov group, whose divisible part is a $p-$group, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, G/K is a finite soluble group such that $|G/K| \leq f_4(r_p)$ and $scl(G/T) \leq f_5(r_p)$. In particular, $r_{hz}(G)$ is finite. Moreover, $r_{hz}(G) \leq \frac{r_p(r_p+3)}{2}$.*

D. J. S. Robinson [RD1968, 6.1] defined the classes $\mathfrak{A}_0$ and $\mathfrak{S}_0$ as follows:

An abelian group A belongs to the class $\mathfrak{A}_0$ if and only if $r_0(A)$ is finite and $r_p(A)$ is finite for all primes p.

A soluble group G belongs to the class $\mathfrak{S}_0$ if and only if G has a finite subnormal series, every factor of which is an abelian $\mathfrak{A}_0$ –group.

Generalizing this, we say that a *group G has finite section rank* if $r_p(G)$ is finite for all primes p and also $p = 0$.

The study of groups of finite section rank splits into two parts: the study of the maximal normal periodic subgroup $Tor(G)$, and the study of the factor group $G/Tor(G)$. For the case when G is a locally generalized radical group, $Tor(G)$ is locally finite. We first consider the locally finite case for groups of finite section rank. Using Corollary 3.5, we obtain

3.12. Proposition. *Let G be a locally finite group. Then the following assertion are equivalent:*

(i) G has finite section rank;

(ii) the maximal $p-$subgroups of G are Chernikov for all prime p;

(iii) every elementary abelian $p-$section of G is finite for each prime p.

The following results show that the restriction that the maximal $p-$subgroups are *Chernikov* for all primes p is very strong. These results have been among the highlights of the theory of locally finite groups.

3.13. Theorem [BVV1981]. *Let G be a locally finite group and suppose that G has finite section rank. Then G is almost locally soluble.*

3.14. Corollary [SVP1971]. *Let G be a locally finite group and suppose that G has finite special rank. Then G is almost locally soluble.*

The following strong and important result is concerned with the structure of locally soluble groups of finite section rank.

3.15. Theorem [KMI1961]. *Suppose G is a locally soluble periodic group of finite section rank. Then G contains a normal divisible abelian subgroup R such that G/R is residually finite and the maximal $p-$subgroups of G/R are finite for each prime p.*

From this theorem and Theorem 3.13 we obtain

3.16. Corollary. *Suppose G is a locally finite group of finite section rank. Then G contains a normal divisible abelian subgroup R such that G/R is residually finite and the maximal $p-$subgroups of G/R are finite for each prime p.*

Now we can obtain a more or less complete description of locally generalized radical groups of finite section rank.

3.17. Theorem. *Let G be a locally generalized radical group of finite section rank. Then G has finite Hirsch–Zaitsev rank, moreover $r_{hz}(G) \leq \frac{t(t+3)}{2}$ where $t = min\,\{r_p(G) \mid p \in \mathbb{P} \cup \{0\}\}$. Furthermore, G has normal subgroups*

$$D \leq T \leq L \leq K \leq S \leq G$$

satisfying the following conditions:

(1) T is periodic and almost locally soluble;

(2) the maximal $p-$subgroups of G are Chernikov for all primes p;

(3) D is a divisible abelian subgroup;

(4) the maximal $p-$subgroups of T/D are finite for all primes p and T/D is residually finite;

(5) L/T is torsion-free nilpotent;

(6) K/L is finitely generated torsion-free abelian;

(7) G/K is finite and $|G/K| \leq f_4(t)$;

(8) S/K is soluble and $scl(S/T) \leq f_5(t)$.

3.18. Corollary. *Let G be a locally (soluble-by-finite) group of finite section rank. Then G is almost locally soluble.*

In particular, the following property belongs specifically to radical groups.

3.19. Proposition. *Suppose G is a periodic radical group of finite section rank. Then G is countable.*

The condition that G be radical here is very important since R. Baer [BR1969, Folgerung 5.4] has constructed an example of an uncountable locally soluble periodic group with finite maximal $p-$subgroups for all primes p.

With the aid of Theorem 3.17, we can also see the influence of the locally soluble subgroups of finite section rank on the structure of locally generalized radical groups.

3.20. Theorem. *Let G be a locally (soluble-by-finite) group. If every locally soluble subgroup of G has finite section rank, then G has finite section rank. In particular, G is almost locally soluble.*

4. Groups of finite minimax rank

In this short section we consider a special case of the Hirsch–Zaitsev rank, connecting it with the minimal and the maximal conditions and uniting them in some sense. This idea was introduced by D. I. Zaitsev and it appeared in the study of groups with the weak minimal condition [ZDI1968]. In this article, D.I. Zaitsev used the term "*index of minimality*", a term which turned out to not be quite suitable and did not reflect the situation precisely. Therefore, later in [ZK1987], D. I. Zaitsev proposed another term, "*minimax rank*".

Let G be a group and let,

$$\langle 1 \rangle = H_0 \leq H_1 \leq \dots \leq H_{n-1} \leq H_n = G$$

be a finite chain of subgroups of G. Let $\mathfrak{C} = \{H_j \mid 0 \leq j \leq n\}$ and let $il(\mathfrak{C})$ denote the number of links $H_j \leq H_{j+1}$ such that the index $|H_{j+1} : H_j|$ is infinite. We say that G *has finite minimax rank* $r_{mm}(G) = m$, if $il(\mathfrak{C}) \leq m$ for every finite chain of subgroups $\mathfrak{C}$ and provided there exists a chain $\mathfrak{D}$ for which this number is exactly m. Otherwise we say that G has *infinite minimax rank*. Of course, if G is a finite group, then $r_{mm}(G) = 0$.

Let H, K be subgroups of a group G and $H \leq K$. We say that the link $H \leq K$ *is infinite* if the index $|K : H|$ is infinite. We say that the link $H \leq K$ is *minimal infinite* if it is infinite and for every subgroup L such that $H \leq L \leq K$ one of the indices $|L : H|$ or $|K : L|$ is finite.

Suppose that a group G has finite minimax rank and let $\mathfrak{D}$ be a finite chain of subgroups such that $il(\mathfrak{D}) = r_{mm}(G)$. Let $H \leq K$ be a link of this chain such that the index $|K : H|$ is infinite and let L be a subgroup of G such that $H < L < K$. If both indices $|K : L|$ and $|L : H|$ are infinite then the chain $\mathfrak{D} \cup \{L\}$ is finite and $il(\mathfrak{D} \cup \{L\}) = il(\mathfrak{D}) + 1$, contradicting the choice of $\mathfrak{D}$. This contradiction shows that every link $H \leq K$ of $\mathfrak{D}$ with $|K : H|$ infinite is minimal infinite.

The class of groups of finite minimax rank is very closely connected with minimax groups, where a group G is *minimax* if G has a finite subnormal series whose factors satisfy either the minimal condition or the maximal condition. These groups initially appeared in the paper [BR1953] due to R. Baer. The first fundamental study of soluble minimax groups was done by D. J. S. Robinson [RD1967]. The term "*minimax group*" is also due to D. J. S. Robinson [RD1967]. In the article [BR1968], Baer used the term "*polyminimax groups*", but later all authors began to use the term "*minimax groups*".

The theory of soluble-by-finite minimax groups is well developed now and these groups have been studied by many authors from different points of view, resulting in a large number of publications dedicated to such groups. Minimax groups appear in different group theoretical investigations and themselves would form an interesting subject for a separate survey. In the current article we will observe some connections between minimax groups and groups of finite rank.

From the above mentioned results and results due to D. I. Zaitsev (see articles [ZDI1968, ZDI1971B]) we obtain the following description of the locally generalized radical groups of finite minimax rank.

4.1. Theorem. *Let G be a locally generalized radical group. Then G has finite minimax rank if and only if G is minimax and almost soluble.*

The next result reflects the connections between minimax groups and groups of finite rank. In fact, this result was proved by D. I. Zaitsev [ZDI1971A].

4.2. Theorem. *Let G be a generalized radical group of finite Hirsch – Zaitsev rank. If G is finitely generated, then $G/Tor(G)$ is soluble-by-finite and minimax.*

5. Groups of finite special rank

In this section we will consider the connection of the special rank with the other ranks. Our first two results illustrate these connections for locally finite p-groups and for polyrational groups respectively.

5.1. Proposition. L*et P be a locally finite $p-$group for some prime p. Then P has finite special rank r if and only if $r_p(P)$ is finite. In particular, P has finite special rank r if and only if P is a Chernikov group. Moreover, in this case, $r_p(P) = r(P)$.*

5.2. Theorem [ZDI1971A]. *Let G be a polyrational group. Then $r_{hz}(G) = r(G)$.*

The following corollaries can be read off from some of the earlier results.

5.3. Corollary. *Let G be a torsion-free locally nilpotent group. If G has finite special rank r, then G has finite Hirsch–Zaitsev rank. Moreover, G is nilpotent and $r(G) = r_{hz}(G)$.*

5.4. Corollary [DKP2007]. *Let G be a locally generalized radical group of finite Hirsch – Zaitsev rank. Then $G/Tor(G)$ has finite special rank. Moreover, $r_{hz}(G) \leq r(G/Tor(G))$ and $r(G/Tor(G)) \leq r_{hz}(G) + f_4(r)$.*

5.5. Theorem. *Let G be a locally generalized radical group. If G has finite special rank r, then G has Hirsch–Zaitsev rank at most r. In particular, G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that T is locally finite, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, G/K is finite and S/K is soluble. Moreover, if $r(G) = r$, then $|G/K| \leq f_1(r)$ and $scl(S/T) \leq f_2(r)$.*

5.6. **Corollary** [PBI1958, 16.3.1]. *Let G be a locally radical group. If G has finite special rank r, then G is a radical group.*

By Proposition 5.1, a periodic locally nilpotent group of finite special rank is a direct product of Chernikov $p-$groups, each of which is soluble. In this connection, the following question arises:

Let P be a Chernikov (in particular, finite) $p-$group of finite special rank r. *Is $scl(P)$ bounded in terms of some function of r?* A negative answer to this question was obtained by Yu. I. Merzlyakov [MYuI1964], using the following example.

Let n be a positive integer, n $\geq$3, and let π be an infinite set of odd primes. For each $p \in \pi$ let $t(p)$ be a natural number with the additional property that $t(p) < t(q)$ whenever $p, q \in \pi$ and $p < q$. Let

$$\sigma = \{t(p) \mid t(p) \in \mathbb{N}, p \in \pi \text{ and } t(p) < t(q) \text{ whenever } p < q\}.$$

Let $G_p = \{E + pA \mid A \in \mathbb{M}_n(\mathbb{Z}/p^{t(p)}\mathbb{Z})\}, p \in \pi$, where E is the identity matrix. The finite $p-$group G_p has finite special rank at most n^2 and $scl(G_p) < scl(G_q)$ whenever $p < q$ [MYuI1964]. It follows that the group $G = \mathbf{Dr}_{p\in\pi} G_p$ is not soluble. Clearly, it is hypercentral and has finite special rank at most n^2.

The following assertion will be useful further.

5.7. Proposition. Let p be a prime and G be a finite $p-$group. Suppose that A is a maximal normal abelian subgroup of G. If $r(A) = r$, then $r(G) \leq \frac{r(5r+1)}{2}$.

As we mentioned above, groups of finite special rank have finite section rank. Moreover, for every prime p the section $p-$rank is not greater than the special rank.

We will say that a group G *has bounded section rank* if the set of positive integers $\{r_p(G) \mid p \in \mathbb{P} \cup \{0\}\}$ is bounded. In this case, the positive integer

$$r_{bs}(G) = max\{r_p(G) \mid p \in \mathbb{P} \cup \{0\}\}$$

is called the *bounded section rank* of G. Otherwise we will say that *G has no bounded section rank.*

Thus, if a group G has finite special rank r, then G has bounded finite section rank at most r. For groups of bounded section rank, it is possible to obtain detailed information concerning their structure (see Theorem 3.17).

5.8. Theorem. *Let G be a locally radical group of finite bounded section rank b. Then $Lnr(G)$ is hypercentral and $G/Lnr(G)$ contains a normal abelian subgroup $A/Lnr(G)$ such that G/A is finite. Moreover, there exists an integer valued function f_6 such that $|G/A| \leq f_6(b)$. In particular, G is hyperabelian.*

5.9. Corollary. *Let G be a locally generalized radical group of finite bounded section rank b. Then G contains a normal hypercentral subgroup L such that G/L is abelian-by-finite. In particular, G is hyperabelian-by-finite. Hence a locally generalized radical group of finite special rank is hyperabelian-by-finite.*

Along the same lines as Proposition 3.19 we also have the following result and note that Baer's example once again shows that the hypothesis of having "bounded section rank" cannot be removed.

5.10. Corollary. *Let G be a locally generalized radical group of bounded section rank. Then G is countable.*

The next result enables us to see the relationship between bounded section rank and special rank in the class of locally generalized radical groups.

5.11. Theorem. *Let G be a locally generalized radical group. Then G has bounded section rank if and only if G has finite special rank.*

As in Section 3 we pay attention to the influence of locally radical subgroups of bounded section rank on the structure of locally generalized radical groups.

5.12. Theorem. *Let G be a locally generalized radical group. If every locally radical subgroup of G has bounded section rank, then G has bounded section rank.*

There are several interesting consequences of this result which we now list.

5.13. Corollary. *Let G be a locally generalized radical group. If every locally radical subgroup of G has finite special rank, then G has finite special rank.*

5.14. Corollary [DES1996]. *Let G be a locally (soluble-by-finite) group. If every locally soluble subgroup of G has finite special rank, then G has finite special rank; in particular, G is almost hyperabelian.*

5.15. Corollary [CNS1990]. *Let G be a locally (soluble-by-finite) group of finite special rank. Then G is almost hyperabelian.*

We now define the *total section rank* of an arbitrary group G by the following rule

$$r_{tot}(G) = r_0(G) + \sum_{p \in \pi(G)} r_p(G).$$

For soluble groups this concept was introduced by D. J. S. Robinson [RD1968, 6.2]. The class of soluble groups of finite total rank is denoted by $\mathfrak{S}_1$. The class $\mathfrak{S}_1$ is exactly the class of soluble A_3 – groups, which was introduced by A. I. Maltsev [MAI1951].

Clearly a group G has finite total section rank, if $r_p(G)$ is finite for all primes p and $r_p(G) = 0$ for all but finitely many primes p. In particular, every group of finite total section rank has bounded finite section rank. Thus we obtain

5.16. Theorem. *Let G be a locally generalized radical group of finite total section rank. Then G has finite Hirsch–Zaitsev rank and, moreover, $r_{hz}(G) \le \frac{t(t+3)}{2}$ where $t = min\{r_p(G) \mid p \in \mathbb{P} \cup \{0\}\}$. In particular, G has normal subgroups $T \le L \le K \le S \le G$ such that T is a Chernikov subgroup, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, G/K is finite and S/K is soluble. Moreover, $|G/K| \le f_4(t)$ and $scl(S/T) \le f_5(t)$.*

For locally generalized radical groups of finite total section rank we can obtain the following strengthening of Corollary 5.9.

5.17. Theorem. *Let G be a locally generalized radical group of finite total section rank. Then $Fitt(G)$ is nilpotent and $G/Fitt(G)$ is almost abelian.*

For soluble groups of finite total section rank (that is groups from the class $\mathfrak{S}_1$ or from the class of soluble A_3-groups), this result was proved by A. I. Maltsev [MAI1951].

The following subclass of $\mathfrak{S}_1$ has been introduced in [MAI1951]. We say that a group G is a soluble A_4-group if it is a soluble A_3-group and $Tor(G)$ is finite.

5.18. Proposition [MAI1951]. Let G be a soluble A_4–group. Then $Fitt(G)$ is nilpotent and $G/Fitt(G)$ is almost abelian and finitely generated.

Finally we consider finitely generated groups of finite special rank.

5.19. Theorem [DKP2007]. *Let G be a finitely generated generalized radical group. If G has bounded section rank, then G is minimax and almost soluble.*

For soluble-by-finite groups this result was obtained by D. J. S. Robinson [RD1969]. Moreover, in [RD1982A, (3.3)] D. J. S. Robinson proved that finitely generated soluble groups of finite section rank are minimax.

6. Groups whose abelian subgroups have finite ranks

The influence of the abelian subgroups on the structure of groups is very important in many classes of groups, including locally generalized radical groups in particular. Thus, for example, if every abelian subgroup of a locally generalized radical group is finite, then the group itself is finite. This follows from a theorem due to M. I. Kargapolov [KMI1963] and P. Hall and C. R. Kulatilaka [HK1964]. Similarly, it follows from a theorem due to V. P. Shunkov and O. H. Kegel and B. A. F. Wehrfritz (see, for example, [KW1973], Theorem 5.8) that if all abelian subgroups of a locally generalized radical group satisfy the minimal condition, then the group is Chernikov. We will not consider in detail the theme of the influence of abelian subgroups on the properties of a group. One can find this information in the book of S. N. Chernikov [CSN1980] or in the surveys [CZ1988] and [ZKCh1972]. In this section we want to present the structure of groups whose abelian subgroups have finite ranks.

We begin with the following result about the structure of groups, whose locally nilpotent radical has finite total section rank. This result was obtained by D. I. Zaitsev [ZDI1977] and J. C. Lennox and D. J. S. Robinson [LR1980, Corollary to Theorem H]. This assertion plays an important role in justifying further results of this section. Note that D. I. Zaitsev developed a group-theoretical proof, while the proof of J. C. Lennox and D. J. S. Robinson uses homological methods.

6.1. Theorem [ZDI1977], [LR1980]. *Let G be a group and suppose that it contains a nilpotent normal subgroup A of finite total section rank such that G/A is nilpotent. Then either G is almost nilpotent or G contains a nilpotent subgroup L such that AL has finite index in G.*

6.2. Theorem [DKP2007]. *Let G be a generalized radical group and suppose that $Tor(G) = \langle 1 \rangle$.*

(i) If every abelian subgroup of G has finite Hirsch–Zaitsev rank, then there is a positive integer r such that every abelian subgroup of G has finite Hirsch–Zaitsev rank at most r.

(ii) If there is a positive integer r such that every abelian subgroup of G has finite Hirsch–Zaitsev rank at most r, then G has a finite Hirsch–Zaitsev rank. Moreover, there is the function f_5:$\mathbb{N} \longrightarrow \mathbb{N}$ such that $r_{hz}(G) \leq f_5(r)$.

6.3. Theorem [DKP2007]. *Let G be a group and suppose that every finitely generated subgroup of G is minimax and soluble-by-finite. If there is a positive integer r such that every abelian subgroup of G has finite section $0-$rank at most r, then G has finite Hirsch–Zaitsev rank at most $f_5(r)$.*

These results have the following important implications.

6.4. Corollary [DKP2007]. *Let G be a group and suppose that every finitely generated subgroup of G is minimax and soluble-by-finite. If there* *is a positive integer r_p for some prime p, such that every abelian subgroup of G has finite section $p-$rank at most r_p, then G has finite Hirsch–Zaitsev rank at most $\frac{f_6(r_p(r_p+3))}{2}$. Moreover, the maximal $p-$subgroup of G are Chernikov and have special rank at most $f_7(r_p)$.*

6.5. Corollary [DKP2007]. *Let G be a generalized radical locally minimax group. If every abelian subgroup of G has finite section $0-$rank, then G has finite Hirsch–Zaitsev rank.*

6.6. Corollary [DKP2007]. *Let G be a generalized radical locally minimax group. If every abelian subgroup of G has finite section $p-$rank for some prime p, then G has finite section $p-$rank.*

6.7. Corollary. *Let G be a generalized radical group. If every abelian subgroup of G has finite section rank, then G has finite section rank.*

6.8. Corollary [BRHH1972]. *Let G be a radical group. If every abelian subgroup of G has finite section rank, then G has finite section rank.*

Consider now locally generalized radical groups, whose abelian subgroups have finite special rank. We begin with the following important result that together with Theorems 6.2 and 6.3 allows us to describe these groups.

6.9. Theorem [GYu1964]. *Let G be a periodic locally soluble group. If every abelian subgroup of G has finite special rank, then G has finite special rank.*

With the help of V.V. Belyaev's theorem (Theorem 3.13), we can derive from here the following results.

6.10. Corollary. *Let G be a generalized radical group. If every abelian subgroup of G has finite special rank, then G has finite special rank.*

6.11. Corollary [KMI1962]. *Let G be a radical group. If every abelian subgroup of G has finite special rank, then G has finite special rank.*

6.12. Corollary [SVP1971B]. *Let G be a locally finite group. If every abelian subgroup of G has finite special rank, then G has finite special rank.*

Note, that it is impossible to extend the condition "generalized radical group" to the condition "locally generalized radical group". Yu.I. Merzlyakov [MYuI1984] has constructed an example of a locally polycyclic group G satisfying the following conditions:

(i) every abelian subgroup of G has finite special rank;

(ii) G has infinite special rank and infinite Hirsch – Zaitsev rank.

However the following assertion is valid.

6.13. Theorem [DES1996]. *Let G be a locally (soluble-by-finite)* *group. Suppose that G satisfies the following conditions:*

(i) if A is an abelian subgroup of G, then $r_p(A)$ is finite for all primes p;

(ii) there is a positive integer b such that $r_0(A) \leq b$ for each abelian subgroup A of G.

Then G has finite special rank.

6.14. Theorem [DKP2007]. *Let G be a locally generalized radical group. If there is a positive integer r, such that every abelian subgroup of G has finite special rank at most r, then G has finite special rank. Moreover, there is a function $f_8 : \mathbb{N} \longrightarrow \mathbb{N}$ such that $r(G) \leq f_8(r)$.*

6.15. Theorem [MYuI1964]). *Let G be a locally soluble group. If there is a positive integer r, such that every abelian subgroup of G has finite special rank at most r, then G has finite special rank. Moreover, $r(G) \leq f_9(r)$.*

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