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On linear algebraic groups over pseudoglobal fields

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Dedicated to Professor V. V. Kirichenko on the occasion of his 65th birthday

Abstract. Some properties of R–equivalence and weak approximation in linear algebraic group over global field are generalized to the case of linear algebraic group over an algebraic function field in one variable with pseudofinite constant field.

Let X be a smooth algebraic variety defined over a field K. Recall that two points $x, y \in X(K)$ are R-equivalent if there is a sequence of points $z_i \in X(K), x = z_1, \dots, y = z_n$, such that for each pair z_i, z_{i+1} there exists a K-rational map $f_i : \mathbb{P}^1 \to X$, regular at 0 and 1, with $f_i(0) = z_i, f_i(1) = z_{i+1}, 1 \leq i \leq n-1$). We shall denote the set of R-equivalence classes on $X(K)$ by $X(K)/R$. If G is a linear algebraic group defined over a field K, the set $G(K)/R$ can be endowed with a natural group structure.

Let V^K be the set of all valuations of a field K, S be a finite subset of V^K, and G be a connected linear algebraic group defined over K. Denote by $\overline{G(K)}$ the closure of $G(K)$ in the product topology, where $G(K)$ is embedded diagonally into the direct product $\prod_{v \in S} G(K_v)$, and $G(K_v)$ is endowed with the v-adic topology induced from that of K_v. One says that G has weak approximation with respect to S if $\overline{G(K)} = \prod_{v \in S} G(K_v)$, and G has weak approximation over K if it is so for any finite subset $S \subset V^K$.

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Let

$$A_S(G) = \prod_{v \in S} G(K_v)/\overline{G(K)}, \quad A(G) = \prod_{v \in V^K} G(K_v)/\overline{G(K)},$$

be the defect of weak approximation in S and over K respectively.

J.-L. Colliot-Thélène, P. Gille and R. Parimala [8] showed that many arithmetical properties of linear algebraic groups and of their homogeneous spaces over totally imaginary number fields have counterparts over the fields of one of the following types:

(gl) a function field K in two variables over an algebraically closed field k of characteristic zero, i.e. the function field of a smooth, projective, connected surface over K;

(ll) the field of fractions K of a henselian, excellent, two-dimensional local domain A with an algebraically closed residue field k of characteristic zero;

(sl) the Laurent series field $l((t))$ over a field l of characteristic zero and cohomological dimension 1.

All these fields K have the following properties:

(I) their cohomological dimension is two;

(II) index and exponent of central simple algebras over K coincide;

(III) $H^1(K, G) = 1$ for any semisimple simply connected group G over K.

J.-L. Colliot-Thélène, P. Gille and R. Parimala also proved in [8] that properties (I) and (II) imply (III) for groups without E_8-factors. Moreover, they proved that for any semisimple simply connected linear algebraic group G defined over a field K satisfying properties (I) and (II) (and cohomological dimension $\mathrm{cd}(K^{\mathrm{ab}})$ of maximal abelian extension of K satisfies $\mathrm{cd}(K^{\mathrm{ab}}) \leq 1$ if factors of type E_8 are allowed), the group $G(K)/R$ is trivial.

In this paper we consider the linear algebraic groups over an algebraic function field K in one variable with a pseudofinite [5] constant field k. We call such a field K pseudoglobal. Recall that a perfect field k is called pseudofinite if it has exactly one extension of degree n for every natural number n, and if every absolutely irreducible affine variety defined over k has a k-rational point. For example, the following fields are pseudofinite [12]:

i) the infinite extension of a finite field having exactly one extension of each degree n;

ii) the fixed field $\overline{\mathbb{Q}}(\sigma)$ in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ for almost all (in the sense of the canonical Haar measure on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) automorphisms $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;

iii) the infinite models of the first-order theory of finite fields.

In particular, these examples show that there exist the pseudoglobal fields of characteristic zero as well as of positive characteristic.

The important feature of pseudoglobal fields is that for these fields there is an analogue (cf. [15], [14], [2]) of the classical global class field theory (see also [11] for such an analogue for more wide class of fields, namely for algebraic function fields in one variable over pseudo-algebraically closed constant fields). In particular, the Hasse principle for Brauer group holds for a pseudoglobal field K, i.e. the canonical map $\mathrm{Br}K \to \prod_{v\in V^K} \mathrm{Br}K_v$ is injective, where $\mathrm{Br}K$ (resp. $\mathrm{Br}K_v$) is the Brauer group of K (resp. the Brauer group of K_v, K_v being the completion of K at a valuation v), v runs over the set V^K of all (trivial on the constant field) valuations of K. Also, the Tate-Nakayama theorems for algebraic tori hold for a pseudoglobal field and all its completions.

In order to adapt the results of J.-L. Colliot-Thélène, P. Gille and R. Parimala [8] to the case of pseudoglobal ground field we need some relevant facts about pseudoglobal fields. Let us enumerate them in the following theorem.

Theorem 1. *A pseudoglobal field field K has the following properties.*

(a) K has the C_2 property: for any $r \geq 1$ and any system of homogeneous forms $f_j(X_1,\dots,X_n) \in K[X_1,\dots,X_n]$, $j = 1,\dots,r$, if $n > \sum_j d_j^2$, where d_j is the degree of f_j, then there is a nontrivial common zero in K^n for these r forms.

(b) The cohomological dimension $\mathrm{cd}(K)$ of K is 2.

(c) The reduced norm map $\mathrm{Nrd}:\ D^* \to L^*$ is surjective for any finite field extension L/K and any central simple algebra D/L.

(d) Every central simple algebra A over K is cyclic, and the index and exponent of A coincide.

(e) For a given prime p, over any finite field extension L of K, the tensor product of two central simple algebras of index p has index at most p.

(f) Over any finite field extension L/K, any quadratic form in at least 5 variables has a nontrivial zero.

(g) The cohomological dimension $\mathrm{cd}(K^{\mathrm{ab}})$ of the maximal abelian extension of K is 1.

Proof. There are some relations between properties stated in Theorem 1. In particular (cf. [8], Theorem 1.1) for a field K of characteristic zero the following implications hold: (i) (a) implies (b) and (f); (ii) (b) is equivalent to (c); (iii) (d) is equivalent to (e). So it is sufficient to prove (a), (d), (f) and (g).

The properties (a) and (d) were proved in [3], and (f) was proved in [4]. Recall briefly the argument from [3]. In [5] J. Ax showed that every pseudo algebraically closed field with abelian absolute Galois group is C_1. Next, it is known that if k is a C_i-field, and K is an extension of k of transcendence degree n, then K is a C_{i+n}-field. Hence a pseudoglobal field is a C_2-field.

To prove (d) we apply the argument which essentially follows the classical case of algebras over global fields. Let A be a finite-dimensional central simple algebra over a pseudoglobal field K. Let $v_1, \dots, v_r$ be all the valuations of K at which A has nontrivial local invariants. Set $n_i = \mathrm{ind}A_{v_i}$, $A_{v_i} = A \otimes_K K_{v_i}$, where K_{v_i} is the completion of K at v_i. Let m be a smallest common multiple of $n_1, \dots, n_r$. For all $i, 1 \le i \le r$, $n_i | n$, where $n = \mathrm{deg}A = [A : K]^{\frac{1}{2}}$, thus we have $m|n$. By Saltman's theorem ([16], Theorem 5.10) for any abelian group $\mathcal{G}$ if L_i/K_{v_i} are $\mathcal{G}$ Galois extension, then there is a $\mathcal{G}$ Galois extension L/K such that $L \otimes_K K_{v_i} = L_i$. Thus we may suppose that there are cyclic extensions L/K and M/K of degree m and n respectively, such that L/K and L_i/K_{v_i} are cyclic of degree m, and M/K and M_i/K_{v_i} are cyclic of degree n. We can take L_i and M_i to be the unramified extension of K_{v_i} of degrees m and n respectively. Then taking into account the class field theory for general local field [18] one sees that n_i is the invariant of the algebra A_{v_i}, $1 \le i \le r$. In view of class field theory for pseudoglobal field (see [3] for more details) A splits over L and M, and the field M is isomorphic to a strongly maximal subfield of A, hence the algebra A is cyclic.

It remains to prove that $\mathrm{ind}A = \mathrm{exp}A$. Since $\mathrm{exp}A | \mathrm{ind}A$, it is enough to prove that $\mathrm{ind}A \le \mathrm{exp}A$. Since the field L splits A, $\mathrm{ind}A \le m$. Further, if $e = \mathrm{exp}A$, then $i(e \cdot [A]) = e \cdot (i[A]) = 0$, where $[A] \in \mathrm{Br}K$ is the corresponding element of the Brauer group of K, and $i : \mathrm{Br}K \hookrightarrow \oplus_v \mathrm{Br}K_v$ is the monomorphism $[A] \mapsto \sum_v [A \otimes_K K_v]$ [3]. Denoting by $\mathrm{inv}_v A$ the local invariant of A at v, we have $e \cdot \mathrm{inv}_v A = 0$ for all valuation v of V^K. Hence $n_i | e, 1 \le i \le r$, and $m|e$. Thus $\mathrm{ind}A \le m \le e = \mathrm{exp}A$, and this completes the proof of (d).

As for (g), we note that since all finite extensions of the constant field k of K are abelian (in fact cyclic), the maximal abelian extension of K contains the subfield $k_s K$ (k_s is the separable closure of k in K^{ab}), $\mathrm{cd}(k_s K) = 1$ by Tsen's theorem. Thus K^{ab} being an algebraic extension of $k_s K$ has cohomological dimension 1. $\square$

Taking into account these results, we get the following results about linear algebraic groups over a pseudoglobal field stated as Corollaries 1 – 3 below.

Corollary 1. *Let K be a pseudoglobal field of characteristic zero. Let G be a connected linear algebraic group defined over K.*

(i) If G is a simply connected group, then $H^1(K,G)=1$.

(ii) Let G be a semisimple simply connected group, and let $1\to\mu\to G\to G^{\mathrm{ad}}\to 1$ be the central isogeny associated to the center μ of G. Then:

(a) the boundary map $\delta:\ H^1(K,G^{\mathrm{ad}})\to H^2(K,\mu)$ is a bijection;

(b) if the group G is not purely of type A, then it is isotropic.

Proof. Taking into account Theorem 1 (d) and (f), the property (i) follows from Theorem 1.2 (v) in [8] which says that if index and exponent coincide for 2-primary and 3-primary algebras over finite field extension of K and the cohomological dimension of K^{ab} is at most one, then $H^1(K,G)=0$. Further, Theorem 2.1 in [8] asserts that if G is a semisimple simply connected group defined over a field K of characteristic zero, $cd(K)\leq 2$, and $\operatorname{ind} A=\exp A$ for every central simple algebra A over any finite field extension of K, then G has the properties stated in (ii). Hence, applying Theorem 1 (b) and (d) completes the proof. □

Let K be a field, T be a K-torus, and let

$$1\longrightarrow F\longrightarrow P\longrightarrow T\longrightarrow 1 \tag{1}$$

be a flasque resolution of T, where P is a quasitrivial torus and F is a flasque torus.

The next two corollaries concern the linear algebraic groups G admitting a *special covering*, i.e. there exists an exact sequence

$$1\longrightarrow \mu\longrightarrow G'\longrightarrow G\longrightarrow 1, \tag{2}$$

where G' is the product of a semisimple simply connected group and a quasitrivial torus, and $G'\to G$ is an isogeny with kernel μ. It is known [9], that the group μ has a flasque resolution

$$1\longrightarrow \mu\longrightarrow F\longrightarrow P\longrightarrow 1, \tag{3}$$

where the torus F is flasque and the torus P is quasitrivial.

Corollary 2. *Let G be a connected linear algebraic group defined over a pseudoglobal field K of characteristic zero. Then*

(i) The quotient $G(K)/R$ is a finite abelian group.

(ii) If G has a special covering (2), then $G(K)/R\simeq H^1(K,F)$, where F is the flasque torus from (3).

Proof. Since the cohomological dimension of a pseudoglobal K field is 2, index and exponent coincide for central simple algebras over K, and $\mathrm{cd}(K^{\mathrm{ab}})=1$ we may apply Corollary 4.10 in [8] which asserts that $G(K)/R\simeq H^1(K,F)$. Thus $G(K)/R$ is a finite abelian group because $H^1(K,F)$ is a finite abelian group [3]. □

Corollary 3. *(i) Let G be a connected linear group defined over a pseudoglobal field K of characteristic zero. Let $S \subset V^K$ be a finite set. Then the closure $\overline{G(K)}$ of the image $G(K)$ under the diagonal map $G(K) \to \prod_{v\in S} G(K_v)$ is a normal subgroup, and the quotient*

$$A_S(G) = \prod_{v\in S} G(K_v)/\overline{G(K)}$$

is a finite abelian group.

(ii) Suppose that G has a special covering (2). The composite maps $G(K) \to H^1(K,\mu) \to H^1(K,F)$ and $G(K_v) \to H^1(K_v,\mu) \to H^1(K_v,F)$ induce isomorphisms of finite abelian groups

$$A_S(G) \simeq \operatorname{Coker}\big[H^1(K,F) \longrightarrow \prod_{v\in S} H^1(K_v,F)\big]$$

and

$$A_S(G) \simeq \operatorname{Coker}[G(K)/R \longrightarrow \prod_{v\in S} G(K_v)/R].$$

Proof. The examination of the proof of Theorem 4.13 in [8] shows that the stated properties hold whenever the field K satisfies the conditions: $\operatorname{cd}(K) \le 2$, $\operatorname{cd}(K^{ab}) \le 1$, index and exponent coincide for 2-primary and 3-primary central simple algebras All these conditions hold for a pseudoglobal field by Theorem 1. The Corollary follows. □

As in the case of a number ground field, for a connected reductive group G defined over a pseudoglobal field K the defect $A(G)$ of weak approximation and the Tate-Shafarevich group

$$Ш(G) = \operatorname{Ker}\Big(H^1(K,G) \longrightarrow \prod_{v\in V^K} H^1(K_v,G)\Big)$$

can be inserted in a short exact sequence. In order to state and prove the corresponding result we will need some more properties of pseudoglobal fields. In particular, we will need the following result which can be regarded as an analogue of Čebotarev density theorem for pseudoglobal fields. Its proof presented in [V. Andriychuk, An analogue of Tchebotarev's density theorem for pseudoglobal fields // Visnyk Kyiv. un-tu, Seriya phiz.-mat. (2000), 4, p. 11-16] is shaped from argument used by M. Fried [13] for algebraic function field with finite constant field.

Theorem 2. *Let L/K be a finite Galois extension of a pseudoglobal field K, $\mathcal{G} = \operatorname{Gal}(L/K)$ and $\tau \in \mathcal{G}$. There exists an infinite set of non equivalent valuation of the field L, whose decomposition group is the cyclic subgroup generated by τ.*

Let M be a finite $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-module over a pseudoglobal field K, $\widehat{M} = \mathrm{Hom}(M, K^*_{\mathrm{sep}})$ be its dual module, $S \subset V^K$ be a finite subset of valuations of the field K,

$$Ш^1_S(M) = \mathrm{Ker}(H^1(K,M) \longrightarrow \prod_{v \notin S} H^1(K_v, M)),$$

$$Ч^1_S(M) = \mathrm{Coker}(H^1(K,M) \longrightarrow \prod_{v \in S} H^1(K_v, M)),$$

$$Ш^1_\omega(M) = \varinjlim_S Ш^1_S(M),\ Ч^1_\omega(M) = \varprojlim_S Ч^1_S(M),\ Ш^1(M) = Ш^1_\emptyset(M).$$

The following theorem 3 enumerates some properties of these groups which are counterparts of corresponding properties of finite modules over a global field.

Theorem 3. *Let M be a finite module over a pseudoglobal field K, $(|M|, \mathrm{char} K) = 1$.*

1) The finite group $Ч^1_S(\widehat{M})$ is isomorphic to the dual group of the group $Ш^1_S(M)/Ш^1(M)$.

2) The finite groups $Ч^1_\omega(\widehat{M})$ and $Ш^1_\omega(M)/Ш^1(M)$ are dual each another.

3) There exists the exact sequence of finite groups

$$0 \longrightarrow Ч^1_\omega(\widehat{M}) \longrightarrow Ш^1_\omega(M)^* \longrightarrow Ш^1(M)^* \longrightarrow 0,$$

where $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

4) $Ч^1_\omega(M) = Ч^1_{S_0}(M)$, where S_0 is the finite subset of valuations of the field K consisting of ramified valuations with non cyclic decomposition groups in a finite Galois extension, over which the module $\widehat{M}$ becomes trivial.

5) If S consists of the valuations having cyclic decomposition groups in a finite Galois extension, over which the module $\widehat{M}$ becomes trivial, then $Ч_S(M) = 0$.

To prove this theorem we need the following fact about valuations of a pseudoglobal field.

Lemma 1. *Let L/K is a finite Galois extension, and $L \neq K$. Then there exists an infinite set of valuations v of K which does not split completely in L. In other words, if $L_w = K_v$ for almost all valuations of K (w is the extension of v to L), then $L = K$.*

Proof. For a global field this is Theorem 2 in [6], see also [1], Corollary 8.8. The proof reduces to the case where L/K is abelian. Let $G = \mathrm{Gal}\,(L/K)$. Denote by C_K and C_L the idele class groups of K and L respectively. Since the idele classes of a pseudoglobal field form a class formation, ([2], Theorem 1), we have $C_K/N_{L/K}C_L \simeq G$. Like the case of global ground field ([1], p. 275) it follows that for every finite subset S of valuations of K containing all valuations ramified in L, the group G is generated by elements $\sigma_v \in \mathrm{Gal}\,(L_w/K_v) \subset G$ for $v \notin S$, w is an extension of v to L. Here σ_v is the image of $\sigma^{[k(v):k]}|_{k(w)} \in \mathrm{Gal}\,(k(w)/k(v))$ under isomorphism $\mathrm{Gal}\,(k(w)/k(v)) \equiv \mathrm{Gal}\,(L_w/K_v) \subset G$, σ is the generator of the absolute Galois group of constant field k, and $k(w)/k(v)$ is the corresponding extension of residue field of K_v. Suppose, contrary to our assertion, that there exist only finitely many valuations of K which not split completely in L, then adding them to S, we would have that all σ_v are trivial for $v \notin S$, so they cannot generate G. □

Proof of Theorem 3. Note, that the proof of this theorem is carried out by the argument analogous to used in the case of a number ground field. First we show, that all the group indicated in the statement of Theorem 3 are finite. For any valuation $v \in V^K$ the completion K_v is a general local field. Hence the group $H^1(K_v, M)$ and $H^1(K_v, \widehat{M})$ are finite by II.5.2 in [18], and the finiteness of groups $\text{Ч}^1_S(M)$ and $\text{Ч}^1_S(\widehat{M})$ follows immediately from their definitions.

Consider the group $\text{Ш}^1_S(M)$. If M is a trivial $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-module, then $\text{Ш}^1_S(M) = 0$. Indeed, since the groups $H^1(K, M)$ and $H^1(K_v, M)$ are inductive limits relative to the inflation homomorphisms of groups $H^1(\mathcal{G}, M)$ and $H^1(\mathcal{G}^v, \widehat{M})$, where $\mathcal{G}$ (respectively $\mathcal{G}^v$) is the Galois group of finite Galois subextensions L/K (respectively L_w/K_v, w is an extension of valuation v to the field L), it suffices to prove that the groups

$$\text{Ш}(L/K, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Ker}(H^1(\mathcal{G}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \prod_{v \notin S} H^1(\mathcal{G}^v, \mathbb{Z}/n\mathbb{Z})) =$$

$$= \mathrm{Ker}(\mathrm{Hom}(\mathcal{G}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \prod_{v \notin S} \mathrm{Hom}(\mathcal{G}^v, \mathbb{Z}/n\mathbb{Z}))$$

are trivial for all finite Galois extensions L/K. If a homomorphism f lies in $\text{Ш}^1(L/K, \mathbb{Z}/n\mathbb{Z})$, then f defines a cyclic subextension $K' = L^{\mathrm{Ker} f}$ of the field L. The restrictions f_v of the homomorphism f to the groups $\mathcal{G}^v$ are trivial. This means that $K'_{w'} = K_v$, where w' is an extension of the valuation v to K'. Lemma 1 implies that $K' = K$, so $\mathrm{Ker} f = \mathcal{G}$ and $f = 0$. Now let M be an arbitrary finite $\mathrm{Gal}(K_{\mathrm{sep}}/K)$-module.

The module M becomes trivial over some finite Galois extension L/K. Consider the following commutative diagram

$$\begin{tikzcd}
H^1(L/K,M) \arrow[r, hook] \arrow[d, "\alpha"'] & H^1(K,M) \arrow[r] \arrow[d] & H^1(L,M) \arrow[d] \\
\prod_{v\notin S} H^1(L_w/K_v,M) \arrow[r, hook] & \prod_{v\notin S} H^1(L_w/K_v,M) \arrow[r, "\mathrm{res}"] & \prod_{v\notin S} H^1(L_w,M)
\end{tikzcd}$$

with exact rows (here w is an extension of valuation v to the field L). In this diagram the left vertical homomorphism has trivial kernel by the preceding argument, and therefore $Ш^1_S(M) = \mathrm{Ker}\alpha = Ш^1_S(L/K,M)$ is a finite group.

Theorem 2 implies that after eliminating from the set S a finite subset of valuations with cyclic decomposition group, for the obtained set of valuations S' we will have $Ш^1_{S'}(M) = Ш^1_S(L/K,M) = Ш^1_S(M)$, and therefore $Ш^1_\omega(M) = Ш^1_{S_0}(M)$, where S_0 is the finite set of valuations ramified in L, and with non cyclic decomposition group. In particular, $Ш^1_\omega(M)$ is a finite group.

Denote by $\prod'_{v\in V^K} H^1(K_v,M)$ the restricted topological product of the groups $H^1(K_v,M)$ relative to the subgroups $H^1_{\mathrm{un}}(K_v,M)$. It is proved in [10] that when the constant field of algebraic function field K is quasi-finite field of formal power series $k_0((t))$ over an algebraically closed field k_0 of characteristic 0, then there exists an exact sequence

$$H^1(K,M) \longrightarrow {\prod_{v\in V^K}}' H^1(K_v,M) \longrightarrow H^1(K,\widehat{M})^*,$$

where $H^1(K,\widehat{M})^* = \mathrm{Hom}_{\mathrm{cont}}(H^1(K,\widehat{M}),\mathbb{Q}/\mathbb{Z})$. By using the argument, analogous to that of [10] we get the exact sequence

$$H^1(K,M) \xrightarrow{\beta} {\prod_{v\notin S}}' H^1(K_v,M) \times \prod_{v\in S} H^1(K,M) \xrightarrow{\gamma} H^1(K,\widehat{M})^*$$

for finite modules M over a pseudoglobal field K.

As in the classical case of a number field [17], it follows that the image of the group $Ш^1_S(M)$ under homomorphism β, is isomorphic to the group $Ш^1_S(M)/Ш^1(M)$ which is dual to the kernel of homomorphism

$$H^1(K,\widehat{M}) \xrightarrow{\tilde{\gamma}} \prod_{v\in S} H^1(K_v,M)$$

Here we use the duality of groups $H^1(K_v,M)$ and $H^1(K_v,\widehat{M})$ (see [19], p. 113, exercise 2). Therefore, the group $\mathrm{Coker}\tilde{\gamma} \simeq Ч^1_S(\widehat{M})$ is dual to the kernel of the restriction homomorphism γ to the group $\prod_{v\in S} K_v, M)$, which is isomorphic to the group $Ш^1_S(M)/Ш^1(M)$, and assertion 1) of Theorem 3 is proved. Now we have the following exact sequence

$$0 \longrightarrow Ч^1_S(\widehat{M}) \longrightarrow Ш^1_S(M)^* \longrightarrow Ш^1(M)^* \longrightarrow 0.$$

Passing in this sequence to the projective limit we obtain assertions 2) and 3) of Theorem 3. The equality $Ш^1_\omega(M) = Ш^1_{S_0}(M)$ follows from Theorem 2, and assertion 2), and the change of M by $\widehat{M}$ implies assertion 4). Finally, it follows from Theorem 2 that for every $v \in V^K$ unramified in a Galois extension L/K there exist infinitely many valuations of K having the same decomposition group as v, which proves assertion 5). □

Corollary 4. *Let S be a finite subset of the set of all valuations of a pseudoglobal field K. Then*

1) $Ш^1_S(L/K, M)$ is a finite group;

2) $H^1(K_S, M)$ is a finite group;

3) $Ш^1_S(K, \mathbb{Z}/n\mathbb{Z}) = Ш^1_S(K, \mu_n) = Ш^1(K, \mu_n) = 0$, where $\mathbb{Z}/n\mathbb{Z}$ is a trivial G_K-module, and μ_n is the group of n-th roots of 1 in the field K, $(n, \mathrm{char}K) = 1$.

Proof. 1) The finiteness of the group $Ш^1_S(L/K, M)$ was proved in the course of the proof of Theorem 3.

2) By [[14], Lemma 4.8, p. 68-69] the kernel of the group $H^1(K_S, M)$ under localization homomorphism is contained in the group $\prod_{v\in S} H^1(K_v, M)$. Since the set S is finite, $H^1(K_v, M)$ is a finite group and the kernel of $Ш^1_S(K_S/K, M)$ of localization homomorphism is finite, thus the group $H^1(K_S/K, M)$ is finite as well.

3) The equality $Ш^1_S(K, \mathbb{Z}/n\mathbb{Z}) = 0$ was proved in the course of the proof of Theorem 3. The triviality of the group $Ш^1_S(K, \mu_n)$ follows from assertion 5) of Theorem 3. Finally, the triviality of the group $Ш^2(K, \mu_n)$ follows from the monomorphism $0 \to \mathrm{Br}K \to \prod_v \mathrm{Br}K_v$ (see [4]). □

Theorem 4. *Let G be a connected reductive group defined over a pseudoglobal field K. Suppose that G is endowed with a special covering $G' \to G$ with kernel μ, for example a semisimple connected group with fundamental group μ. The covering $G' \to G$ define an exact sequence of finite group*

$$0 \longrightarrow A(G) \longrightarrow Ш^1(k, \widehat{\mu})^* \longrightarrow Ш(G) \longrightarrow 0. \qquad (4)$$

As in the classical case (see [17]) the proof of Theorem 4 is based on the following three facts:

1. $A(G) = Ч^1_\omega(\mu)$ for connected reductive group defined over a pseudoglobal field K and endowed with a special covering $G' \to G$ with kernel μ
2. The map $Ш(K,G) \xrightarrow{\partial} Ш^2(K,\mu)$, defined by special k-covering $G' \to G$ with kernel μ, is a bijection.
3. There is an exact sequence

$$0 \longrightarrow Ш^1_\omega(K,\widehat{\mu}) \longrightarrow Ш^1_\omega(K,\mu)^* \longrightarrow Ш^1(k,\mu)^* \longrightarrow 0. \qquad (5)$$

Proof. The equality $A(G) = Ч^1_\omega(\mu)$ follows from Corollary 3 taking into account the exact sequences (2) and (3). The bijection $Ш(K,G) \xrightarrow{\partial} Ш^2(K,\mu)$ follows from Corollary 2, and the exact sequence (5) follows from Theorem 3. □

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