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RESEARCH ARTICLE

Filtered and graded Procesi extensions of rings

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Abstract. In this paper, we introduce filtered and graded Procesi extensions of filtered and graded rings as a natural modification of Procesi extensions of rings. We show that these extensions behave well from the geometric point of view.

1. Introduction

For basic notions, conventions and generalities, which we need here in this paper, we refer to [7] and [8], see also [1-6, 13, 14, 15].

There are many different ways of describing the ring extensions and their applications; several of these were thought to be different in some papers, see [2,4,5,6,9,10,11,12,14].

If R and S are not necessarily commutative rings, then the prime ideal structures of R and of S are, in general, rather poorly related. It appears that if one restricts to so-called filtered and graded Procesi extensions of rings R and S, as we see in this paper, then things improve considerably over the filtered and graded levels.

In fact, the usefulness of the topological spec-map appears in the study of ring extensions, the theory of schemes which centers around sheaves and in geometrical applications.

We study the Procesi extensions of filtered and graded rings and show that these extensions behave well in constructing geometric spaces. First, we introduce the effect of existence of the extensions $\varphi : R \to S$; filtered ring homomorphism, with $S = \varphi(R).S^R$ on a filtered ring S. This will allow to study the transfer of properties from the filtration of FS to the

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filtration of $F'S$. Next, we prove that if S is a filtered Procesi extension of R then, by using the Rees-level, $G(S)$ is again a graded Procesi extension of $G(R)$. Finally, we show that our extension, over the associated graded level can be applied to the affine schemes.

2. Filtered Procesi Extensions

Throughout this paper R and S will denote filtered rings with unit and $\mathbb{Z}-$ filtrations $FR = \{F_nR\}_{n\in\mathbb{Z}}, FS = \{F_nS\}_{n\in\mathbb{Z}}$ respectively. R-filt will denote the category of left filtered unitary R-modules.

Let $\varphi : R \to S$ be a filtered ring homomorphism in R-filt; then φ is said to be filtered Procesi extension if $S = \varphi(R).S^R$ where $S^R = \{s \in S : s\varphi(r) = \varphi(r)s \;\forall\, r \in R\}$. Hence S may be viewed as a filtered ring through φ as: $F'_nS = F_n(\varphi(R)).S^R = (\varphi(R) \cap F_nS).S^R; n \in \mathbb{Z}$. It is equally straightforward to prove the following:

2.1. Lemma. Under the assumption and notation mentioned above we have:

(a) S, with the filtration $F'S$, is a filtered ring,

(b) S, with filtration $F'S$, is left a filtered R-module, and

(c) $HOM_{FR,FS}(R,S) \subseteq HOM_{FR,F'S}(R,S)$. □

A morphism $f : M \to N$ between filtered R-modules is said to be strict if $f(F_nM) = f(M)\cap F_nN$, for all $n \in \mathbb{Z}$. For a complete information on filtered and graded ring theory, the reader is referred to [1, 3, 7, 8].

It is easy to prove the following characterizations:

2.2. Proposition. Let $\varphi : R \to S$ be a filtered Procesi extension as above.

(a) If φ is a monomorphism and R a strongly filtered ring, in the sense that $(F_nR)(F_mR) = F_{n+m}R$, forall $n, m \in \mathbb{Z}$, then $F'_nS \cong F_nR.S^R$, for all $n \in \mathbb{Z}$, and $F'S$ is a strong filtration on S.

(b) If φ is strict and R a strongly filtered ring then S is strongly filtered with respect to $F'S$.

(c) If φ is a monomorphism and I any two sided ideal in R then $IS = SI$.

(d) $\varphi(Z(R)) \subseteq Z(S) \subseteq S^R$, where $Z(R)$ is the commutative subring in R of all the central elements, with the induced filtration.

(e) If φ is an epimorphism and S strongly filtered ring with respect to FS then $S = S.Z(S)$ is strongly filtered with respect to $F'S; F'_nS = F_nS.Z(S)$, for all $n \in \mathbb{Z}$.

Proof. Straightforward. □

A filtration FM on $M \in R$-filt is said to be discrete if there is an integer α such that $F_nM = 0$ for all $n < \alpha$, separated if $\bigcup_{n\in\mathbb{Z}} F_nM = 0$ and exhaustive if $M = \bigcup_{n\in\mathbb{Z}} F_nM$. Finally M is said to be filtered complete if $M \cong M^{\wedge_F} = \varprojlim_{n} M/F_nM$. In other words, M is complete if FM separated and all Cauchy sequences in the FM-topology of M converge, [1, 3].

We now come to the main results of this section:

2.3. Proposition. With notations as above.
(a) If FS on S is discrete then $F'S$ on S discrete.
(b) If FS is separated then $F'S$ separated.
(c) If FS is exhaustive then $F'S$ exhaustive.
(d) If FS is exhaustive, separated and complete then S is filtered and complete at $F'S$.

Proof. (a) We may take $\alpha_F = \alpha_{F'}$. Then we have

$$F'_nS = (F_nS \cap \varphi(R)).S^R = (0 \cap \varphi(R)).S^R = 0,$$

for all $n < \alpha$.

(b) Let $t \in \bigcap_{n\in\mathbb{Z}} F'_nS = \bigcap_{n\in\mathbb{Z}} ((\varphi(R) \cap F_nS).S^R)$. Then

$$t = y_{n1}.x_1; y_{n1} \in F_nS \cap \varphi(R), x_1 \in S^R,$$

for all $n \in \mathbb{Z}$. This implies that $t = 0$.

(c) Since $\bigcup_{n\in\mathbb{Z}} F_nS = S$, then $((\bigcup_{n\in\mathbb{Z}} F_nS) \cap \varphi(R)).S^R = \varphi(R).S^R = S$.

On the other hand,

$$\bigcup_{n\in\mathbb{Z}} F'_nS = \bigcup_{n\in\mathbb{Z}} [(F_nS \cap \varphi(R)).S^R] =$$

$$= [(\bigcup_{n\in\mathbb{Z}} F_nS) \cap \varphi(R)].S^R = \varphi(R).S^R = S.$$

This yields the assertion.

(d) By using (b) and (c) we conclude that all Cauchy sequences in the $F'S$-topology of S converge. From this, the result easily follows. □

2.4. Open question. Two filtrations FM and $F'M$ on an R-module M, are said to be equivalent if there exists some $\alpha \in \mathbb{N}$ such that $F_{n-\alpha}M \subseteq F'_nM \subseteq F_{n+\alpha}M$, for all $n \in \mathbb{Z}$, see [3]. With these notations, is it true that FS is equivalent to $F'S$?

3. Graded Procesi Extensions

If $\varphi : R \to S$ is a filtered Procesi extension with $S = \varphi(R).S^R$ as above then we get the associated graded extension morphisms $\tilde{\varphi} : \tilde{R} \to \tilde{S}$ (respectively, $G(\varphi) : G(R) \to G(S)$) in $\tilde{R}-gr$ (respectively, in $G(R)-gr$) in a natural way. Now, obviously, if $S = \varphi(R).S^R$, then

$$\tilde{S} = \bigoplus_{n\in\mathbb{Z}}(\varphi(R) \cap F_nS).S^R \cong \sum_{n\in\mathbb{Z}}(\varphi(R) \cap F_nS).S^R X_e^n; X_e = X_e.1 \in (\tilde{S})_1,$$

where X_e is the central element of degree one in $\tilde{S}$.
Therefore $\tilde{S} = (\varphi(R).S^R)^{\sim} = \tilde{\varphi}(\tilde{R}).\tilde{S}^{\tilde{R}}$;

$$\begin{aligned} \tilde{S}^{\tilde{R}} &= \{\tilde{s} \in \tilde{S} : \tilde{\varphi}(\tilde{r}) = \tilde{s}\varphi(\tilde{r}) \text{ for all } \tilde{r} \in \tilde{R}\} \\ &= \{s \in S : \varphi(r)s = s\varphi(r) \text{ for all } r \in R\}^{\sim} = (S^R)^{\sim}. \end{aligned}$$

We now come to the main result of this paper.
3.1. Proposition. With notations and conventions introduced above, let $\tilde{S} = \tilde{\varphi}(\tilde{R}).\tilde{S}^{\tilde{R}}$, i.e. is a $\tilde{\varphi}$ graded Procesi extension. Then $\bar{\tilde{\varphi}} : \tilde{R}/X\tilde{R} \to \tilde{S}/X\tilde{S}$ is a graded Procesi extension in $\tilde{R}/X\tilde{R} - gr$. In other words, if $\tilde{S} = \tilde{\varphi}(\tilde{S}).\tilde{S}^{\tilde{R}}$ then $\bar{\tilde{S}} = \bar{\tilde{\varphi}}(\bar{\tilde{R}}).\bar{\tilde{S}}^{\bar{\tilde{R}}}$, where $\bar{\tilde{S}} = \tilde{S}/X\tilde{S}$ and $\bar{\tilde{R}} = \tilde{R}/X\tilde{R}$.

Proof. Consider the following commutative diagram

$$\begin{array}{rcccl} & \tilde{R} & \xrightarrow{\bar{\tilde{\varphi}}} & \tilde{S} & \\ & \eta_{\tilde{R}} \downarrow & & \downarrow \eta_{\tilde{S}} & \\ G(R) & \cong \tilde{R}/X\tilde{R} & \xrightarrow{\bar{\tilde{\varphi}}} & \tilde{S}/X\tilde{S} \cong G(R); & \end{array}$$

with

$$\bar{\tilde{\varphi}}(\bar{\tilde{r}}) = \tilde{\varphi}(\tilde{r}) + X\tilde{S} \text{ , for all } \bar{\tilde{r}} \in \tilde{R}/X\tilde{R}.$$

Now, let $\bar{\tilde{s}} = \tilde{s} + X\tilde{S}, \tilde{s} = s_mX^m = \tilde{\varphi}(\tilde{r}).\tilde{z} = \tilde{z}\tilde{\varphi}(\tilde{r}); s_m \in F_mS$ and $\tilde{z} \in \tilde{S}^{\tilde{R}}$. Hence $\tilde{s}\tilde{\varphi}(\tilde{r}) + X\tilde{S} = \tilde{\varphi}(\tilde{r})\tilde{s} + X\tilde{S}$ and $\tilde{s}\tilde{\varphi}(\tilde{r}) - \tilde{\varphi}(\tilde{r})\tilde{s} \in X\tilde{S}$. This implies that $\bar{\tilde{s}} \in \bar{\tilde{\varphi}}(\bar{\tilde{R}}).\bar{\tilde{S}}^{\bar{\tilde{R}}}$, where

$$\bar{\tilde{S}}^{\bar{\tilde{R}}} = \{\bar{\tilde{s}} \in \bar{\tilde{S}} : \bar{\tilde{\varphi}}(\bar{\tilde{r}}).\bar{\tilde{s}} = \bar{\tilde{s}}.\bar{\tilde{\varphi}}(\bar{\tilde{r}}) \text{ , for all } \bar{\tilde{r}} \in \bar{\tilde{R}}\},$$

and $\bar{\tilde{s}} = \tilde{\varphi}(\tilde{r})\tilde{z} + X\tilde{S}$. Conversely, let $\bar{\tilde{t}} \in \bar{\tilde{\varphi}}(\bar{\tilde{R}}).\bar{\tilde{S}}^{\bar{\tilde{R}}}; \bar{\tilde{t}} = (\tilde{\varphi}((\tilde{r}) + X\tilde{S}).\bar{\tilde{z}} = (\tilde{\varphi}(\tilde{r}) + X\tilde{S}).(\tilde{s} + X\tilde{S}) = \tilde{\varphi}(\tilde{r}).\tilde{s} + X\tilde{S} = \tilde{s}\tilde{\varphi}(\tilde{r}) + X\tilde{S}$. Therefore $\bar{\tilde{t}} \in \bar{\tilde{S}}$. Hence, we conclude that $\bar{\tilde{S}} = \bar{\tilde{\varphi}}(\bar{\tilde{R}}).\bar{\tilde{S}}^{\bar{\tilde{R}}}$ and $\bar{\tilde{\varphi}} : \tilde{R}/X\tilde{R} \to \tilde{S}/X\tilde{S}$ ia a graded Procesi extension in $\tilde{R}/X\tilde{R} - gr$. □

3.2. Remark. With notations and conventions as above, let $S = \varphi(R).S^R$, i.e. $\varphi \in R$-filt is a filtered Procesi extension and consider the commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \sigma_R \downarrow & & \downarrow \sigma_R \\ G(R) & \xrightarrow{G(\varphi)} & G(S) \end{array}$$

Then one may derive

$$G(S) = G(\varphi)(G(R)).G(S)^{G(R)} = G(\varphi(R).S^R),$$

giving the same result as in 3.1.

4. Geometric Implications

In the sequel of this section, R, S will be filtered rings with unit such that $G(R), G(S)$ are Noetherian domains. Now, let $\varphi : R \to S$ be filtered Procesi extension as above such that $S = \varphi(R).S^R$. Hence we get a graded Procesi extension $T = G(\varphi) : G(R) \to G(S)$ such that $G(S) = T(G(R)).G(S)^{G(R)}$. It is straight forward to show that:
(1) $T(Z(G(R))) \subseteq Z(G(S)) \subseteq G(S)^{G(R)}$.
(2) The inverse image $T^{-1}(p)$ of a (graded) prime ideal of $G(S)$ is a (graded) prime ideal of $G(R)$.

Let us endow the graded prime spectrum $X = Spec^g(G(R))$ (similar to $Y = Spec^g(G(S))$) with the so-called Zariski topology, by letting the open sets (then basic affine Noetherian open sets, see [9,10]) for this topology to be the sets $X(f) = \{p \in X : f \notin p\}$, where f runs through the homogeneous elements of $G(R)$.

In general, $Spec^g(G(R))$ is not a scheme. However in case $G(R)$ is positively graded, then we write $Proj(G(R))$ for the Zariski open subset of X consisting of the graded prime ideals not containing $G(R)_+ = \bigoplus_{n>0} G(R)_n$, and in this case the closed set $V(G(R)_+)$ in X is nothing but $Spec(G(R)_0)$. Therefore $P(X) = Proj(G(R)) = \{p \in X : G(R)_n \not\subset p$ for some $n > 0\}$. It is clear that $X = p(X)$ if and only if $G(R)_n.G(R)_{-n} = G(R)$ for all $n > 0$.

We then have the following result:
4.1. Proposition. Any filtered Procesi extension $\varphi : R \to S$ induces a continuous morphism

$${}^aT : Spec^g(G(S)) = Y \to X = Spec^g(G(R)), p \mapsto T^{-1}(p) = G(\varphi)^{-1}(p).$$

Proof. Let $p \in Y = Spec^g(G(S))$, and assume $r_1.G(R).r_2 \subset T^{-1}(p) = {}^aT(p)$, for some $r_1, r_2 \in h(G(R))$ such that $r_2 \notin T^{-1}(p)$. Now, $T(r_1)G(S)T(r_2) = T(r_1)T(G(R))G(S)^{G(R)}T(r_2) = T(r_1)T(G(R))T(r_2)G(S)^{G(R)} \subseteq p\,G(S)^{G(R)} = p$, where $T(r_1), T(r_2) \in h(G(S))$ and $T(r_2) \notin p \in Y$. Then $T(r_1) \in p$ and so $r_1 \in T^{-1}(p)$. Therefore $T^{-1}(p) \in X$. On the other hand, we leave it as straightforward verification, that for any $f \in h(G(R))$ we have ${}^aT^{-1}(X(f)) = Y(T(f)); T(f) \in h(G(S))$, which shows that ${}^aT$ is continuous. This proves our assertion. □

Let us consider the behaviour of the Procesi extensions with respect to affine schemes: If $f \in h(G(R)) \cap Z(G(R))$ then, by the exactness of the localization functors, $G(\varphi) = T : G(R) \to G(S)$ induces a graded Procesi extension, over the localization level, $G(R)_f \xrightarrow{T_f} G(S)_{T(f)}$ such that

$$G(S)_{T(f)} = T_f(G(R)_f).(G(S)_{T(f)})^{G(R)_f},$$

and

$$G(R)_f = Q^g_f(G(R)), G(S)_{T(f)} = Q^g_{T(f)}(G(S));$$

the graded localization at f and $T(f)$, respectively. Thus $T = G(\varphi)$ induces a graded ring extension

$$\underline{O}^g_Y(Y(f)) \longrightarrow \underline{O}^g_X(X(T(f)))$$

which already is compatible with the restriction graded homomorphism of the graded structure sheaves $\underline{O}^g_Y, \underline{O}^g_X$. Hence we have a graded sheaf extension $T^{sheaf} : \underline{O}^g_Y \longrightarrow \underline{O}^g_X$. Again, by the exactness of the localization functors associated to p and ${}^aT(p) = q, T$ induces a graded local ring extension of the stalks;

$$T^{sheaf}_p : \underline{O}^g_{Y,p} \longrightarrow \underline{O}^g_{X,{}^aT(p)}.$$

Therefore we have proved the following useful result.

4.2. Proposition. The filtered Procesi extension $\varphi : R \to S$ such that $S = \varphi(R).S^R$ induces a graded Procesi extension (in the above sense)

$$(Y = Spec^g(G(S)), \underline{O}^g_Y) \to (X = Spec^g(G(R)), \underline{O}^g_X)$$

of graded affine schemes. □

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