Algebra and Discrete Mathematics
Number 4. **(2007).** pp. 123 – 130 RESEARCH ARTICLE

Wreath product of metric spaces

Bogdana Oliynyk

Communicated by M. Ya. Komarnytskyj

Dedicated to Professor V. V. Kirichenko on the occasion of his 65th birthday

Abstract. This paper describes a new construction of wreath product of metric spaces. The group of isometries of the wreath product of metric spaces is calculated.

1. Introduction

In 1959 F.Harrary [1] and G.Sabidussi [2] introduced a new construction of composition of graphs. Later this construction was called the wreath product of graphs.

Definition 1. *The wreath product of two simple graphs G_1 and G_2 is defined to be a graph whose vertices are the ordered pairs (v, w) for which v is a vertex of G_1, and w is a vertex of G_2. There is an arc between (v, w) and (v_1, w_1) if either of following holds:*
- *$v = v_1$ and there is an arc between w and w_1 in G_2*
- *there is an arc between v and v_1 in G_1*

Denote the wreath product of graphs G_1 and G_2 by $G_1 \wr G_2$.

Recall, that a graph G is called locally finite if every its vertex has finite degree.

The main result of Sabidussi about wreath product of graphs is

The research was partially supported by Fundamental Research State Fund, project F25.1/096

2000 Mathematics Subject Classification: *28D15.*

Key words and phrases: *wreath product, metric space, isometry group.*

Theorem 1. *(Sabidussi, 1959)*
If G_1 is a locally finite graph and G_2 is a finite graph, then the wreath product $G_1 \wr G_2$ of these graphs have the automorphism group isomorphic to $(AutG_1) \wr (AutG_2)$.

In 2003 this result was strengthen by E.Dobson and J.Morris.

Theorem 2. *(Dobson–Morris, 2003)*
Let G_1 be some graph and G_2 be a graph not isomorphic to a proper induced subgraph of itself. Then the wreath product $G_1 \wr G_2$ of these graphs have the automorphism group isomorphic to $(AutG_1) \wr (AutG_2)$.

Dobson and Morris proved that one can not omit the condition on the graph G_2.

G.Sabidussi in [2] considered slightly different wreath products for directed graphs end colored directed graphs. These constructions was also considered by E.Dobson and J.Morris.

Some quite different construction of wreath product of graphs was considered by A.Erschler in [3].

The main purpose of this paper is to introduce a notion of the wreath product of metric spaces that is analogous to the Sabidussi's and Harrary's one. We consider also isometry groups of the wreath products of metric spaces.

2. Construction

Following [4] metric spaces (X, d_X) and (Y, d_Y) are called isomorphic if there exists a scale, that is a strictly increasing continuous function $s : \mathbb{R}^+ \to \mathbb{R}^+$, $s(0) = 0$, such that $d_X = s(d_Y)$.

It is easy to observe that if metric spaces (X, d_X) and (Y, d_Y) are isomorphic then their isometry groups $IsomX$ and $IsomY$ are isomorphic.

Let (X, d_X) and (Y, d_Y) be metric spaces. Assume that there exists a positive number r, such that for arbitrary points $x_1, x_2 \in X$, $x_1 \neq x_2$, the inequality $d_x(x_1, x_2) \geqslant r$ holds. Additionally assume that the diameter $diamY$ of the space (Y, d_Y) is finite. Then fix a scale $s(x)$ such that

$$diam(s(Y)) < r. \tag{1}$$

Define a function ρ_s on the cartesian product $X \times Y$ by the rule:

$$\rho_s((x_1, y_1), (x_2, y_2)) = \begin{cases} d_X(x_1, x_2), & \text{if } x_1 \neq x_2 \\ s(d_Y(y_1, y_2)), & \text{if } x_1 = x_2 \end{cases}, \tag{2}$$

where $x_1,\ x_2 \in X$, $y_1,\ y_2 \in Y$.

The function ρ_s is a metric on the set $X \times Y$.

We call $(X \times Y, \rho_s)$ the wreath product of spaces X and Y with scale s and denote by $X wr_s Y$.

Proposition 1. *Let s_1 and s_2 be scales such that the inequality (1) holds. Then spaces $(X \times Y, \rho_{s_1})$ and $(X \times Y, \rho_{s_2})$ are isomorphic.*

Proof. Let $r_0 = diam(s_2(Y))$. Since for s_2 the inequality (1) holds, it follows that $r_0 < r$. Define a new function $\widehat{S}(x)$ on the $\mathbb{R}^+$ by the rule:

- if $x < r_0$, then $\widehat{S}(x) = s_1(s_2^{-1}(x))$;
- if $r_0 \leqslant x \leqslant r$, then $\widehat{S}(x)$ is a line segment joining the points $(r_0, s_1(s_2^{-1}(r_0))$ and (r, r);
- if $x > r$, then $\widehat{S}(x) = x$.

It is clear that $\widehat{S}(x)$ is a scale and $\widehat{S(\rho_{s_2})} = \rho_{s_1}$. Then spaces $(X \times Y, \rho_{s_1})$ and $(X \times Y, \rho_{s_2})$ are isomorphic. □

Another way to describe the wreath product of X and Y with scale s is as follows. Each point of X is replaced by an isometric copy of $s(Y)$, that is by an isomorphic copy of Y. And the distance between two points u and v is defined by the rule:

- if u and v belong to the same copy of $s(Y)$ then the distance between these points is $s(d_Y(u, v))$;
- if u and v belong to different copies of $s(Y)$, $u \in s(Y)_{x_1}$ and $v \in s(Y)_{x_2}$, $x_1 \neq x_2$ then the distance between them is $d_X(x_1, x_2)$.

Since the wreath product of metric spaces is unique up to isomorphism we assume in the sequel that corresponding scale is fixed. Denote the wreath product of metric spaces X and Y by $X wr Y$

It is easy to see that the operation of wreath product of metric spaces is not commutative.

Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Assume that for spaces X and Y a non-zero distances d_X and d_Y are bounded low by a positive number. Additionally assume that the diameters $diamY$ and $diamZ$ of the space (Y, d_Y) and (Z, d_Z) are finite. Then this proposition can be proved by direct calculations.

Proposition 2. *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces as above. Then spaces $(X wr Y) wr Z$ and $X wr (Y wr Z)$ are isomorphic.*

3. Main properties

Let (X, d_X) and (Y, d_Y) be metric spaces as considered before. The following properties of the wreath product of metric spaces hold.

Theorem 3. *1) $XwrY$ is ultrametric space iff X and Y are ultrametric spaces.*

2) $XwrY$ is a separable space iff X and Y are separable metric spaces.

3) $XwrY$ is a complete space iff Y is a complete metric space.

4) $XwrY$ is a compact space iff X is finite and Y is compact.

Proof. 1) Let X and Y be ultrametric spaces. Let us show that for arbitrary points (x_1, y_1), (x_2, y_2), (x_3, y_3) from $X \times Y$ the inequalities

$$\rho_s(x_i, y_i) \leqslant \max\{\rho_s(x_j, y_j), \rho_s(x_k, y_k)\}, \ \ 1 \leqslant i, j, k \leqslant 3 \tag{3}$$

hold. The proof consists of 3 cases.

Case 1: $x_1 \neq x_2$, $x_2 \neq x_3$, $x_1 \neq x_3$, then by definition of ρ_s we obtain that the distance between (x_i, y_i) and (x_j, y_j), $1 \leqslant i, j \leqslant 3$, is equal to the distance between x_i and x_j in (X, d_X). Since this space is ultrametric the inequalities (3) hold.

Case 2. $x_1 = x_2 = x_3$, then by (2) we obtain that the distance between (x_i, y_i) and (x_j, y_j) is equal to the distance between y_i and y_j in $(Y, s(d_Y))$, $1 \leqslant i, j \leqslant 3$. Since Y is ultrametric then $s(Y)$ is ultrametric too and the inequalities (3) hold.

Case 3. Without loss of generality it can be assumed that $x_1 = x_2 \neq x_3$. Using (2) we obtain

$$\rho_s((x_1, y_1), (x_3, y_3)) = d_X(x_1, x_3) = d_X(x_2, x_3) = \rho_s((x_2, y_2), (x_3, y_3)).$$

By (1) and $\rho_s((x_1, y_1), (x_2, y_2)) = s(d_Y(y_1, y_2))$ so that

$$\rho_s((x_1, y_1), (x_2, y_2)) < \rho_s((x_1, y_1), (x_3, y_3)) = \rho_s((x_2, y_2), (x_3, y_3)).$$

Therefore the inequalities (3) hold.

In opposite side. Let y be some point from Y. Assume that X is not ultrametric. Then there exist points x_1, x_2, $x_3 \in X$ such that the inequality

$$d_X(x_1, x_2) \leqslant \max\{d_X(x_1, x_3), d_X(x_2, x_3)\},$$

holds. By (2), it follows that for point (x_1, y), (x_2, y), (x_3, y) from $X \times Y$ the inequalities (3) do not hold. Hence $X \times Y$ is not ultrametric space. If the space Y is not ultrametric the proof is similar.

2) Let X and Y be separable metric spaces. Let $\widehat{X}$ be a countable everywhere dense subset of X. Since there exists a positive number r such that for arbitrary points $x_1, x_2 \in X$, $x_1 \neq x_2$, the inequality $d_x(x_1, x_2) \geqslant r$ holds, it follows that X is countable and $\widehat{X} = X$. Let $\widehat{Y}$ be a countable everywhere dense subset of Y. Then $X \times \widehat{Y}$ is a countable everywhere dense subset of $X \times Y$ and therefore $XwrY$ is a separable space.

Note that projections of the countable dense subset of $XwrY$ on each coordinate give countable dense subsets in both spaces X and Y.

3) Let $\{(x_n, y_n),\ n \geqslant 1\}$ be a fundamental sequence in $XwrY$. Then $\{x_n,\ n \geqslant 1\}$ is fundamental in X, and $\{y_n,\ n \geqslant 1\}$ is fundamental in Y. Due to completeness of Y the latter sequences converge. Since there exists a positive number r such that for arbitrary points $x_1, x_2 \in X$, $x_1 \neq x_2$, the inequality $d_x(x_1, x_2) \geqslant r$ holds, it follows that $\{(x_n, y_n),\ n \geqslant 1\}$ is convergent sequence in $XwrY$ iff $x_i = x_j = x$ for all $i, j \geqslant 1$ and $\{y_n,\ n \geqslant 1\}$ is convergent sequence in $s(Y)$. As Y is a complete metric space, then $s(Y)$ is also complete. Hence, there exists $\lim_{n\to\infty} y_n = y$ in the space $(Y, s(d_Y))$. It follows that $\lim_{n\to\infty}(x_n, y_n) = (x, y)$ in $(X \times Y, \rho_s)$.

Let the space Y is not complete. Then $s(Y)$ is not complete too. Hence there exists a fundamental sequence $\{y_n,\ n \geqslant 1\}$ that has no limit in (Y, d_Y). It follows that for arbitrary point $x \in X$ a fundamental sequence $\{(x, y_n),\ n \geqslant 1\}$ has no limit in $(X \times Y, \rho_s)$. Therefore $XwrY$ is not complete.

4) Let $X = \{x_1, x_2, \dots, x_n\}$, and the space Y is compact. Consider a covering $\{Q_\alpha,\ \alpha \in I\}$ of $XwrY$. Since X is finite, we see that $XwrY = \bigcup_{i=1}^n s(Y)_i$, where $s(Y)_i$ is an isometric copy of $s(Y)$ corresponding x_i, $1 \leqslant i \leqslant n$. Then we can consider the covering $\{Q_\alpha,\ \alpha \in I\}$ as a finite union of $\{Q_{\alpha_i},\ \alpha_i \in I_i\}$, $I = \cup_{i=i}^n I_i$, where $\{Q_{\alpha_i},\ \alpha_i \in I_i\}$ is a covering of $s(Y)_i$. Since Y is a compact space, it follows that $s(Y)_i$ is also compact for $1 \leqslant i \leqslant n$. Therefore, for any $s(Y)_i$, $1 \leqslant i \leqslant n$ there exists a finite subcovering of the covering $\{Q_{\alpha_i},\ \alpha_i \in I_i\}$. It follows that there exists a finite subcovering of covering $\{Q_\alpha,\ \alpha \in I\}$.

Let X be an infinite set. Fix a number r such that for arbitrary points $x_1, x_2 \in X$, $x_1 \neq x_2$, the inequality $d_x(x_1, x_2) \geqslant r$ holds. Denote by $Q_{(x,y)}$ a ball $B((x, y), r_1)$ in $XwrY$ centered in $(x, y) \in X \times Y$ of positive radius $r_1 < r$. Then the set $\{Q_{(x,y)} : (x, y) \in X \times Y\}$ is an infinite covering of $XwrY$ that does not posses any finite subcovering. Then the space $XwrY$ is not compact.

Let $X = \{x_1, x_2, \dots, x_n\}$ and Y is not. Then there exists an infinite covering $\{Q_\alpha,\ \alpha \in I\}$ of Y that does not contain a finite subcovering. Hence for every isomorphic copy $s(Y)_i$, $1 \leqslant i \leqslant n$ there exists a covering $\{Q_{\alpha_i},\ \alpha_i \in I_i\}$, that does not contain a finite subcovering. Then $\{Q_{\alpha_i},\ \alpha_i \in \cup_{i=1}^n I_i\}$ is a covering of $XwrY$, that does not contain a finite subcovering. This completes the proof. □

This theorem immediately implies

Corollary 1. *If X and Y are Polish spaces then $XwrY$ is a Polish space.*

The main result of this report is the following

Theorem 4. *The isometry group of the wreath product of metric spaces X and Y is isomorphic as a permutation group to the wreath product of isometry groups of spaces X and Y*

$$Isom(X wr Y) \simeq IsomX \wr IsomY.$$

Proof. At first let us prove that

$$\varphi = [g, h(x)] \in IsomX \wr IsomY$$

is an isometry of $X wr Y$. By definition of the wreath product of permutation groups ([5]) φ acts on $X \times Y$. We shall see that φ preserves the metric ρ_s. Indeed,

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) = \rho_s((x_1^g, y_1^{h(x_1)}), (x_2^g, y_2^{h(x_2)})) = \begin{cases} d_X(x_1^g, x_2^g), & \text{if } x_1^g \neq x_2^g \\ s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})), & \text{if } x_1^g = x_2^g. \end{cases}$$

Since $g \in IsomX$, it follows that $x_1^g = x_2^g$ iff $x_1 = x_2$. Then $h(x_1) = h(x_2)$, that is $h(x_1)$ and $h(x_2)$ define the same isometry t of Y. Note that t is an isometry of Y iff t is isometry of $s(Y)$. Hence,

$$s(d_Y(y_1^{h(x_1)}, y_2^{h(x_2)})) = s(d_Y(y_1^{h(x_1)}, y_2^{h(x_1)}) = s(d_Y(y_1, y_2)).$$

Since $g \in IsomX$, it means that $d_X(x_1^g, x_2^g) = d_X(x_1, x_2)$. Therefore

$$\rho_s(\varphi(x_1, y_1), \varphi(x_2, y_2)) = \begin{cases} d_X(x_1, x_2), & \text{if } x_1 \neq x_2 \\ s(d_Y(y_1, y_2)), & \text{if } , x_1 = x_2. \end{cases}$$

This means that φ is an isometry of $X wr Y$.

Now let us prove that for any isometry φ of $X wr Y$ there exists $g \in IsomX$ and $h(x) \in IsomY^X$, where $[g, h(x)]$ acts on $X \times Y$ as φ does. Let the function φ maps some point (x_1, y_1) to (x_2, y_2). Using (2), we obtain that the function φ maps any point of the form $(x_1, \star)$ to point of the form $(x_2, \star)$. It follows that φ acts as an isometry on each isometric copy $s(Y)_x$, $x \in X$. In each copy $s(Y)_x$ choose a point y_x. Then φ is an isometry on $\{y_x, \ x \in X\}$. This implies that there exists $g \in IsomX$ and $h(x) \in Isoms(Y)^X$, where $[g, h(x)]$ acts on $X \times Y$ as φ does. Since $Isom(s(Y)) \simeq IsomY$, it follows that we can consider $[g, h(x)]$ as $[g, h(x)] \in IsomX \wr IsomY$

□

4. Examples

Observe, that constructions of wreath product of graphs and wreath product of metric spaces are different. We can consider arbitrary simple graph as a metric space. The distance between two point of such a graph is the length of the shortest path between them in this graph. For arbitrary simple graphs G_1 and G_2 the metric spaces $G_1 \wr G_2$ and $G_1 wr G_2$ are different (indeed not isomorphic).

Example 1. *Let G_1 and G_2 be two isomorphic simple graphs. The sets of vertices of these graphs consist of two points v_1 and v_2. The points v_1 and v_2 are connected by an edge. Then $G_1 \wr G_2$ is a complete graph on 4 points. If we consider G_1 and G_2 as a metric spaces, then $G_1 wr G_2$ is a metric spaces with the following matrix of distances*

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & 0 & 1 & 1 \\ 1 & 1 & 0 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} & 0 \end{pmatrix}.$$

Observe, that metric spaces $G_1 \wr G_2$ and $G_1 wr G_2$ are not isomorphic.

Example 2. *The stairs into the sky.*

Conceder two metric spaces: $\mathbb{Z}$ and $[a, b]$ both with the euclidean metric. Let $X = \mathbb{Z}$, $Y = [a, b]$ and denote $Stairs = \mathbb{Z} wr [a, b]$. This space consist of countable enumerated family of pairwise disjoint segments $\bigcup_{i=-\infty}^{\infty} [c_i, d_i]$. The length of each segment is less than 1 $|c_i - d_i| < 1$, $i \in Z$. The distance between points of this space is defined by the following rule:

- *if points belong to the same segment then the distance between them is simply the euclidean distance,*
- *if points belong to the segments with number k and l, $k \neq l$ then the distance between them is equal to $|k - l|$.*

This space is complete. The isometry group of Stairs is isomorphic to the permutational wreath product of the infinite dihedral group and the cyclic group of order 2:

$$Isom(Stairs) \simeq D_\infty \wr C_2.$$

References

[1] Harrary F. On the group of composition of two graphs // Duke Math J. 26(1959),p. 29–34

[2] Subidussi G. The composition of graphs // Duke Math J. 26(1959),p. 693–696

[3] Eshler A. Generalized wreath products // IMRN, volume 2006, article ID57835, p.1–14

[4] Shoenberg I.J. Metric spaces and completely monotone functions// Ann. Math.- 1938.- v.39 N4.- p.811-841

[5] Kaloujnine L.A., Beleckij P.M., Feinberg V.T. Kranzprodukte. Leipzig: BSB B.G. Teubner Verlagsgesellschaft, 1987.- 168p.

Contact information

B. V. Oliynyk Department of Informatics, National University "Kyiv-Mohyla Academy", Skovorody 2, 04070 Kyiv Ukraine
E-Mail: bogd@ukma.kiev.ua

Received by the editors: 28.02.2008
and in final form 09.04.2008.