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RESEARCH ARTICLE

Hereditary stable tubes in module categories

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Dedicated to Kunio Yamagata on the occasion of his sixtieth birthday

Abstract. The concepts of a self-hereditary stable tube $\mathcal{T}$ and a hereditary stable tube $\mathcal{T}$ in a module category $\operatorname{mod} A$ are introduced, where A is a finite dimensional algebra over an algebraically closed field. Characterisations of self-hereditary stable tubes and hereditary stable tubes are given, and illustrative examples of such tubes are presented. Some open problems are presented.

1. Introduction

Throughout we denote by K an algebraically closed field, by A a finite dimensional K-algebra, and by $\operatorname{mod} A$ the category of finite dimensional right A-modules. Given an algebra A, we denote by $\Gamma(\operatorname{mod} A)$ the Auslander-Reiten quiver of $\operatorname{mod} A$, viewed as a translation quiver with respect to the Auslander-Reiten translation $\tau_A = D\mathrm{Tr}$, where $D = \operatorname{Hom}_K(-, K) : \operatorname{mod} A^{op} \to \operatorname{mod} A$ is the standard K-duality and Tr is the transpose operator of Auslander, see [1], [2], and [16].

In the representation theory of algebras an essential role is playing by special type of components of the Auslander-Reiten quivers, called stable tubes. In particular, by a result of Crawley-Boevey [4], for every algebra A of tame representation type, all but finitely many indecomposable A-modules of any fixed K-dimension lie in homogeneous tubes of $\Gamma(\operatorname{mod} A)$.

We recall from [5] and [15] that a connected component $\mathcal{T}$ of $\Gamma(\operatorname{mod} A)$ is said to be a **stable tube of rank** $r = r_{\mathcal{T}} \geq 1$ if there is an isomorphism

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of translation quivers $\mathcal{T} \cong \mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, see also [13] and [14]. A stable tube of rank $r = 1$ is defined to be a **homogeneous tube**, see also [17]. It turns out that it is of importance to study hereditary tubes and self-hereditary tubes in the following sense.

Definition 1.1. *Let $\mathcal{T}$ be a stable tube of $\Gamma(\mathrm{mod}\, A)$.*

(a) *The tube $\mathcal{T}$ is defined to be* **hereditary**, *if* $\mathrm{pd}_A X \leq 1$ *and* $\mathrm{id}_A X \leq 1$, *for any indecomposable module X in $\mathcal{T}$.*

(b) *The tube $\mathcal{T}$ is defined to be* **self-hereditary**, *if* $\mathrm{Ext}^2_A(X, Y) = 0$, *for all indecomposable modules X and Y in $\mathcal{T}$.*

(c) *A sequence $(X_1, \ldots, X_r)$ of indecomposable A-modules in $\mathcal{T}$ is said to be a* τ_A**-cycle** *of length $r \geq 1$, if*

$$\tau_A X_1 \cong X_r, \tau_A X_2 \cong X_1, \ldots, \tau_A X_r \cong X_{r-1}.$$

Obviously, every stable tube $\mathcal{T}$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of a hereditary algebra A is hereditary and, clearly, every hereditary stable tube $\mathcal{T}$ is self-hereditary, but the converse implication does not hold in general, see Section 4. Note also that, given a self-hereditary (resp. hereditary) stable tube $\mathcal{T}$, the minimal additive subcategory $\mathrm{add}\, \mathcal{T}$ of $\mathrm{mod}\, A$ containing $\mathcal{T}$ is self-hereditary (resp. hereditary).

Furthermore, it follows from [1, VIII.4.5], [15], [17, XI.3.3], and the structure theorem [17, XII.3.4] that, given a concealed algebra B of Euclidean type, the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, B)$ of B has a disjoint union decomposition

$$\Gamma(\mathrm{mod}\, B) = \mathcal{P}(B) \cup \boldsymbol{\mathcal{T}}^B \cup \mathcal{Q}(B),$$

where $\mathcal{P}(B)$ is the unique postprojective component of $\Gamma(\mathrm{mod} B)$ containing all the indecomposable projective B-modules, $\mathcal{Q}(B)$ is the unique preinjective component containing all the indecomposable injective A-modules, and

$$\boldsymbol{\mathcal{T}}^B = \{\mathcal{T}^B_\lambda\}_{\lambda \in \mathbb{P}_1(K)}$$

is a $\mathbb{P}_1(K)$-family of pairwise orthogonal hereditary standard stable tubes $\mathcal{T}^A_\lambda$. In particular, every tube $\mathcal{T}$ in $\Gamma(\mathrm{mod}\, B)$ is stable, standard, and hereditary.

We recall from [15] that a stable tube $\mathcal{T}$ in $\Gamma(\mathrm{mod} A)$ is said to be **standard**, if there is an equivalence of K-categories

$$K(\mathcal{T}) = K\mathcal{T}/M_{\mathcal{T}} \cong \mathrm{ind}\, \mathcal{T}, \tag{1.2}$$

where $\mathrm{ind}\, \mathcal{T}$ is the full K-subcategory of $\mathrm{mod}\, A$ whose objects are representatives of the isomorphism classes of the indecomposable modules in $\mathcal{T}$ and $K(\mathcal{T}) = K\mathcal{T}/M_{\mathcal{T}}$ is the mesh category of $\mathcal{T}$, see also [1] and [17].

One of the main results of the paper (Theorem 3.8) asserts that given a standard stable tube $\mathcal{T}$ of $\Gamma(\operatorname{mod} A)$ such that the mouth A-modules of the tube $\mathcal{T}$ are pairwise orthogonal bricks and the quotient algebra $B = A/\mathrm{Ann}_A\mathcal{T}$ is a projective right A-module and a projective left A-module then the following four conditions are equivalent:

(a) the tube $\mathcal{T}$ is hereditary,
(b) $\mathrm{pd}_A X \leq 1$, for any indecomposable A-module X of $\mathcal{T}$,
(c) $\mathrm{id}_A X \leq 1$, for any indecomposable A-module X of $\mathcal{T}$, and
(d) the tube $\mathcal{T}$ is self-hereditary.

Here $\mathrm{Ann}_A\mathcal{T}$ is the annihilator of $\mathcal{T}$, that is, the intersection of the annihilators of all modules from $\mathcal{T}$.

In Section 3, we also study generalised standard stable tubes in the sense of Skowroński [20], that is, the stable tubes $\mathcal{T}$ of $\Gamma(\operatorname{mod} A)$ such that $\mathrm{rad}_A^{\infty}(X, Y) = 0$, for each pair of indecomposable modules X and Y in $\mathcal{T}$, where $\mathrm{rad}_A^{\infty} = \mathrm{rad}^{\infty}(\operatorname{mod} A)$ is the infinite radical of the category $\operatorname{mod} A$, see [1, Appendix], [2, Section V.7], and [16, Section 11.1].

It was shown in [21], [22], and [24] that a stable tube $\mathcal{T}$ of $\Gamma(\operatorname{mod} A)$ is standard if and only if $\mathcal{T}$ is generalised standard. By applying this equivalence, we prove that, given a faithful stable tube $\mathcal{T}$ of $\Gamma(\operatorname{mod} A)$, the following three conditions are equivalent:

(a) $\mathcal{T}$ is standard,

(b) $\mathcal{T}$ is hereditary and the mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks,

(c) $\mathcal{T}$ is self-hereditary and the mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks.

We recall that a tube $\mathcal{T}$ is said to be **faithful** if $\mathrm{Ann}_A\mathcal{T} = 0$.

In Section 4 we present four examples of standard stable tubes. In the first one we construct an algebra C and a standard stable tube $\mathcal{T}$ of rank $r = 2$ in $\Gamma(\operatorname{mod} C)$ with the mouth modules E and S such that

(i) $\mathrm{pd}_C S = 2$, $\mathrm{id}_C S = 1$, $\mathrm{pd}_C E = 1$, and $\mathrm{id}_C E = 1$,
(ii) $\mathrm{id}_C X = 1$, for any indecomposable module X lying in the tube $\mathcal{T}$,
(iii) $\mathrm{gl.dim}\, C = 2$, and
(iv) the tube $\mathcal{T}$ is self-hereditary, but it is not hereditary.

In the second example we construct an algebra B such that $\mathrm{gl.dim}\, B = 3$ and $\Gamma(\operatorname{mod} B)$ admits a (generalised) standard stable tube that is self-hereditary, but is neither faithful nor hereditary. In the third one we construct an algebra R such that $\mathrm{gl.dim}\, R = \infty$ and $\Gamma(\operatorname{mod} R)$ admits a standard stable tube $\mathcal{T}$ that is not self-hereditary. In the fourth example, we construct a self-injective algebra Λ (with $\mathrm{gl.dim}\,\Lambda = \infty$) such that $\Gamma(\operatorname{mod}\Lambda)$ admits a self-hereditary (generalised) standard stable tube consisting entirely of modules of infinite projective and infinite injective dimension.

The reader is referred to [17, Chapters X and XI] and [18, Chapters XV-XVII] for further results on the hereditary tubes and application of the results presented here. We freely use the terminology and notation introduced in [1], [6], [17] and [18].

2. The extension category and self-hereditary tubes

We start by recalling a construction of Ringel [15] that shows how hereditary and self-hereditary stable tubes occur as components of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of an arbitrary algebra A.

We recall that a **brick** E in $\operatorname{mod} A$ is an indecomposable right A-module E such that $\operatorname{End} E \cong K$. Two bricks E and E' in $\operatorname{mod} A$ are called **orthogonal** if $\operatorname{Hom}_A(E, E') = 0$, and $\operatorname{Hom}_A(E', E) = 0$.

Assume that $\mathbf{E} = \{E_1, \ldots, E_r\}$ is a family of pairwise orthogonal bricks in $\operatorname{mod} A$. Following Ringel [15], we consider the full subcategory

$$\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_A(E_1, \ldots, E_r) \tag{2.1}$$

of $\operatorname{mod} A$ (called an **extension category**) whose non-zero objects are all the modules M such that there exists a chain of submodules $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_l = 0$, for some $l \geq 1$, with M_i/M_{i+1} isomorphic to one of the bricks $E_1, \ldots, E_r$, for all $i \in \{0, 1, \ldots, l-1\}$. In other words, $\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_A(E_1, \ldots, E_r)$ is the smallest additive subcategory of $\operatorname{mod} A$ containing the bricks $E_1, \ldots, E_r$ and closed under extensions.

We recall that an algebra A is hereditary if $\operatorname{Ext}^2_A(X, Y) = 0$, for each pair of A-modules X and Y in $\operatorname{mod} A$, or equivalently, if $\operatorname{pd}_A X \leq 1$ and $\operatorname{id}_A X \leq 1$, for any indecomposable (even simple) module X in $\operatorname{mod} A$. Following this definition, we introduce some new concepts that are useful in the study of hereditary and self-hereditary tubes.

Definition 2.2. *Let A be a finite dimensional K-algebra and let $\mathcal{H}$ be a family of modules in* $\operatorname{mod} A$*, or a full subcategory of* $\operatorname{mod} A$*.*

(a) *$\mathcal{H}$ is defined to be a* **hereditary family** *of* $\operatorname{mod} A$*, if* $\operatorname{pd}_A X \leq 1$ *and* $\operatorname{id}_A X \leq 1$*, for any module X in $\mathcal{H}$.*

(b) *$\mathcal{H}$ is defined to be a* **self-hereditary family** *of* $\operatorname{mod} A$*, if* $\operatorname{Ext}^2_A(X, Y) = 0$*, for each pair of A-modules X and Y in $\mathcal{H}$.*

(c) *A connected component $\mathcal{C}$ of the Auslander-Reiten quiver* $\Gamma(\operatorname{mod} A)$ *of* $\operatorname{mod} A$ *is defined to be* **self-hereditary***, if* $\operatorname{Ext}^2_A(X, Y) = 0$*, for each pair of A-modules X and Y in $\mathcal{C}$. A component $\mathcal{C}$ is defined to be* **hereditary***, if* $\operatorname{pd}_A X \leq 1$ *and* $\operatorname{id}_A X \leq 1$*, for any module X in $\mathcal{C}$.*

It is clear that $\mathcal{H} = \operatorname{mod} A$ is a hereditary subcategory of $\operatorname{mod} A$ if and only if $\mathcal{H} = \operatorname{mod} A$ is self-hereditary. Note also that any hereditary family $\mathcal{H}$ of $\operatorname{mod} A$ is self-hereditary. The examples of Section 4 show that the converse implication does not hold in general.

Basic properties of the extension category $\mathcal{EXT}_A(E_1,\dots,E_r)$ are collected in the following lemma.

Lemma 2.3. *Let A be an algebra, and $\mathbf{E}=\{E_1,\dots,E_r\}$ a finite family of pairwise orthogonal bricks in* $\operatorname{mod} A$. *The extension category $\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_A(E_1,\dots,E_r)$* (2.1) *has the following properties.*

(a) *$\mathcal{E}_{\mathbf{E}}$ is an exact abelian subcategory of* $\operatorname{mod} A$ *that is closed under extensions in* $\operatorname{mod} A$, *and $\mathbf{E}=\{E_1,\dots,E_r\}$ is a complete set of pairwise non-isomorphic simple objects in $\mathcal{E}_{\mathbf{E}}$.*

(b) *The finite set $\mathbf{E}=\{E_1,\dots,E_r\}$ is a hereditary family of* $\operatorname{mod} A$ *if and only if $\mathcal{E}_{\mathbf{E}}$ is a hereditary subcategory of* $\operatorname{mod} A$.

(c) *The finite set $\mathbf{E}=\{E_1,\dots,E_r\}$ is a self-hereditary family of* $\operatorname{mod} A$ *if and only if $\mathcal{E}_{\mathbf{E}}$ is a self-hereditary subcategory of* $\operatorname{mod} A$.

Proof. The statement (a) is proved in [15], see also[17, Chapter X].

(b) The sufficiency is obvious. To prove the necessity, we assume that the set $\mathbf{E}=\{E_1,\dots,E_r\}$ is a hereditary family of $\operatorname{mod} A$, that is, $\operatorname{pd}_A E_j \le 1$ and $\operatorname{id}_A E_j \le 1$, for any $j\in\{1,\dots,r\}$, or equivalently, $\operatorname{Ext}^2_A(E_j,Y)=0$ and $\operatorname{Ext}^2_A(X,E_j)=0$, for any $j\in\{1,\dots,r\}$ and for all modules X and Y in $\operatorname{mod} A$.

We fix a module Y in $\operatorname{mod} A$ and we prove that $\operatorname{Ext}^2_A(X,Y)=0$, for any module X in $\mathcal{E}_{\mathbf{E}}=\mathcal{EXT}_A(E_1,\dots,E_r)$. If X is any of the modules $E_1,\dots,E_r$, the assumption gives the result. Assume that X is an arbitrary non-zero object of $\mathcal{E}_{\mathbf{E}}$. Then X contains a submodule X_0 isomorphic to one of the modules $E_1,\dots,E_r$, because they form a complete list of simple objects of the category $\mathcal{E}_{\mathbf{E}}$, up to isomorphism, by (a). Then there exists an exact sequence

$$0\to X_0\to X\to \overline{X}\to 0$$

in the category $\mathcal{E}_{\mathbf{E}}$ and $\dim_K\overline{X}<\dim_K X$. Hence we derive the induced exact sequence

$$\operatorname{Ext}^2_A(\overline{X},Y)\longrightarrow \operatorname{Ext}^2_A(X,Y)\longrightarrow \operatorname{Ext}^2_A(X_0,Y).$$

Since, by induction, the left hand term and the right hand term of the sequence are zero then we get $\operatorname{Ext}^2_A(X,Y)=0$, and the required result follows by induction on $\dim_K X$. The equality $\operatorname{Ext}^2_A(X,Y)=0$, for any module X in $\operatorname{mod} A$ and any module Y in $\mathcal{E}_{\mathbf{E}}$, follows in a similar way.

(c) Apply the arguments used in the proof of (b). The details are left to the reader. $\square$

The following important theorem is essentially due to Ringel [15].

Theorem 2.4. *Let A be an algebra, and $(E_1, \dots, E_r)$, with $r \geq 1$, be a τ_A-cycle of pairwise orthogonal bricks in* mod A *such that* $\mathbf{E} = \{E_1, \dots, E_r\}$ *is a self-hereditary family of* mod A. *The abelian category*

$$\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_A(E_1, \dots, E_r)$$

(see (2.1)) *has the following properties.*

(a) $\mathcal{E}_{\mathbf{E}}$ *has almost split sequences.*

(b) *All indecomposable objects of the category $\mathcal{E}_{\mathbf{E}}$ are uniserial of finite length in $\mathcal{E}_{\mathbf{E}}$ and they form a self-hereditary component $\mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$ of* $\Gamma(\mathrm{mod}\, A)$.

(c) *The component $\mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$ is a self-hereditary standard stable tube of rank r.*

(d) *The modules $E_1, \dots, E_r$ form a complete set of modules lying on the mouth of the tube $\mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$.*

Proof. By our assumption, $r \geq 1$ and the modules $E_1, \dots, E_r$ are pairwise orthogonal bricks such that

$$\tau_A E_1 \cong E_r, \tau_A E_2 \cong E_1, \dots, \tau_A E_r \cong E_{r-1}$$

and $\mathrm{Ext}_A^2(E_i, E_s) = 0$, for all $i, s \in \{1, \dots, r\}$. Then, in view of Lemma 2.3 (c), the theorem is a consequence of [15], see also [17, Chapter X]. □

Corollary 2.5. *Let A be an algebra, $\mathcal{T}$ a stable tube of rank $r \geq 1$ of* $\Gamma(\mathrm{mod}\, A)$, *and* $\mathbf{E} = \{E_1, \dots, E_r\}$ *a self-hereditary family of pairwise orthogonal bricks forming the mouth of the tube $\mathcal{T}$.*

(a) *The tube $\mathcal{T}$ is self-hereditary, $\mathcal{EXT}_A(E_1, \dots, E_r)$ is a self-hereditary uniserial abelian subcategory of* mod A, *and*

$$\mathrm{add}\, \mathcal{T} = \mathcal{EXT}_A(E_1, \dots, E_r).$$

(b) *If* $\mathbf{E} = \{E_1, \dots, E_r\}$ *is a hereditary family then the tube $\mathcal{T}$ is hereditary,* add $\mathcal{T} = \mathcal{EXT}_A(E_1, \dots, E_r)$ *is a hereditary uniserial abelian subcategory of* mod A, *and* $\mathcal{T} = \mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$.

Proof. Because $\mathcal{T}$ is a stable tube and $\mathbf{E} = \{E_1, \dots, E_r\}$ is a self-hereditary family of pairwise orthogonal bricks forming the mouth of the tube $\mathcal{T}$ then, up to permutation, $(E_1, \dots, E_r)$ is a τ_A-cycle and Lemma 2.3 and Theorem 2.4 apply to the extension category $\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_A(E_1, \dots, E_r)$. Hence, the intersection $\mathcal{T} \cap \mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$ of the components $\mathcal{T}$ and $\mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$ contains the modules $E_1, \dots, E_r$ and we get $\mathcal{T} = \mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$. Furthermore, in view of Theorem 2.4, it follows that

$$\mathrm{add}\, \mathcal{T} = \mathrm{add}\, \mathcal{T}_{\mathcal{E}_{\mathbf{E}}} = \mathcal{EXT}_A(E_1, \dots, E_r)$$

and the corollary follows from Lemma 2.3. □

In the notation of Corollary 2.5, given an indecomposable module M in $\mathcal{T}$, we denote by $\ell_{\mathcal{T}}(M)$ the $\mathcal{T}$-length of M, that is, the length of M in the uniserial abelian subcategory $\operatorname{add}\mathcal{T} = \mathcal{EXT}_A(E_1, \ldots, E_r)$ of $\operatorname{mod} A$.

Corollary 2.6. *Let A be an algebra and $\mathcal{T}$ a stable tube of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of $\operatorname{mod} A$. If $\mathcal{T}$ is self-hereditary and the mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks then $\mathcal{T}$ is standard.*

Proof. Assume that $\mathcal{T}$ is a self-hereditary stable tube of rank $r \geq 1$. Then there exits a τ_A-cycle $(E_1, \ldots, E_r)$ of mouth modules of $\mathcal{T}$ such that $\mathbf{E} = \{E_1, \ldots, E_r\}$ is a self-hereditary family of $\operatorname{mod} A$. Then Corollary 2.5 yields $\mathcal{T} = \mathcal{T}_{\mathcal{E}_{\mathbf{E}}}$ and, according to Theorem 2.4, the tube $\mathcal{T}$ is standard. □

3. Basic properties of self-hereditary and hereditary stable tubes

We start with a lemma showing that the hereditariness and the self-hereditariness of a stable tube $\mathcal{T}$ of rank $r \geq 1$ of $\Gamma(\operatorname{mod} A)$ is decided on the level of the set of all mouth modules of $\mathcal{T}$.

Lemma 3.1. *Let A be an algebra and $\mathcal{T}$ a stable tube in $\Gamma(\operatorname{mod} A)$.*

(i) *The tube $\mathcal{T}$ is self-hereditary if and only if the finite family of mouth modules of $\mathcal{T}$ is self-hereditary.*

(ii) *The tube $\mathcal{T}$ is hereditary if and only if the finite family of mouth modules of $\mathcal{T}$ is hereditary.*

Proof. We only prove (i), because the proof of (ii) is similar. The necessity is obvious. To prove the sufficiency, we assume that $\mathcal{T}$ is a stable tube of rank $r \geq 1$ in $\Gamma(\operatorname{mod} A)$ and $(E_1, \ldots, E_r)$ is a τ_A-cycle of mouth modules of $\mathcal{T}$ such that $\operatorname{Ext}^2_A(E_i, E_j) = 0$, for all $i, j \in \{1, \ldots, r\}$. By [15] and [17, Lemma X.1.4], for each $i \in \{1, \ldots, r\}$, there exists a unique ray

$$E_i = E_i[1] \longrightarrow E_i[2] \longrightarrow E_i[3] \longrightarrow \ldots \longrightarrow E_i[m] \longrightarrow E_i[m{+}1] \longrightarrow \ldots$$

of indecomposable modules in $\mathcal{T}$ starting at E_i. Moreover, for each $m \geq 1$, there exists an almost split sequence

$$(*) \qquad 0 \longrightarrow E_i[m] \longrightarrow E_i[m{+}1] \oplus E_{i+1}[m{-}1] \longrightarrow E_{i+1}[m] \longrightarrow 0$$

in $\operatorname{mod} A$, where we set $E_j[0] = 0$ and $E_{i+kr}[t] = E_i[t]$, for all $i \in \{1, \ldots, r\}$, $t \geq 1$, and $k \in \mathbb{Z}$.

We show that $\mathrm{Ext}_A^2(X,Y) = 0$, for each pair of indecomposable A-modules X and Y lying on the tube $\mathcal{T}$. Assume that X and Y are such modules. By a description of the indecomposable modules lying in $\mathcal{T}$ given in [15] and [17, Lemma X.1.4], there are isomorphisms $X \cong E_i[m]$ and $Y \cong E_j[n]$, for some $i, j \in \{1,\dots,r\}$ and some integers $m, n \geq 1$. First we assume that $n = 1$, that is, there is an isomorphism $Y \cong E_j[1] = E_j$.

We prove by induction on $m \geq 1$ that $\mathrm{Ext}_A^2(E_i[m], Y) = 0$. If $m = 1$ then $E_i[m] = E_i$ and we are done by our assumption. Assume that $m \geq 1$ is such that $\mathrm{Ext}_A^2(E_i[m], Y) = 0$ and $\mathrm{Ext}_A^2(E_i[m{-}1], Y) = 0$, for all $i \in \{1,\dots,r\}$. From the exact sequence $(*)$ we derive the induced exact sequence

$$\mathrm{Ext}_A^2(E_{i+1}[m], Y) \longrightarrow \mathrm{Ext}_A^2(E_i[m{+}1] \oplus E_{i+1}[m{-}1], Y) \longrightarrow \mathrm{Ext}_A^2(E_i[m], Y).$$

The induction hypothesis yields $\mathrm{Ext}_A^2(E_{i+1}[m{-}1], Y) = 0$, the left hand term and the right hand term of the sequence is zero. It follows that $\mathrm{Ext}_A^2(E_{i+1}[m{+}1], Y) = 0$, for $i \in \{0,\dots,r{-}1\}$. By the induction principle, we conclude that $\mathrm{Ext}_A^2(X,Y) = 0$, for Y and any indecomposable A-module X lying on the tube $\mathcal{T}$.

Applying the same arguments, we prove by induction on $n \geq 1$ that $\mathrm{Ext}_A^2(X, E_j[n]) = 0$, for any indecomposable A-module X lying on the tube $\mathcal{T}$. This finishes the proof. □

In the following section we give examples of self-hereditary stable tubes $\mathcal{T}$ in $\Gamma(\mathrm{mod}\, A)$ that are not hereditary, where A is an algebra of global dimension 2. We also show that neither of the two conditions $\mathrm{pd}_A X \leq 1$ and $\mathrm{id}_A X \leq 1$ in the Definition 1.1 (a) of hereditary tube can be dropped. However, we would like to answer the following interesting questions.

Questions 3.2. Let A be an algebra.

(i) Characterise the stable tubes $\mathcal{T}$ in $\Gamma(\mathrm{mod}\, A)$ such that one of the two conditions $\mathrm{pd}_A X \leq 1$ and $\mathrm{id}_A X \leq 1$ in the Definition 1.1 (a) of hereditary tube can be dropped.

(ii) When a self-hereditary stable tube $\mathcal{T}$ (resp. every self-hereditary stable tube $\mathcal{T}$) in $\Gamma(\mathrm{mod}\, A)$ is hereditary?

(iii) When a standard stable tube $\mathcal{T}$ (resp. every standard stable tube $\mathcal{T}$) in $\Gamma(\mathrm{mod}\, A)$ is hereditary (or self-hereditary)?

Remarks 3.3. (a) The examples of Section 4 show that the answer to the above questions is not affirmative in general. Below only partial answers are given.

(b) Algebras A for which all stable tubes $\mathcal{T}$ in $\Gamma(\operatorname{mod} A)$ are standard and hereditary are provided by quasitilted algebras of canonical type (see [7] and [8]), or more generally, by the algebras with separating families of almost cyclic coherent components studied in [9] and [11]. We also note that the class of quasitilted algebras of canonical type contains the representation-infinite tilted algebras of Euclidean type (see [15], [17] and [18]) and the tubular algebras [15].

(c) The class of algebras A having a standard hereditary stable tube $\mathcal{T}$ in $\Gamma(\operatorname{mod} A)$ is very large and contains algebras of arbitray high global dimension, see [10] and [24].

(d) On the other hand, if A is a connected algebra and $\mathcal{T}$ is a sincere stable tube without external short cycles, then $\mathcal{T}$ is a faithful standard stable tube (hence hereditary stable tube, see Theorem 3.4) and A is a concealed canonical algebra (of global dimension 2), see [12] and [24].

(e) The representation-infinite self-injective algebras of polynomial growth having strictly positive Galois covering form a large class of algebras whose all stable tubes $\mathcal{T}$ are self-hereditary, but not hereditary, see [19] and [27]-[29].

Throughout this paper, we use the notion of a generalised standard component introduced in [20]. We recall that a component $\mathcal{C}$ of $\Gamma(\operatorname{mod} A)$ is **generalised standard** if $\operatorname{rad}_A^\infty(X, Y) = 0$, for each pair of indecomposable modules X and Y in $\mathcal{C}$.

The following characterisation of generalised standard stable tubes given in [24, Lemma 1.3] is of importance, see also [20, Corollary 5.3] and [22, Lemma 3.2].

Theorem 3.4. *Let A be an algebra and let $\mathcal{T}$ be a stable tube of $\Gamma(\operatorname{mod} A)$. The following four statements are equivalent.*

(a) *$\mathcal{T}$ is standard.*

(b) *The mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks.*

(c) *$\mathcal{T}$ is generalised standard.*

(d) *$\operatorname{rad}_A^\infty(E, E') = 0$, for each pair of mouth modules E and E' of the tube $\mathcal{T}$.* □

We recall that the **right annihilator** of a right A-module M is the two-sided ideal $\operatorname{Ann}_A M = \{a \in A;\ Ma = 0\}$ of A. Recall also that the module M is said to be **faithful** if the ideal $\operatorname{Ann}_A M$ is zero.

The following result proved in [20, Lemma 5.9] shows that any faithful generalised standard stable tube is hereditary.

Theorem 3.5. *Let A be an algebra and $\mathcal{T}$ a faithful standard stable tube of $\Gamma(\mathrm{mod}\, A)$. Then $\mathrm{pd}_A X \leq 1$ and $\mathrm{id}_A X \leq 1$, for any indecomposable module X of $\mathcal{T}$, that is, the tube $\mathcal{T}$ is hereditary.*

Proof. For the sake of completeness, we include here the proof given in [20]. First we show that, given a component $\mathcal{C}$ of $\Gamma(\mathrm{mod}\, A)$,

1° there exists a module M in $\mathrm{add}\, \mathcal{C}$ such that $\mathrm{Ann}_A \mathcal{C} = \mathrm{Ann}_A M$, and

2° $\mathcal{C}$ is faithful if and only if the category $\mathrm{add}\, \mathcal{C}$ admits a faithful A-module.

To see 1°, for a given module X in add $\mathcal{C}$, we fix an isomorphism of A-modules $X \cong X_1 \oplus X_2 \oplus \ldots \oplus X_s$, where $X_1, X_2, \ldots, X_s$ are indecomposable A-modules in $\mathcal{C}$. Then

$$\mathrm{Ann}_A X = \mathrm{Ann}_A(X_1 \oplus X_2 \oplus \ldots \oplus X_s) = \mathrm{Ann}_A X_1 \cap \mathrm{Ann}_A X_2 \cap \ldots \cap \mathrm{Ann}_A X_s.$$

Moreover, $\mathrm{Ann}_A Y \supseteq \mathrm{Ann}_A Z$ if Y is a submodule of Z. Then the ideals of the form $\mathrm{Ann}_A(X_1 \oplus X_2 \oplus \ldots \oplus X_s)$, where $X_1, X_2, \ldots, X_s$ are indecomposable A-modules in the component $\mathcal{C}$, form a partially ordered set, with respect to the inclusion. Because the algebra A is finite dimensional then the family contains a minimal element $\mathrm{Ann}_A(M_1 \oplus M_2 \oplus \ldots \oplus M_\ell)$, for some indecomposable A-modules $M_1, M_2, \ldots, M_\ell$ in $\mathcal{C}$. It follows that $\mathrm{Ann}_A \mathcal{C} = \mathrm{Ann}_A M$, where $M = M_1 \oplus M_2 \oplus \ldots \oplus M_\ell$ is a module in $\mathrm{add}\, \mathcal{C}$. The statement 2° follows immediately from 1°.

Now assume that $\mathcal{T}$ is a faithful standard stable tube of $\Gamma(\mathrm{mod}\, A)$ and let X be an indecomposable A-module in $\mathcal{T}$. We only prove that $\mathrm{pd}_A X \leq 1$, because the proof of the inequality $\mathrm{id}_A X \leq 1$ is similar.

Assume, to the contrary, that $\mathrm{pd}_A X \geq 2$. Then, it follows from [1, IV.2.7] that $\mathrm{Hom}_A(D({}_AA), \tau_A X) \neq 0$. Let $f : D({}_AA) \longrightarrow \tau_A X$ be a non-zero homomorphism in mod A. Because the stable tube $\mathcal{T}$ is faithful then, by 2°, the category add $\mathcal{T}$ admits a faithful A-module M and it follows from [1, VI.2.2] that the A-module $D({}_AA)$ is cogenerated by M, that is, there exist an integer $t \geq 1$ and an epimorphism $h : M^t \longrightarrow D({}_AA)$ of A-modules. Hence, there exists an indecomposable direct summand Z of M^t such that the composite homomorphism

$$Z \xrightarrow{\quad g \quad} D({}_AA) \xrightarrow{\quad f \quad} \tau_A X$$

is non-zero, where g is the restriction of h to the summand Z of M^t. Note that the A-module $D({}_AA)$ is injective and the tube $\mathcal{T}$ contains no indecomposable injective A-modules. Then the indecomposable modules Z and $\tau_A X$ are not injective, because they lie in the tube $\mathcal{T}$. This yields

- $\mathrm{rad}_A(Z, D({}_AA)) = \mathrm{rad}_A^\infty(Z, D({}_AA))$, and

- $\mathrm{rad}_A(D(_AA), \tau_A X) = \mathrm{rad}_A^\infty(D(_AA), \tau_A X)$.

Consequently, we get $0 \neq fg \in \mathrm{rad}_A^\infty(Z, \tau_A X)$. This contradicts the assumption that the tube $\mathcal{T}$ is standard, because we know from Theorem 3.4 that any standard stable tube is generalised standard. The proof of the theorem is complete. □

Corollary 3.6. *Let A be an algebra, $\mathcal{T}$ a standard stable tube of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A, and $B = A/\mathrm{Ann}_A\mathcal{T}$. Then $\mathcal{T}$ is a hereditary standard stable tube of $\Gamma(\mathrm{mod}\, B)$, under the fully faithful exact embedding $\mathrm{mod}\, B \hookrightarrow \mathrm{mod}\, A$ induced by the canonical algebra surjection $A \longrightarrow B$.*

Proof. Apply Theorems 3.4 and 3.5. □

Proposition 3.7. *Let A be an algebra and $\mathcal{T}$ a faithful stable tube of $\Gamma(\mathrm{mod}\, A)$. The following five conditions are equivalent.*

(a) *$\mathcal{T}$ is standard.*

(b) *$\mathcal{T}$ is hereditary and the mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks.*

(c) *$\mathrm{pd}\, X \leq 1$, for each X in $\mathcal{T}$, and the mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks.*

(c′) *$\mathrm{id}\, X \leq 1$, for each X in $\mathcal{T}$, and the mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks.*

(d) *$\mathcal{T}$ is self-hereditary and the mouth modules of $\mathcal{T}$ are pairwise orthogonal bricks.*

Proof. The implication (a)⇒(b) is a consequence of Theorem 3.5. The implications (b)⇒(c)⇒(d) and (b)⇒(c′)⇒(d) are obvious. Since the implication (d)⇒(a) is a consequence of Corollary 2.6, the proof is complete. □

The following theorem gives a partial answer to Question 3.2 (i).

Theorem 3.8. *Let A be an algebra, $\mathcal{T}$ a stable tube of rank $r \geq 1$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A such that the mouth A-modules of the tube $\mathcal{T}$ are pairwise orthogonal bricks.*

If the quotient algebra $B = A/\mathrm{Ann}_A\mathcal{T}$ is a projective right A-module and a projective left A-module then the following four conditions are equivalent.

(a) *The tube $\mathcal{T}$ is hereditary.*

(b) *$\mathrm{pd}_A X \leq 1$, for any indecomposable A-module X of $\mathcal{T}$.*

(c) *$\mathrm{id}_A X \leq 1$, for any indecomposable A-module X of $\mathcal{T}$.*

(d) *The tube $\mathcal{T}$ is self-hereditary.*

Proof. The implications (a)⇒(b)⇒(d) and (a)⇒(c)⇒(d) are obvious. It remains to prove that (d) implies (a).

Assume that $\mathcal{T}$ is a stable tube of rank $r \geq 1$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A, and $(E_1, \dots, E_r)$ is a τ_A-cycle of the mouth A-modules of the tube $\mathcal{T}$ that are pairwise orthogonal A-bricks. Since $B = A/\mathrm{Ann}_A\mathcal{T}$ then the algebra surjection $A \to B$ induces an embedding $\mathrm{mod}\, B \hookrightarrow \mathrm{mod}\, A$. Hence every A-module X of $\mathcal{T}$ can be viewed as a B-module, $\mathcal{T}$ is a stable tube of rank $r \geq 1$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, B)$ of B, and $(E_1, \dots, E_r)$ is a τ_B-cycle of the mouth B-modules of the tube $\mathcal{T}$ that are pairwise orthogonal B-bricks. Moreover, $\mathcal{T}$ is a faithful tube of $\Gamma(\mathrm{mod}\, B)$.

Assume that $B = A/\mathrm{Ann}_A\mathcal{T}$ is a projective right A-module and a projective left A-module. It follows that the epimorphism ${}_AA \to {}_AB$ of left A-modules spits and induces a splitting monomorphism $D({}_AB) \to D({}_AA)$ of right A-modules, where $D = \mathrm{Hom}_K(-, K) : \mathrm{mod}\, A^{op} \to \mathrm{mod}\, A$ is the standard K-duality. Hence, the injective cogenerator $D({}_AB)$ of $\mathrm{mod}\, B$ is an injective A-module.

Since the canonical embedding $\mathrm{mod}\, B \hookrightarrow \mathrm{mod}\, A$ is an exact functor then, given a module X in $\mathcal{T}$ viewed as a B-module, the minimal projective resolution of X in $\mathrm{mod}\, B$ is a projective resolution of X in $\mathrm{mod}\, A$, and the minimal injective resolution of Y in $\mathrm{mod}\, B$ is an injective resolution of Y in $\mathrm{mod}\, A$. Hence, we concude that

- $\mathrm{pd}_A X = \mathrm{pd}_B X$, for any module X in $\mathcal{T}$,
- $\mathrm{id}_A Y = \mathrm{id}_B Y$, for any module Y in $\mathcal{T}$,
- $\mathrm{Ext}^2_A(X, Y) = \mathrm{Ext}^2_B(X, Y)$, for any pair of modules X and Y in $\mathcal{T}$,

and

- the tube $\mathcal{T}$ is A-self-hereditary if and only if $\mathcal{T}$ is B-self-hereditary.

To prove that (d) implies (a), assume that the tube $\mathcal{T}$ is B-self-hereditary. Then $\mathcal{T}$ viewed as a tube of $\Gamma(\mathrm{mod}\, B)$ is faithful, A-self-hereditary, $(E_1, \dots, E_r)$ is a τ_B-cycle of the mouth B-modules of the tube $\mathcal{T}$, and $E_1, \dots, E_r$ are pairwise orthogonal B-bricks. Hence, by Proposition 3.7, the tube $\mathcal{T}$ is hereditary, when viewed as a tube of $\Gamma(\mathrm{mod}\, B)$. Moreover, by the remarks made above, the tube $\mathcal{T}$ is hereditary, when viewed as a tube of $\Gamma(\mathrm{mod}\, A)$. This finishes the proof of the implication (d)⇒(a) and of the theorem. □

We recall that an algebra A is called **symmetric** if there is an A-A-bimodule isomorphism $A \cong D(A)$. A large class of symmetric algebras is provided by the trivial extensions

$$T(B) = B \ltimes D(B)$$

of arbitrary algebras B by their injective cogenerator $D(B)$.

The following proposition shows that non-trivial components of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of any symmetric algebra A are not self-hereditary.

Proposition 3.9. *Let A is a symmetric algebra and $\mathcal{C}$ a connected component of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A containing an indecomposable non-projective module X. Then*

(i) *the indecomposable module $Y = \tau_A X$ lies in $\mathcal{C}$ and $\operatorname{Ext}^2_A(X, Y) \cong \underline{\operatorname{Hom}}_A(Y, Y) \neq 0$,*

(ii) *the component $\mathcal{C}$ is not self-hereditary.*

Proof. Assume that A is a symmetric algebra. Then A is self-injective, that is, all projective A-modules are injective. Let $\underline{\operatorname{mod}}\, A$ be the stable category of $\operatorname{mod} A$. Recall that the objects of $\underline{\operatorname{mod}}\, A$ are the modules M in $\operatorname{mod} A$ without non-zero projective direct summands and, for any pair of objects M and N in $\underline{\operatorname{mod}}\, A$, the space $\underline{\operatorname{Hom}}_A(M, N)$ of all morphisms from M to N in $\underline{\operatorname{mod}}\, A$ is the quotient

$$\underline{\operatorname{Hom}}_A(M, N) = \operatorname{Hom}_A(M, N)/\mathcal{P}(M, N),$$

where $\mathcal{P}(M, N)$ is the subspace of $\operatorname{Hom}_A(M, N)$ consisting of the A-module homomorphisms $M \to N$ that have a factorisation through a projective module.

We denote by $\Omega_A : \underline{\operatorname{mod}}\, A \longrightarrow \underline{\operatorname{mod}}\, A$ the Heller syzygy functor which assigns to any object M of $\underline{\operatorname{mod}}\, A$ the kernel of the projective cover $P_A(M) \to M$ of M in $\operatorname{mod} A$. The functor Ω_A is an equivalence and there is a functorial isomorphism $\tau_A \cong \Omega_A^2$ of functors on $\operatorname{mod} A$, see [2, IV.3.8].

Assume that $\mathcal{C}$ is a connected component of $\Gamma(\operatorname{mod} A)$ containing an indecomposable non-projective module X. Then $Y = \tau_A X$ is an indecomposable non-projective module in $\mathcal{C}$ and there exist two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_A(X) & \longrightarrow & P_A(X) & \longrightarrow & X & \longrightarrow & 0 \\ 0 & \longrightarrow & \Omega_A^2(X) & \longrightarrow & P_A(\Omega_A(X)) & \longrightarrow & \Omega_A(X) & \longrightarrow & 0 \end{array}$$

that induce the isomorphisms

$$\begin{aligned} \operatorname{Ext}^2_A(X, Y) &\cong \operatorname{Ext}^1_A(\Omega_A(X), Y) \\ &\cong \underline{\operatorname{Hom}}_A(\Omega_A^2(X), Y) \\ &\cong \underline{\operatorname{Hom}}_A(\tau_A(X), Y) \\ &\cong \underline{\operatorname{Hom}}_A(Y, Y) \end{aligned}$$

of vector spaces. Since the identity homomorphism $\mathrm{id}_Y : Y \to Y$ defines a non-zero morphism $\underline{\mathrm{id}}_Y \in \underline{\operatorname{Hom}}_A(Y, Y)$ then $\operatorname{Ext}^2_A(X, Y) \neq 0$ and the component $\mathcal{C}$ is not self-hereditary. $\square$

Remark 3.10. Proposition 3.9 applies to any stable tube $\mathcal{T}$ of the representation-infinite tame symmetric algebras exhibited in [26]. Note also that the trivial extension algebras

$$T(B) = B \ltimes D(B)$$

of all generalised canonical algebras B and, more generally, of concealed generalised canonical algebras B, contain standard stable tubes (see [10] and [25]), which are not self-hereditary, according to Proposition 3.9. Note also that the symmetric algebras for which the Auslander-Reiten quiver admits a standard stable tube have the Cartan matrix singular, see [3].

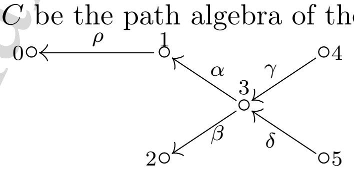
4. Examples

In relation with Questions 3.2, we end the paper with four examples of (generalised) standard stable tubes. In the first one we construct an algebra C and a standard stable tube $\mathcal{T}$ of rank $r = 2$ in $\Gamma(\mathrm{mod} C)$ such that $\mathrm{gl.dim}\, C = 2$ and the tube $\mathcal{T}$ is self-hereditary, but it is not hereditary. In the second one we construct an algebra B such that $\mathrm{gl.dim}\, B = 3$ and $\Gamma(\mathrm{mod}\, B)$ admits a (generalised) standard stable tube that is self-hereditary, but is neither faithful nor hereditary. In the third one we construct an algebra R such that $\mathrm{gl.dim}\, R = \infty$ and $\Gamma(\mathrm{mod}\, R)$ admits a standard stable tube $\mathcal{T}$ that is not self-hereditary. In the fourth example, we construct a self-injective algebra Λ (with $\mathrm{gl.dim}\, \Lambda = \infty$) such that $\Gamma(\mathrm{mod}\, \Lambda)$ admits a self-hereditary (generalised) standard stable tube consisting entirely of modules of infinite projective and infinite injective dimension.

The following example shows that, in the Definition 1.1 (a) of hereditary tube, neither of the two conditions $\mathrm{pd}_A X \leq 1$ and $\mathrm{id}_A X \leq 1$ can be dropped.

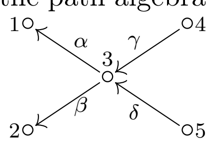
Example 4.1. Let C be the path algebra of the quiver

Q' :



bound by one zero relation $\alpha\rho = 0$. Then the quotient algebra $A = C/Ce_0C$ of C is isomorphic to the path algebra KQ of the full subquiver

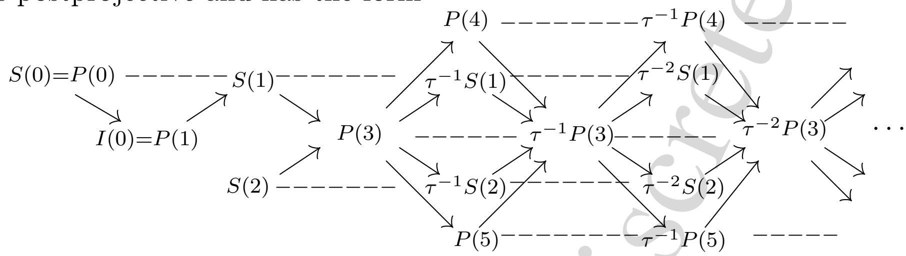
Q :



of Q' given by the vertices 1, 2, 3, 4, and 5. The canonical algebra surjection $C \longrightarrow A$ induces fully faithful exact embedding $\operatorname{mod} A \hookrightarrow \operatorname{mod} C$. It is easy to check that the indecomposable projective C-modules

$$P(0) = e_0 C = S(0) \text{ and } P(1) = e_1 C \cong I(0)$$

are the unique indecomposable C-modules that do not lie in $\operatorname{mod} A \hookrightarrow \operatorname{mod} C$. The component $\mathcal{P}(C)$ of $\Gamma(\operatorname{mod} C)$ containing the module $P(0)$ is postprojective and has the form

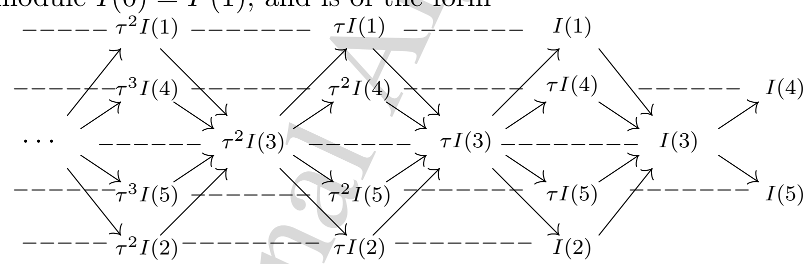


It is obtained from the unique postprojective component $\mathcal{P}(A)$ of the Auslander-Reiten quiver $\Gamma(\operatorname{mod} A)$ by adding two points connected by one arrow corresponding to the irreducible embedding

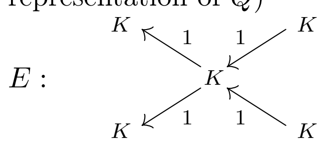
$$P(0) = \operatorname{rad} P(1) \hookrightarrow P(1).$$

The postprojective component $\mathcal{P}(C)$ contains all the indecomposable projective C-modules, up to isomorphism. Note that the indecomposable projective C-module $I(0) = P(1)$ is also injective.

It follows that the unique preinjective component $\mathcal{Q}(A)$ of $\Gamma(\operatorname{mod} A)$ is also the unique preinjective component $\mathcal{Q}(C)$ of $\Gamma(\operatorname{mod} C)$. It contains all the indecomposable injective C-modules, except the projective-injective C-module $I(0) = P(1)$, and is of the form



Note that $\Gamma(\operatorname{mod} A)$ contains a standard stable tube $\mathcal{T}$ of rank $r = 2$ with the mouth A-modules E and $S = S(3)$, where $S(3)$ is the simple module corresponding to the vertex 3 and E is the following indecomposable A-module (viewed as a representation of Q)



see [17, Example X.2.12]. The tube $\mathcal{T}$ remains a standard stable tube in $\Gamma(\operatorname{mod} C)$, under the fully faithful exact embedding $\operatorname{mod} A \hookrightarrow \operatorname{mod} C$.

Now we show that

(i) $\mathrm{pd}_C S(3) = 2$, $\mathrm{id}_C S(3) = 1$, $\mathrm{pd}_C E = 1$, and $\mathrm{id}_C E = 1$,

(ii) $\mathrm{id}_C X = 1$, for any indecomposable module X lying in the tube $\mathcal{T}$,

(ii) $\mathrm{gl.dim}\, C = 2$, and

(iv) the tube $\mathcal{T}$ of $\operatorname{mod} C$ is self-hereditary, but it is not hereditary.

To prove (i), we note that A-modules $S(3)$ and E have minimal projective resolutions in $\operatorname{mod} C$ of the forms

$$0 \longrightarrow P(0) \longrightarrow P(1) \oplus P(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0,$$
$$0 \longrightarrow P(3) \longrightarrow P(4) \oplus P(5) \longrightarrow E \longrightarrow 0,$$

and the minimal injective resolutions of $S(3)$ and of E are of the forms

$$0 \longrightarrow S(3) \longrightarrow I(3) \longrightarrow I(4) \oplus I(5) \longrightarrow 0,$$
$$0 \longrightarrow E \longrightarrow I(1) \oplus I(2) \longrightarrow I(3) \longrightarrow 0.$$

To prove (ii), we observe that the postprojective component $\mathcal{P}(C)$ of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} C)$ contains all the indecomposable projective C-modules, up to isomorphism, and $\mathcal{P}(C)$ is closed under predecessors in $\operatorname{mod} C$, by [1, VIII.2.5]. It then follows that

$$\mathrm{Hom}_C(\tau_C^{-1} X, C_C) = 0,$$

for any indecomposable C-module X that is not postprojective. Thus, [1, IV.2.7(b)] yields $\mathrm{id}_C X \leq 1$, for any indecomposable C-module X that is not postprojective, and (ii) follows. Because one easily shows that $\mathrm{pd}_C S(j) \leq 1$, for all $j \neq 3$, then $\mathrm{gl.dim}\, C = 2$. Because (ii) yields $\mathrm{Ext}^2_C(X, Y) = 0$, for any indecomposable modules X and Y in $\mathcal{T}$ then the tube $\mathcal{T}$ is self-hereditary.

Let $C' = C^{\mathrm{op}}$ be the algebra opposite to C. Because there is an isomorphism $A^{\mathrm{op}} \cong A$ of algebras then the preceding consideration implies that the Auslander–Reiten quiver $\Gamma(\operatorname{mod} C')$ dual to $\Gamma(\operatorname{mod} C)$ admits a standard stable tube $\mathcal{T}'$ of rank $r = 2$, with the mouth modules $E' = D(E)$ and $S'(3) = e_3 C' \cong D(S)$, that has the following properties

(i′) $\mathrm{id}_{C'} S'(3) = 2$, $\mathrm{pd}_{C'} S'(3) = 1$, $\mathrm{id}_{C'} E' = 1$, and $\mathrm{pd}_{C'} E' = 1$,

(ii′) $\mathrm{pd}_{C'} Y = 1$, for any indecomposable module Y lying in the tube $\mathcal{T}'$,

(iii′) $\mathrm{gl.dim}\, C' = 2$, and

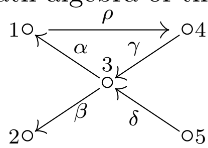
(iv′) the tube $\mathcal{T}'$ is self-hereditary, but it is not hereditary.

that are dual to (i)-(iv).

It follows from (i)-(iv) and (i′)-(iv′) that, in Definition 1.1 of hereditary tube, neither of the two conditions $\mathrm{pd}_A X \leq 1$ and $\mathrm{id}_A X \leq 1$ can be dropped.

Example 4.2. Let B be the path algebra of the quiver

$\Delta :$

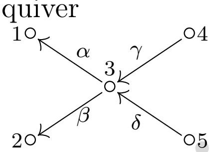


bound by two zero relations $\alpha\rho = 0$ and $\rho\gamma = 0$. It is easy to see that the quotient algebra

$$A = B/\mathcal{I},$$

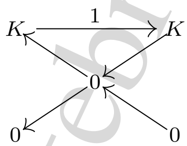
where $\mathcal{I}$ is the two sided ideal of B generated by the arrow ρ, is isomorphic to the path algebra KQ of the quiver

$Q :$



considered in Example 4.1. Then we have a fully faithful exact embedding $\operatorname{mod} A \hookrightarrow \operatorname{mod} B$ induced by the canonical algebra surjection $B \longrightarrow A$. It is easy to see that there is precisely one indecomposable A-module X, up to isomorphism, such that X does not lie in the subcategory $\operatorname{mod} A$ of $\operatorname{mod} B$. The module X is isomorphic with the unique projective-injective B-module

$P(1) = I(4) :$



It follows that the standard stable tube $\mathcal{T}$ of rank 2 of $\Gamma(\operatorname{mod} A)$ constructed in Example 4.1 remains a standard stable tube $\mathcal{T}$ of $\Gamma(\operatorname{mod} B)$ and the annihilator $\operatorname{Ann}_B \mathcal{T}$ of $\mathcal{T}$ is just the ideal $\mathcal{I}$ of B generated by the arrow ρ, that is, $A \cong B/\operatorname{Ann}_B \mathcal{T}$.

The simple B-module $S = S(3)$ at the vertex 3 lying on the mouth of the tube $\mathcal{T}$ has a minimal projective resolution in $\operatorname{mod} B$ of the form

$$0 \longrightarrow P(3) \longrightarrow P(4) \longrightarrow P(1) \oplus P(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0,$$

and a minimal injective resolution in $\operatorname{mod} B$ of the form

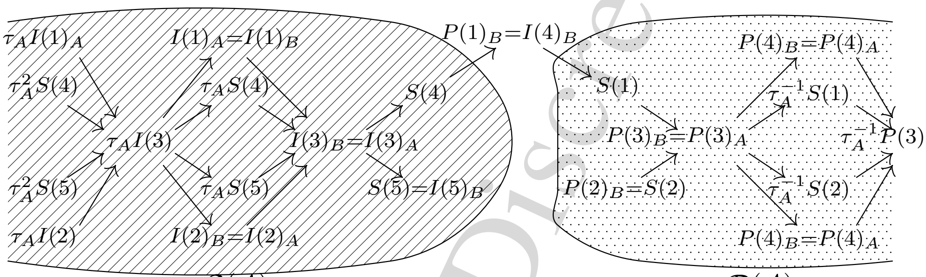
$$0 \longrightarrow S(3) \longrightarrow I(3) \longrightarrow I(4) \oplus I(5) \longrightarrow I(1) \longrightarrow I(3) \longrightarrow 0.$$

Hence, $\operatorname{pd}_B S(3) = 3$ and $\operatorname{id}_B S(3) = 3$. It follows that $\operatorname{gl.dim} B = 3$, because the simple B-module $S(2)$ is projective and the remaining simple B-modules $S(1)$, $S(4)$, and $S(5)$ have minimal projective resolutions in $\operatorname{mod} B$ of the forms

$$0 \longrightarrow P(3) \longrightarrow P(4) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0,$$

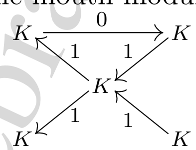
$$0 \longrightarrow P(3) \longrightarrow P(4) \longrightarrow S(4) \longrightarrow 0,$$
$$0 \longrightarrow P(3) \longrightarrow P(5) \longrightarrow S(5) \longrightarrow 0.$$

Observe also that the quiver $\Gamma(\mathrm{mod}\, B)$ admits a component $\mathcal{C}$ containing all the indecomposable projective B-modules and all the indecomposable injective B-modules. The component $\mathcal{C}$ is obtained from the unique postprojective component $\mathcal{P}(A)$ of A and the unique preinjective component $\mathcal{P}(A)$ of A by a glueing with the projective-injective B-module $P(1)_B = I(4)_B$ as follows



$\mathcal{Q}(A)$ $\mathcal{P}(A)$

Now we show that $\mathcal{T}$ is a self-hereditary stable tube of $\Gamma(\mathrm{mod}\, B)$. Recall from Example 4.1 that the mouth module E of $\mathcal{T}$ has the form

E : 

and $\tau_A S(3) \cong E \cong \tau_B S(3)$. Then the minimal projective resolution of E in $\mathrm{mod}\, B$ is of the form

$$0 \longrightarrow P(3) \longrightarrow P(4) \oplus P(5) \longrightarrow E \longrightarrow 0,$$

and the minimal injective resolution of E in $\mathrm{mod}\, B$ is of the form

$$0 \longrightarrow E \longrightarrow I(1) \oplus I(2) \longrightarrow I(3) \longrightarrow 0.$$

It follows that $\mathrm{pd}_B E = 1$ and $\mathrm{id}_B E = 1$. Hence we derive the equalities

$$\mathrm{Ext}^2_B(E, S(3) \oplus E) = 0 \quad \text{and} \quad \mathrm{Ext}^2_B(S(3) \oplus E, E) = 0.$$

Moreover, the short exact sequence

$$0 \longrightarrow S(1) \oplus S(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$$

yields isomorphisms of vector spaces

$$\mathrm{Ext}^2_B(S(3), S(3)) \cong \mathrm{Ext}^1_B(S(1) \oplus S(2), S(3)) \cong \mathrm{Ext}^1_B(S(1), S(3)),$$

because the B-module $S(2)$ is projective. By applying the Auslander–Reiten formula and the shape of the component of $\Gamma(\operatorname{mod} B)$ containing the module $S(1)$ (see the figure presented above), we get the isomorphisms

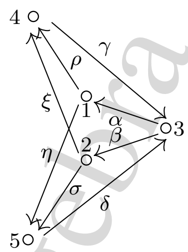
$$\operatorname{Ext}^1_B(S(1), S(3)) \cong D\overline{\operatorname{Hom}}_B(S(3), \tau_B S(1)) \cong D\overline{\operatorname{Hom}}_B(S(3), S(4)) = 0,$$

and, consequently, $\operatorname{Ext}^2_B(S(3), S(3)) = 0$. It follows that the two element family $\{S(3), E\}$ of mouth B-modules of the tube $\mathcal{T}$ is self-hereditary and consists of pairwise orthogonal bricks. Then, by Lemmas 2.3 and 3.1, the tube $\mathcal{T}$ of $\Gamma(\operatorname{mod} B)$ is self-hereditary. Because $\operatorname{pd}_B S(3) = 3$ and $S(3)$ lies on $\mathcal{T}$ then the tube $\mathcal{T}$ is not hereditary. Note also that $\mathcal{T}$ viewed as a tube of $\Gamma(\operatorname{mod} A)$ is faithful, standard, and hereditary, because $A \cong B/\operatorname{Ann}_B \mathcal{T}$. This finishes the example.

Now we give an example of an algebra R, with $\operatorname{gl.dim} R = \infty$, such that $\Gamma(\operatorname{mod} R)$ admits a standard stable tube $\mathcal{T}$ that is not self-hereditary.

Example 4.3. Let $R = K\Omega/\mathcal{I}$ be the bound quiver algebra, where Ω is the quiver

$\Omega :$



and $\mathcal{I}$ is the two-sided ideal of the path algebra $K\Omega$ generated by the elements

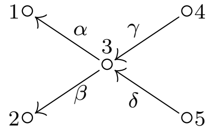
$\rho\gamma - \eta\delta$, $\xi\gamma - \sigma\delta$,
$\alpha\rho - \beta\xi$, $\alpha\eta - \beta\sigma$,
$\rho\gamma\beta$, $\gamma\beta\sigma$,
$\sigma\delta\alpha$, and $\delta\alpha\rho$.

Denote by J the two-sided ideal of R generated by the cosets

$\rho + \mathcal{I}$, $\sigma + \mathcal{I}$, $\xi + \mathcal{I}$, and $\eta + \mathcal{I}$

of the arrows ρ, σ, ξ, and η of Ω. Then the quotient algebra $A = R/J$ is isomorphic to the path algebra KQ of the quiver

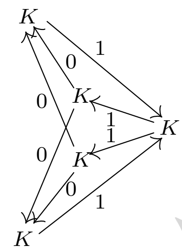
$Q :$



and the canonical algebra surjection $R \longrightarrow A$ induces a fully faithful exact embedding $\operatorname{mod} A \hookrightarrow \operatorname{mod} R$.

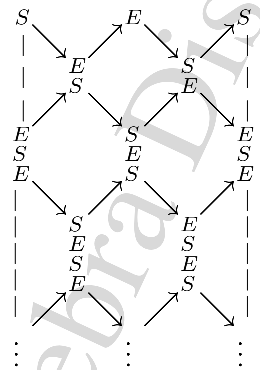
Let $S = S(3)$ be the simple R-module at the vertex 3 of Ω, and let E be the indecomposable R-module

E :



Because $SJ = 0$ and $EJ = 0$ then the R-modules S and E lie in the exact subcategory $\operatorname{mod} A$ of $\operatorname{mod} R$. It is easy to see that the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A admits a standard stable tube $\mathcal{T}$ of rank 2 of the form (see [17, Example 2.12])

$\mathcal{T}$:



where the indecomposable modules are represented by their composition factors (see [1, V.2.7]) in the extension subcategory $\mathcal{EXT}_A(E, S)$ of $\operatorname{mod} A$, and one identifies them along the vertical dotted lines. In particular, the R-modules S and E are orthogonal bricks in $\operatorname{mod} A$, hence in $\operatorname{mod} R$, and (E, S) is a τ_A-cycle, that is, $\tau_A E \cong S$ and $\tau_A S \cong E$.

Now we show that (E, S) is a τ_R-cycle, that is, $\tau_R E \cong S$ and $\tau_R S \cong E$. Observe that the module $S = S(3)$ admits the minimal projective presentation

$$P(1) \oplus P(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$$

in $\operatorname{mod} R$. Applying the Nakayama functor

$$\nu_R = D\mathrm{Hom}_R(-, R)$$

and using the isomorphism $\nu_R P(a) \cong I(a)$, for any vertex $a \in \Omega_0$, we get, by [1, IV.2.4], an exact sequence

$$0 \longrightarrow \tau_R S(3) \longrightarrow I(1) \oplus I(2) \longrightarrow I(3)$$

in $\operatorname{mod} R$. Hence, $\tau_R S(3) \cong E \cong \tau_A S(3)$.

Note also that the minimal projective presentation of E in $\mathrm{mod}\, R$ has the form

$$P(3) \longrightarrow P(5) \oplus P(4) \xrightarrow{\pi} E \longrightarrow 0$$

and $\mathrm{Ker}\, \pi \cong P(3)/S(3)$. Applying the Nakayama functor ν_R yields an exact sequence

$$0 \longrightarrow \tau_R E \longrightarrow I(3) \longrightarrow I(5) \oplus I(4)$$

and hence $\tau_R E \cong S(3) \cong \tau_A E$.

Because the R-modules $S = S(3)$ and E are orthogonal bricks in $\mathrm{mod}\, A$ they are also orthogonal bricks in $\mathrm{mod}\, R$ and we can form the extension subcategory $\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_R(E, S)$ of $\mathrm{mod}\, R$, where $\mathbf{E} = \{E, S\}$. By Lemma 2.3, $\mathcal{E}_{\mathbf{E}}$ is abelian and closed under extensions in $\mathrm{mod}\, C$. It follows that

$$\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_R(E, S) = \mathcal{EXT}_A(E, S),$$

because the subcategory $\mathrm{mod}\, A \hookrightarrow \mathrm{mod}\, R$ of $\mathrm{mod}\, R$ is exact and closed under extensions. In particular, $\mathcal{E}_{\mathbf{E}}$ consists entirely of A-modules and every simple object in $\mathcal{E}_{\mathbf{E}}$ is isomorphic to E or to S. It follows that almost split sequences in $\mathrm{mod}\, A$ starting from indecomposable modules lying in $\mathcal{T}$ remain almost split in $\mathrm{mod}\, R$ and all indecomposable summands of their terms lie also in $\mathcal{T}$. Hence we conclude that the standard stable tube $\mathcal{T}$ of $\Gamma(\mathrm{mod}\, A)$ remains a standard stable tube $\mathcal{T}$ of $\Gamma(\mathrm{mod}\, R)$.

Now we show that the tube $\mathcal{T}$ of $\Gamma(\mathrm{mod}\, R)$ is not self-hereditary, by proving that $\mathrm{Ext}^2_R(E, S) \neq 0$. By applying the functor $\mathrm{Hom}_R(-, S)$ to the short exact sequence

$$0 \longrightarrow P(3)/S(3) \longrightarrow P(5) \oplus P(4) \xrightarrow{\pi} E \longrightarrow 0$$

we derive an isomorphism

$$\mathrm{Ext}^2_R(E, S) \cong \mathrm{Ext}^1_R(P(3)/S(3), S).$$

Because the canonical exact sequence

$$0 \longrightarrow S(3) \longrightarrow P(3) \xrightarrow{\pi} P(3)/S(3) \longrightarrow 0$$

in $\mathrm{mod}\, R$ does not split then $\mathrm{Ext}^1_R(P(3)/S(3), S) \neq 0$ and, consequently, $\mathrm{Ext}^2_R(E, S) \neq 0$.

The preceding two short exact sequences give a minimal projective presentation

$$0 \longrightarrow S(3) \longrightarrow P(3) \longrightarrow P(5) \oplus P(4) \xrightarrow{\pi} E \longrightarrow 0$$

of length 3 of the module E. Hence, using the minimal projective presentation of $S(3)$, we get a non-split exact sequence

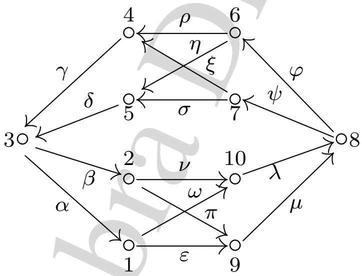
$$0 \longrightarrow E \longrightarrow P(1) \oplus P(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$$

in $\operatorname{mod} R$. By combining these two exact sequences, we get a periodic infinite minimal projective resolution of the module E. This shows that $\mathrm{gl.dim}\, R = \infty$. One can also show that the algebra R is self-injective, see [1, Section V.3]. This finishes the example.

In the following example, we construct a self-injective algebra Λ (with $\mathrm{gl.dim}\, \Lambda = \infty$) such that $\Gamma(\operatorname{mod} \Lambda)$ admits a self-hereditary (generalised) standard stable tube consisting entirely of modules of infinite projective and infinite injective dimension.

Example 4.4. Let $\Lambda = K\Delta/I$ be the bound quiver algebra, where

Δ :



and I is the two-sided ideal of the path K-algebra $K\Delta$ of Δ generated by the elements:

$\rho\gamma - \eta\delta$, $\xi\gamma - \sigma\delta$, $\varphi\rho - \psi\xi$,
$\varphi\eta - \psi\sigma$, $\varepsilon\mu - \omega\lambda$, $\pi\mu - \nu\lambda$,
$\alpha\varepsilon - \beta\pi$, $\alpha\omega - \beta\nu$, $\rho\gamma\beta$,
$\sigma\delta\alpha$, $\mu\psi\sigma$, $\lambda\varphi\rho$,
$\varepsilon\mu\psi$, $\nu\lambda\varphi$, $\gamma\beta\nu$, and $\delta\alpha\varepsilon$.

It is easy to check that Λ is a self-injective algebra and, hence, the global dimension $\mathrm{gl.dim}\, \Lambda$ of Λ is infinite.

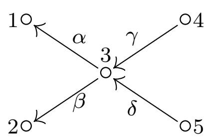
Denote by J the two-sided ideal of Λ generated by the cosets

$\rho + I$, $\sigma + I$, $\eta + I$, $\xi + I$, $\varphi + I$, $\psi + I$,
$\lambda + I$, $\mu + I$, $\nu + I$, $\varepsilon + I$, $\pi + I$, and $\omega + I$

of the arrows ρ, σ, η, ξ, φ, ψ, λ, μ, ν, ε, π, and ω in Δ.

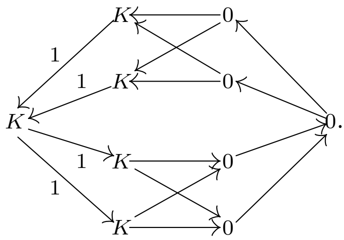
Obviously the quotient algebra $A = \Lambda/J$ is isomorphic to the path algebra KQ of the quiver

Q :

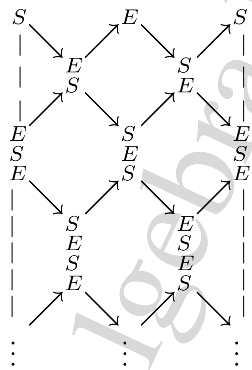


The algebra surjection $\Lambda \to A$ induces a fully faithful exact embedding $\operatorname{mod} A \hookrightarrow \operatorname{mod} \Lambda$.

Let $S = S(3)$ be the simple Λ-module at the vertex 3 of Δ, and let E be the indecomposable Λ-module

$\Delta :$ 

Because $SJ = 0$ and $EJ = 0$ then the Λ-modules S and E lie in the exact subcategory $\operatorname{mod} A$ of $\operatorname{mod} \Lambda$ and, as in Example 4.3, the Auslander-Reiten quiver $\Gamma(\operatorname{mod} A)$ of $\operatorname{mod} A$ admits the standard stable tube $\mathcal{T}$ of rank 2 of the form

$\mathcal{T} :$ 

where the indecomposable modules are represented by their composition factors (see [1, V.2.7]) in the extension subcategory $\mathcal{EXT}_A(E, S)$ of $\operatorname{mod} A$, and one identifies them along the vertical dotted lines. In particular, the R-modules S and E are orthogonal bricks in $\operatorname{mod} A$, hence in $\operatorname{mod} R$, and (E, S) is a τ_A-cycle, that is, $\tau_A E \cong S$ and $\tau_A S \cong E$.

Now we show that $\mathcal{T}$ is a self-hereditary standard stable tube in $\Gamma(\operatorname{mod} \Lambda)$. First we show that (E, S) is a τ_Λ-cycle, that is, $\tau_\Lambda E \cong S$ and $\tau_\Lambda S \cong E$. For, observe that the module $S = S(3)$ admits a minimal projective presentation

$$P(1) \oplus P(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$$

in $\operatorname{mod} \Lambda$. By applying the Nakayama functor $\nu_\Lambda = D\mathrm{Hom}_\Lambda(-, \Lambda)$, see [1, III.2.8], and using the isomorphism $\nu_\Lambda P(a) \cong I(a)$, for any vertex a

of Δ, we get an exact sequence

$$0 \longrightarrow \tau_\Lambda S(3) \longrightarrow I(1) \oplus I(2) \longrightarrow I(3)$$

in $\operatorname{mod}\Lambda$, see [1, IV.2.4]. Hence $\tau_\Lambda S(3) \cong E \cong \tau_A S(3)$.

The minimal projective presentation of E in $\operatorname{mod}\Lambda$ has the form

$$P(3) \longrightarrow P(4) \oplus P(5) \xrightarrow{h} E \longrightarrow 0$$

and $\operatorname{Ker} h \cong P(3)/S(8)$. By applying the Nakayama functor ν_Λ, we get an exact sequence

$$0 \longrightarrow \tau_\Lambda E \longrightarrow I(3) \longrightarrow I(4) \oplus I(5)$$

in $\operatorname{mod}\Lambda$ and, hence, $\tau_\Lambda E \cong S(3) \cong \tau_A E$.

In view of Theorem 2.4, to prove that $\mathcal{T}$ is a self-hereditary tube in $\Gamma(\operatorname{mod}\Lambda)$ it is enough to show that $\{S, E\}$ is a self-hereditary family of $\operatorname{mod}\Lambda$. Applying the functors $\operatorname{Hom}_\Lambda(-, S)$ and $\operatorname{Hom}_\Lambda(-, E)$ to the canonical exact sequence

$$0 \longrightarrow \operatorname{rad} P(3) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0$$

we derive the isomorphisms

$$\operatorname{Ext}^2_\Lambda(S, S) \cong \operatorname{Ext}^1_\Lambda(\operatorname{rad} P(3), S) \text{ and } \operatorname{Ext}^2_\Lambda(S, E) \cong \operatorname{Ext}^1_\Lambda(\operatorname{rad} P(3), E).$$

Hence, by applying the Auslander-Reiten formula [1, IV.2.13], we get the isomorphisms

$$\begin{array}{rcll} \operatorname{Ext}^1_\Lambda(\operatorname{rad} P(3), S) & \cong & D\underline{\operatorname{Hom}}_\Lambda(\tau_\Lambda^{-1} S, \operatorname{rad} P(3)) & \\ & \cong & D\underline{\operatorname{Hom}}_\Lambda(E, \operatorname{rad} P(3)) & = 0, \\ \operatorname{Ext}^1_\Lambda(\operatorname{rad} P(3), E) & \cong & D\underline{\operatorname{Hom}}_\Lambda(\tau_\Lambda^{-1} E, \operatorname{rad} P(3)) & \\ & \cong & D\underline{\operatorname{Hom}}_\Lambda(S, \operatorname{rad} P(3)) & = 0, \end{array}$$

because $E/\operatorname{rad} E \cong S(4) \oplus S(5)$ and the simple Λ-modules $S = S(3)$, $S(4)$, and $S(5)$ are not composition factors of $\operatorname{rad} P(3)$. Consequently, we get

$$\operatorname{Ext}^2_\Lambda(S, S) = 0 \quad \text{and} \quad \operatorname{Ext}^2_\Lambda(S, E) = 0.$$

Applying the functors $\operatorname{Hom}_\Lambda(-, S)$ and $\operatorname{Hom}_\Lambda(-, E)$ to the exact sequence

$$0 \longrightarrow P(3)/S(8) \longrightarrow P(4) \oplus P(5) \xrightarrow{h} E \longrightarrow 0$$

in $\operatorname{mod}\Lambda$, we derive the isomorphisms

$$\begin{aligned}\operatorname{Ext}^2_\Lambda(E,S) &\cong \operatorname{Ext}^1_\Lambda(P(3)/S(8),S),\\ \operatorname{Ext}^2_\Lambda(E,E) &\cong \operatorname{Ext}^1_\Lambda(P(3)/S(8),E).\end{aligned}$$

Hence again, by applying the Auslander-Reiten formula, we get the isomorphisms

$$\begin{aligned}\operatorname{Ext}^1_\Lambda(P(3)/S(8),S) &\cong D\underline{\operatorname{Hom}}_\Lambda(\tau^{-1}_\Lambda S, P(3)/S(8))\\ &\cong D\underline{\operatorname{Hom}}_\Lambda(E, P(3)/S(8)) \qquad = 0,\\ \operatorname{Ext}^1_\Lambda(P(3)/S(8),E) &\cong D\underline{\operatorname{Hom}}_\Lambda(\tau^{-1}_\Lambda E, P(3)/S(8))\\ &\cong D\underline{\operatorname{Hom}}_\Lambda(S, P(3)/S(8)) \qquad = 0,\end{aligned}$$

because $E/\operatorname{rad} E \cong S(4)\oplus S(5)$ and the simple Λ-modules $S(4)$ and $S(5)$ are not composition factors of $P(3)/S(8)$, and the simple module $S = S(3)$ is not isomorphic to a submodule of $\operatorname{soc}(P(3)/S(8)) \cong S(9)\oplus S(10)$. Consequently, we get

$$\operatorname{Ext}^2_\Lambda(E,S) = 0 \quad \text{and} \quad \operatorname{Ext}^2_\Lambda(E,E) = 0.$$

Finally, we note that we have a minimal exact sequence

$$\begin{aligned}0 \longrightarrow S(3) \longrightarrow P(8) &\longrightarrow P(9)\oplus P(10)\\ &\longrightarrow P(1)\oplus P(2) \longrightarrow P(3) \longrightarrow S(3) \longrightarrow 0,\end{aligned}$$

which induces a minimal infinite (periodic) projective resolution of $S = S(3)$ and a minimal infinite (periodic) injective resolution of $S = S(3)$ in $\operatorname{mod}\Lambda$.

This shows that $\operatorname{pd} S = \infty$, $\operatorname{id} S = \infty$, and $\operatorname{gl.dim}\Lambda = \infty$. Similarly, we show that $\operatorname{pd} E = \infty$ and $\operatorname{id} S = \infty$. It follows that $\operatorname{pd} Z = \infty$ and $\operatorname{id} Z = \infty$, for every Λ-module Z lying in the tube $\mathcal{T}$, because the R-modules S and E are orthogonal bricks in $\operatorname{mod}\Lambda$, every module Z of the tube $\mathcal{T}$ lies in the extension subcategory $\mathcal{E}_{\mathbf{E}} = \mathcal{EXT}_R(E,S)$ of $\operatorname{mod}\Lambda$, where $\mathbf{E} = \{E,S\}$, and every simple object in $\mathcal{E}_{\mathbf{E}}$ is isomorphic to E or to S, see Lemma 2.3.

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