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Serial group rings of finite groups. p-solvability

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Abstract. We prove that for any finite p-solvable group G with a cyclic p-Sylow subgroup and any field F of characteristic p, the group ring FG is serial. As a corollary for an arbitrary field we will produce a list of all groups of order ≤ 100 whose group rings are serial.

Introduction

The following problem is classical (see [17, Question 16.9] for a more general form of this question).

Question 1. *To describe all pairs (F, G), where F is an arbitrary field and G is a finite group, such that the group ring FG is serial.*

If the characteristic of F does not divide the order of G, then by Maschke's theorem (see [2, p. 157]) the ring FG is artinian semisimple, in particular serial. Omitting this 'trivial' case, everywhere in this paper we will assume that the characteristic p of F divides the order of G (and therefore the ring FG is not semisimple).

If the ring FG is serial, it follows from Higman's theorem [7] that the p-Sylow subgroup G_p of G must be cyclic. This condition is not necessary — for instance (see [1]) the group $\mathrm{SL}(2,5)$ has a cyclic 5-Sylow subgroup, but over (any) algebraically closed field F of characteristic 5 the ring FG is not serial.

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If F is algebraically closed, then (see [1, Chapter 5]) the ring FG is serial if and only if the Brauer tree of G is a 'star'. Unfortunately even for a given finite group calculating its Brauer tree is a difficult task, probably essentially more difficult that our original question. Furthermore not so much is known on Brauer trees of classes of groups.

The most powerful known sufficiency condition for seriality is due to Morita [10] (later reproved by Srinivasan [16]). Namely if F is an algebraically closed field of characteristic p and G is a p-solvable finite group with a cyclic subgroup G_p, then the group ring FG is serial. Furthermore Morita shows that multiplicities of indecomposable projective modules in the same block of FG coincide (FG is quasi-primary decomposable in his terminology).

In this paper we will extend Morita's result to the case of an arbitrary field of characteristic p. It follows that the group ring of a solvable group with a cyclic p-Sylow subgroup over a field of characteristic p is serial, in particular this is the case for groups of odd orders.

Note that in [9] the authors showed (extending earlier Murase's result [12]) that for an arbitrary field F and any p-nilpotent finite group G with a cyclic p-Sylow subgroup, the group ring FG is a principal ideal ring, in particular is serial. Our proof in p-solvable case is based upon Kirichenko's theorem [6] on seriality of an artinian ring whose Jacobson radical is principal as a left and right ideal.

Modulo this result the proof of the main theorem of the paper is not difficult. But it allows us an effective checking of seriality for groups of small orders. In this paper we will render this for groups of order ≤ 100. Namely we will show that in this case Higman's necessary condition is almost sufficient: the only exception is given by the alternating group A_5 over any field F of characteristic 5 (in this case the 5-Sylow subgroup is cyclic, but the group ring FA_5 is not serial).

We will demonstrate on particular examples calculational difficulties that occur, most of them were overcomed using GAP [5] and MAGMA [4] packages. The most challenging problem is splitting idempotents (or simple modules) when extending the ground field. We will prove that, if the field extension $F \subseteq F'$ does not split idempotents, then seriality of the group ring FG is equivalent to seriality of the ring $F'G$. Note that whether the seriality of a group ring is preserved under scalar extensions is an open problem (see [3, p. 275, Question 6]).

1. Basic notions

All rings in this paper are assumed to be associative with unity, and all modules are unitary and (with few exceptions) right. An idempotent e of a ring R is said to be *indecomposable* if for any decomposition $e = f + g$ into a sum of orthogonal idempotents, either $f = 0$ or $f = e$ holds. It follows from orthogonality in this equality that $ef = fe = f$ and $eg = ge = g$, in particular $f, g \in eRe$.

A ring R is said to be *semiperfect*, if $1 = e_1 + \cdots + e_n$ for some family of pairwise orthogonal indecomposable idempotents. It is known that this set (a complete orthogonal system of indecomposable idempotents) is uniquely determined up to conjugation by a unit in R. For instance every (left or right) artinian ring is semiperfect. For a semiperfect ring R we obtain a decomposition $R_R = \oplus_{i=1}^n e_i R$ into a direct sum of indecomposable projective modules (often called *principal projective modules*), whose endomorphism rings $R_i = \mathrm{End}(e_i R) = e_i R e_i$ are local. Note also that the set $R_{ij} = e_i R e_j$ is an R_i-R_j-bimodule.

A module M is said to be *chain*, if every two submodules of M are comparable by inclusion; and M is called *serial*, if M is a direct sum of chain modules. We say that a ring R is *right (left) serial* if the module R_R (${}_R R$) is serial. Finally a ring is said to be *serial* if it is right serial and left serial. Every serial ring is semiperfect. Furthermore, for a semiperfect ring there exists the following useful criterion for checking seriality.

Fact 2 ([15, Lemma 1.22]). *Let R be a semiperfect ring with a complete system of orthogonal indecomposable idempotents $e_1, \ldots, e_n$. Then R is right serial if and only if for every $r \in R_{ij}$, $s \in R_{ik}$ there are $u \in R_{jk}$, $v \in R_{kj}$ such that $ru = s$ or $sv = r$ holds.*

Let F be a field of characteristic p and G is a finite group of order divisible by p. The group ring $R = FG$ is artinian, therefore the seriality criterion (Fact 2) is applicable to R. Furthermore because R is quasi-Frobenius, therefore right seriality of R implies its seriality. It is also known (see [15, Lemma 8.1]) that every artinian serial ring has finite representation type. Taking into account Higman's result [7] we conclude that seriality of the ring FG yields that (any) p-Sylow subgroup G_p of G is cyclic. Because we are interested in seriality of FG, we will further assume that the group G_p is cyclic.

The following sufficient condition for seriality is due to Murase.

Fact 3 ([11, Theorem 3]). *Let F be an arbitrary field of characteristic p and G is a finite group with a cyclic normal p-Sylow subgroup. Then the group ring FG is a principal ideal ring, in particular is serial.*

Recall that a finite group G is said to be *p-nilpotent* if its p-Sylow sugbroup G_p admits a normal complement. For instance, every finite group G with a cyclic 2-Sylow subgroup is 2-nilpotent. Thus the following result completely describes serial rings of finite groups over a field of characteristic 2.

By $\mathrm{Jac}(M)$ we will denote the Jacobson radical of a module M. In particular the Jacobson radical of a ring R is a two-sided ideal.

Fact 4 ([9, Theorem 11]). *Let F be an arbitrary field of characteristic p and let G be a finite p-nilpotent group with a cyclic p-Sylow subgroup. Then FG is a (left and right) principal ideal ring, in particular is serial.*

Note that (by an old result of Nakayama [13, Lemma 1]) if I is a two-sided ideal of an artinian ring R such that $I = Ra = bR$, then $I = aR = Rb$ holds true. In particular, in Fact 4 we have $\mathrm{Jac}(R) = rR = Rr$ for some $r \in R$.

2. Main result

A finite group G is said to be *p'-group* if its order is coprime to p. We say that a finite group G is *p-solvable* if it has a subnormal series $\{e\} \subset G_1 \subset \ldots \subset G_l = G$ such that all successive factors G_{i+1}/G_i are either p-groups or p'-groups. For instance every p-nilpotent group is p-solvable, so as any solvable group. Furthermore the symmetric group S_4 is 3-solvable but not 3-nilpotent. Also the group $A_5 \times C_7$ is 7-solvable, but not solvable.

For groups with cyclic p-Sylow subgroups we have the following useful criterion for checking p-solvability.

Fact 5 ([18]). *Let G be a finite group with a cyclic p-Sylow subgroup. Then G is p-solvable if and only if it has a chain $\{e\} \subset H \subset K \subset G$ of completely characteristic (therefore normal) subgroups such that H is a p'-group, K/H is isomorphic to G_p, and G/K is a cyclic p'-group.*

Note that in this fact $K = HG_p$, therefore K is p-nilpotent.

Let R be a semiperfect ring. Gathering together isomorphic principal projectives, we obtain a direct sum decomposition $R_R = \oplus_{i=1}^{m}(e_iR)^{m_i}$. The number m_i is called the *multiplicity* of the module $P_i = e_iR$.

The following proposition is a rephrase of Kirichenko [6, Lemma 12.4.4].

Proposition 1. *Let R be an artinian ring whose Jacobson radical is principal as a left and right ideal. Then R is serial and the multiplicities of indecomposable projective modules in each block of R coincide.*

Proof. Without loss of generality we may assume that R is indecomposable. If $J = \mathrm{Jac}(R)$ then (see [6, Corollary 11.4.3]) the ring $R' = R/J^2$ is indecomposable. Since $J' = \mathrm{Jac}(R') = J/J^2$ is principal as a left and right ideal, applying [6, Lemma 12.4.4] we obtain that R' is serial and $R' = \oplus_{i=1}^m (e_i' R')^k$, where $e_i' = e_i + J^2$ are indecomposable non-isomorphic idempotents (that is multiplicities of principal projectives in R' all equal k). Since R is artinian, by [6, Theorem 12.3.10] it follows that R is serial, and clearly R possesses a similar decomposition $R = \oplus_{i=1}^m (e_i R)^k$ into a direct sum of non-isomorphic indecomposable projective modules. □

The following main result of the paper (generalizing to arbitrary fields Morita's theorem [10]) is now just a synch.

Theorem 1. *Let F be an arbitrary field of characteristic p and let G be a finite p-solvable group with a cyclic p-Sylow subgroup. Then the group ring FG is serial. Furthermore in each block of FG the multiplicities of principal projectives coincide.*

Proof. Choose normal subgroups H and K of G as in Fact 5 and let $R = FK$. Since K is a p-nilpotent group, Fact 4 yields that $J = \mathrm{Jac}(R) = Rr = rR$ for some $r \in R$. Since G/K is a p'-group, it follows from Passmann [14, 1.2.7] that $\mathrm{Jac}(FG) = J \cdot FG = FG \cdot J$, that is $\mathrm{Jac}(FG) = rFG = FGr$.

It remains to apply Proposition 1. □

If the field F is algebraically closed, then the multiplicity of each principal projective module $P_i = e_i R$ equals the dimension of the corresponding simple module (its socle or top) $S_i = P_i/\mathrm{Jac}(P_i)$. Thus in this case all simple modules in a given block have equal dimension.

Corollary 1. *Let F be an arbitrary field of characteristic p and G is a solvable group with a cyclic p-Sylow subgroup. Then the group ring FG is serial. In particular this is the case for groups of odd orders.*

Proof. Every solvable group is p-solvable for any prime p. Furthermore by Thompson's result each group of odd order is solvable. □

3. Groups of order ≤ 100

In this section we will prove the following theorem describing all fields F and groups G of order ≤ 100 such that the group ring FG is serial. The proof will occupy the whole section and more information on the structure of those rings could be found along the lines of the proof.

Theorem 2. *Let F be an arbitrary field of characteristic p and G is a group of order ≤ 100 divisible by p. Then the group ring FG is serial if and only if (any) p-Sylow subgroup of G is cyclic and the pair (G, p) is different from the pair $(A_5, 5)$.*

Since A_5 is the only nontrivial perfect group of order ≤ 100, each group of this size, but A_5, is solvable, hence p-solvable for any p. We will also use the information collected in a table from [9]. For instance, there are 1048 non-isomorphic groups of order ≤ 100. As usual (by Higman's result) we may assume that the p-Sylow subgroup G_p of G is cyclic. We will prove the theorem by considering different values for p.

The case $p = 2$ has been already considered in [9].

Fact 6 ([9, Theorem 11])**.** *Let F be an arbitrary field of characteristic 2 and let G be a finite group of even order. Then the group ring FG is serial if and only the 2-Sylow subgroup of G is cyclic. In this case the group G is 2-nilpotent.*

Note (see table in [9]) that there are exactly 168 non-isomorphic groups of order ≤ 100 with a cyclic 2-Sylow subgroup, and all these groups are 2-nilpotent.

A similar result takes place for $p \geq 11$.

Lemma 1. *Let F be an arbitrary field of characteristic $p \geq 11$ and G is a group whose order ≤ 100 is divisible by p. Then the p-Sylow subgroup G_p of G is cyclic and normal, therefore the group ring FG is serial.*

Proof. Since $|G| \leq 100$ and $p \geq 11$ it easily follows that G_p has p elements (hence cyclic), in particular any two p-Sylow subgroups of G has a trivial intersection. By Sylow theorem (the number of p-Sylow subgroups equals 1 modulo p) we conclude that G_p is a normal subgroup, therefore seriality follows from Fact 3. □

For instance, for $p = 11$ there are 29 non-isomorphic groups or order ≤ 100 with a non-trivial cyclic 11-Sylow subgroup.

Clearly the lemma 1 holds true for any group G of order $< p^2$, with no restriction on p.

Now we consider the case $p = 7$.

Lemma 2. *Let F be an arbitrary field of characteristic 7 and let G be a group of order ≤ 100 divisible by 7. The the group ring FG is serial if and only if (any) 7-Sylow subgroup of G is cyclic. In this case G is 7-solvable.*

Proof. Every group of order ≤ 100 with a nontrivial 7-Sylow subgroup is 7-solvable. If moreover the 7-Sylow subgroup is cyclic then the ring FG is serial by Theorem 1. □

Note that there are 57 non-isomorphic groups of order ≤ 100 with a nontrivial cyclic 7-Sylow subgroup. For all, but one, of these groups the subgroup G_7 is normal. The only exception is the group $[56, 11]$ (of order 56) in the classification of GAP, which is a semidirect product of C_2^3 and C_7 (where C_2^3 is normal).

Now let us consider more interesting case $p = 3$.

Proposition 2. *Let F be an arbitrary field of characteristic 3 and let G be a group of order ≤ 100 divisible by 3. Then the group ring FG is serial if and only if the 3-Sylow subgroup of G is cyclic. In this case either G is 3-solvable or G is isomorphic to A_5.*

The proof of this proposition will occupy few pages. From the table in [9] we see that there are 405 non-isomorphic groups of order ≤ 100 with a nontrivial cyclic 3-Sylow subgroup. Furthermore 404 from them are 3-solvable, therefore the group ring FG is serial by Theorem 1.

The remaining case is the alternating group A_5. Below we will show that for every field F of characteristic 3 the group ring FA_5 is serial and clarify the structure of this ring. Consider first the principal case of finite fields $\mathbb{F}_3$ and $\mathbb{F}_9$.

Lemma 3. *The group ring $R = FA_5$ over the prime field $F = \mathbb{F}_3$ is serial.*

Proof. Calculating in MAGMA we decompose R into a direct sum of indecomposable projective modules: $R_R = P_1^3 \oplus P_2^4 \oplus P_3$, where P_1 is 6-dimensional, P_2 has dimension 9, and P_3 is 6-dimensional.

Furthermore R has one (trivial) 1-dimensional simple module S_1 of multiplicity 1, one simple module S_4 of dimension 4 and multiplicity 4; and one 6-dimensional module S_6 of multiplicity 3. Also the module $P_1 = S_6$ is simple, and the composition series of modules P_2 and P_3 are shown on the following diagram:

$$P_2 : \begin{pmatrix} S_4 \\ S_1 \\ S_4 \end{pmatrix} \quad \text{and} \quad P_3 : \begin{pmatrix} S_1 \\ S_4 \\ S_1 \end{pmatrix},$$

in particular the block containing projectives P_2 and P_3 is principal (that is, contains the trivial module).

It is not difficult to check that that endomorphism rings of modules S_1 and S_4 are isomorphic to $\mathbb{F}_3$ (that is, these modules are absolutely irreducible); but the endomorphism ring of S_6 is isomorphic to $\mathbb{F}_9$ (and this module is not absolutely irreducible). From this it is straightforward to show that the block structure of the ring R has the form $R = M_3(\mathbb{F}_9) \oplus B_1$, where $B_1 = \begin{pmatrix} Q & Q & Q & X \\ Q & Q & Q & X \\ Q & Q & Q & X \\ Y & Y & Y & T \end{pmatrix}$ is an indecomposable serial ring of dimension 42. Furthermore $Q = \mathrm{End}(P_2)$ and $T = \mathrm{End}(P_3)$ are chain rings of length 2 isomorphic to $\mathbb{F}_3[x]/x^2 = 0$ and X, Y are 1-dimensional bimodules.

Thus B_1 'blows up' from its $(3,4)$-minor $B = \begin{pmatrix} Q & X \\ Y & T \end{pmatrix}$, and B is isomorphic to the factor of the hereditary noetherian prime ring $\begin{pmatrix} F[x] & F[x] \\ xF[x] & F[x] \end{pmatrix}$ by the ideal $\begin{pmatrix} x^2F[x] & xF[x] \\ x^2F[x] & x^2F[x] \end{pmatrix}$. □

Lemma 4. *The group ring $R = FA_5$ over the field $F = \mathbb{F}_9$ is serial.*

Proof. Using MAGMA, it is easily calculated that $R \cong M_3(F) \oplus M_3(F) \oplus B_1$, that is the block $M_3(\mathbb{F}_9)$ from Lemma 3 has got splitted into a direct sum of two blocks (that is the simple 6-dimensional module S_6 has got splitted into a direct sum of two non-isomorphic simple 3-dimensional modules S_3 and S_3' both of multiplicity 3). □

To consider the case of an arbitrary field, we need to introduce some formalism.

Let F be a field of characteristic p and let G be a finite group whose order is divisible by p. We say that a field extension $F \subseteq F'$ is *G-non-essential* (or just *non-essential*, if the group G is fixed), if every simple FG-module remains simple when extending scalars to F' (that is, when tensoring with F'). Clearly this condition can be reformulated in terms of idempotents: if e_i is an indecomposable idempotent in FG, then it remains indecomposable as an element of $F'G$.

For instance, it is known (see [8]) that every simple representation of the symmetric group S_n is absolutely irreducible, therefore any field extension is S_n-non-essential.

The following proposition shows that seriality is preserved with respect to non-essential extensions.

Proposition 3. *Let G be a finite group and let $F \subseteq F'$ be a G-non-essential extension of fields. Then the group ring $R = FG$ is serial if and only if the ring $R' = F'G$ is serial.*

Proof. Let $e_1, \ldots, e_n$ be a complete system of orthogonal indecomposable idempotents of R. By the assumption this system has a similar property in R'.

$\Leftarrow$. Suppose that the ring R' is serial and prove the seriality of R. Choose two arbitrary elements $r \in R_{ij}$, $s \in R_{ik}$ and verify the criterion for (right) seriality from Fact 2. Applying this criterion to ring R', by symmetry we may assume that $ru' = s$ for some $u' \in R'_{jk}$.

We will be looking for an element $u \in R_{ij}$ satisfying $ru = s$ in the form $u = \sum_i x_i g_i$, where g_i runs over all elements of G and $x_i \in F$ are variables. Decomposing analogously r and s we obtain a system of linear equations with coefficients in F. Because this system is solvable in F', it is solvable in F by Kronecker–Capelli theorem.

$\Rightarrow$. Suppose that R is serial with the Jacobson radical J. It follows that each principal projective R-module $P_i = e_i R$ has a unique composition series. Furthermore, every simple R-module is isomorphic to the module $S_i = P_i/e_i J$ (the top of P_i).

By the assumption every principal projective R'-module is of the form $P'_i = e_i R'$, therefore every simple R'-module is isomorphic to the module $S'_i = P'_i/e_i J'$, where J' denotes the Jacobson radical of R'.

Tensoring the short exact sequence $0 \to e_i J \to P_i \to S_i \to 0$ with F' we obtain the short exact sequence $0 \to F' \otimes e_i J \to F' \otimes P_i \to F' \otimes S_i \to 0$. But clearly $F' \otimes P_i \cong P'_i$ which yields $F' \otimes S_i \cong S'_i$ (because both modules are simple and are homomorphic images of the same principal projective).

It easily follows that $F' \otimes J = J'$. Indeed the inclusion $F' \otimes J \subseteq J'$ is obvious (for instance, because J is nilpotent). For the opposite inclusion it suffices to check that $e_i J' \subseteq F' \otimes e_i J$ for every i. But this follows considering the above short exact sequence and the sequence $0 \to e_i J' \to P'_i \to S'_i \to 0$ by comparing dimensions.

To prove that R' is serial by [6, Theorem 12.3.10] it suffices to check that every R'-module $e_i J'/e_i J'^2$ is simple. By what we have proved this factor module is isomorphic to the module $F' \otimes (e_i J/e_i J^2)$ which is simple by our assumption. □

Thus if F is a field of characteristic p and G is a finite group, then starting from a prime subfield $\mathbb{F}_p \subseteq F$ one could investigate seriality of the ring FG using the chain of G-essential extensions $\mathbb{F}_p \subset F_1 \subset \ldots \subset F$. The length of this chain reflects splitting of simple modules when expanding the field. For the group A_5 over the field of characteristic 3 every chain of essential extensions has length at most two.

Lemma 5. *Let $G = A_5$ and let $F \subseteq F'$ be a pair of fields of characteristic 3. This extension is G-essential if and only if the equation $x^2 = 2$ has a solution in F' but not in F.*

Using the standard realization of $\mathbb{F}_9$ as a factor $\mathbb{F}_3[x]/f(x)$, where $f(x) = x^2 - 2$ is an irreducible over $\mathbb{F}_3$ polynomial, the last condition is equivalent to the property that F' contains a copy of $\mathbb{F}_9$, but F is not.

Proof. Let $F = \mathbb{F}_3$ and decompose the ring $R = FG$ into a direct sum of indecomposable projective modules as in Lemma 3. Only the simple (projective) module $S_6 = P_1$ is not absolutely irreducible. If this module is decomposable in some extension $F \subseteq F'$, then the idempotent e_1 such that $e_1R \cong P_1$ decomposes: $e_1 = f + g$ for some orthogonal idempotents $f, g \in R_1 = e_1Re_1$ such that $f \neq 0, e_1$.

Using calculations in GAP it is not difficult to find a basis $v_1 = e_1, v_2$ in R_1, where the multiplication table is given by the rules $v_1^2 = v_1$, $v_1v_2 = v_2v_1 = v_2$ and $v_2^2 = 2v_1$. Writing f as $a_1v_1 + a_2v_2$ with $a_i \in F'$ we obtain the system of equations $a_1 = a_1^2 + 2a_2^2$ and $a_2 = 2a_1a_2$. Throwing away trivial solutions $a_1 = a_2 = 0$ and $a_1 = 1$, $a_2 = 0$ (which give $f = 0$ and $f = e_1$ correspondingly) we obtain $a_1 = 2$ and $a_2^2 = 2$.

This system has a solution if and only if the equation $x^2 = 2$ is solvable in F', that is if F' contains $\mathbb{F}_9$ as a subfield. Clearly these arguments work for any pair of fields of characteristic 3. □

Taking into account Lemmas 3, 4 and Proposition 3, we have completed the proof of Proposition 2. To complete the proof of Theorem 2, it remains to consider the case $p = 5$.

Proposition 4. *Let F be an arbitrary field of characteristic 5 and let G be a finite group of order ≤ 100 divisible by 5. Then the group ring $R = FG$ is serial if and only if the 5-Sylow subgroup of G is cyclic and $G \neq A_5$. In this case G is 5-solvable.*

Proof. From the table in [9] we see that there are 123 groups of order ≤ 100 with a nontrivial cyclic 5-Sylow subgroup and 122 of them are 5-solvable. Thus, by Theorem 1 it remains to show that for the remaining group $G = A_5$ its group ring is not serial for any field of characteristic 5.

If $F = \mathbb{F}_5$ then calculating in MAGMA we obtain the decomposition $R = P_1^5 \oplus P_2^3 \oplus P_3$, where (non-isomorphic) projectives P_1 and P_3 have dimension 5, and P_2 is 10-dimensional. Furthermore, R has one 1-dimensional (trivial) module S_1 of multiplicity 1, one simple module S_3 of dimension 3 and multiplicity 3; and one 5-dimensional simple module

S_5 of multiplicity 5. Furthermore, the module $P_1 = S_5$ is simple, and the composition series for the remaining projectives are the following:

$$P_2 : \begin{pmatrix} & S_3 & \\ S_1 & & S_3 \\ & S_3 & \end{pmatrix} \quad \text{and} \quad P_3 : \begin{pmatrix} S_1 \\ S_3 \\ S_1 \end{pmatrix},$$

in particular P_2 is not a chain module. □

This can also be seen from the structure of R. Namely $R = M_5(F) \oplus C$ is a block decomposition of R. Here C is an indecomposable ring $\begin{pmatrix} Q & Q & Q & X \\ Q & Q & Q & X \\ Q & Q & Q & X \\ Y & Y & Y & T \end{pmatrix}$ of dimension 35, where $Q = \mathrm{End}(P_3) \cong F[x]/x^3 = 0$ is a chain ring of length 3, $T \cong F[x]/x^2 = 0$ is a chain ring of length 2; and X, Y are 1-dimensional bimodules.

Choose nonzero elements $x \in C_{14}$ and $y \in C_{41}$ (they are defined up to a multiplicative constant). If q generates the Jacobson radical of $Q = C_{11}$, this choice can be made such that $xy = q^2$. Then the seriality criterion (Fact 2) fails for $r = q$ and $s = x$. Indeed the equality $qu = x$ for some $u \in C_{14}$ implies $u = ax$ for some $a \in F$. But then $qx = 0$ yields $x = 0$, a contradiction.

Otherwise $xv = q$ for some $v \in C_{14}$, that is $v = ay$ for some $a \in F$. But this yields $aq^2 = q$, a contradiction.

It is not difficult to check that all simple $\mathbb{F}_5 A_5$-modules are absolutely irreducible. By Proposition 3 the group ring FA_5 is not serial for any field F of characteristic 5.

In conclusion of this section we will repeat (slightly modified) an open problem from [3].

Question 7. *Let $F \subseteq F'$ be a field extension and let G be a finite group. Is it true that seriality of the ring $R = FG$ implies seriality of $R' = F'G$? Conversely, does seriality of R' yields seriality of R?*

4. One example

In this section we consider in more detail the group ring $R = FG$, where F is an arbitrary field of characteristic 3 and G is the *binary octahedron group* $2O$ (the group $[48, 28]$ in the classification of GAP). This group is a double cover $2{\cdot}S_4^-$ of the symmetric group S_4, but also there is a non-splitting short exact sequence $\{e\} \to \mathrm{SL}(2,3) \to G \to C_2 \to \{e\}$. It possesses the following subnormal series $\{e\} \subset C_2 \subset C_4 \subset Q_8 \subset \mathrm{SL}(2,3) \subset G$, therefore is 3-solvable. Here $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group. Because G has no normal subgroups of order 16 it is not 3-nilpotent. By Theorem 1 for any field of characteristic 3 the group ring FG is serial. In the following proposition we will reveal the structure of this ring.

Proposition 5. *Let G be the binary octahedron group $2O$ and let F be a field of characteristic* 3.

1) If F contains a square root of 2*, then $R = FG = B \oplus M_3(F) \oplus M_3(F) \oplus B \oplus B$, where B is the ring $\left(\begin{smallmatrix} F[x]/x^2 & F[x]/x \\ xF[x]/x^2 & F[x]/x^2 \end{smallmatrix}\right)$ (cp. Lemma 3).*

2) If F contains no square root of 2*, then $R = FG = B \oplus M_3(F) \oplus M_3(F) \oplus M_2(W)$, where W is a non-commutative chain ring of length* 3 *such that $W/\mathrm{Jac}(W)$ is a field which is* 2*-dimensional over F.*

In the remaining part of this section we will proceed with the proof of this proposition. Let $H = Q_8$ be given by generators a, b (say, $a = i$, $b = j$) and relations $a^4 = 1$, $a^2 = b^2$ and $ba = a^3b$.

Extend this presentation to a presentation of $K = \mathrm{SL}(2,3)$ by adding a new generator d (say, $d = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$) and new relations $d^3 = 1$, $dad^{-1} = b$, and $dbd^{-1} = ab$ describing the action of d by conjugation on Q.

Finally G is obtained from K by adding a new generator s and new relations $a^2 = s^2$, $sas^{-1} = a^2b$, $sbs^{-1} = a^3$ and $sds^{-1} = d^2$.

We start by recalling from [9] that the ring FQ_8 has the following block decomposition

$$FQ_8 = F \oplus F \oplus F \oplus F \oplus M_2(F)\,.$$

These blocks correspond to the following primitive central idempotents:

$$\begin{aligned} e_1 &= 2 + 2a + 2a^3 + 2b + 2a^2b + 2a^2 + 2ab + 2a^3b, \\ e_2 &= 2 + a + a^3 + 2b + 2a^2b + 2a^2 + ab + a^3b, \\ e_3 &= 2 + 2a + 2a^3 + b + a^2b + 2a^2 + ab + a^3b, \\ e_4 &= 2 + a + a^3 + b + a^2b + 2a^2 + 2ab + 2a^3b, \\ e_5 &= 2 + a^2, \end{aligned}$$

where $e_1, \ldots, e_4$ are indecomposable, but e_5 is a sum of two (isomorphic) indecomposable idempotents

$$e_{51} = e + 2a + a^3 + 2b + a^2b + 2a^2 \quad \text{and} \quad e_{52} = e + a + 2a^3 + b + 2a^2b + 2a^2.$$

Now we lift this block decomposition to the block decomposition of the ring FK. It is easily calculated that the element d (the generator of a 3-Sylow subgroup of SL$(2,3)$) acts by conjugation on central primitive idempotents of FQ in the following way: $e_1^d = e_1$, $e_2^d = e_4$, $e_3^d = e_2$, $e_4^d = e_3$ and $e_5^d = e_5$, therefore there are three orbits $\{e_1\}$, $\{e_2, e_4, e_3\}$ and $\{e_5\}$ of this action.

It follows that the central primitive idempotents of FH are $f_1 = e_1$, $f_2 = e_2 + e_3 + e_4$ and $f_3 = e_5$; and this ring has the following block structure

$$FK = V \oplus M_3(F) \oplus M_2(V)\,,$$

where $V = F[x]/x^3 = 0$ is a chain ring of length 3 (see [12, 9] for details).

To understand the block structure of FG, consider the action of s by conjugation on the idempotents e_i. It is easily seen that $e_1^s = e_1$, $e_2^s = e_3$, $e_3^s = e_2$, $e_4^s = e_4$ and $e_5^s = e_5$, hence there are four orbits $\{e_1\}$, $\{e_2, e_3\}$, $\{e_4\}$ and $\{e_5\}$ of this action. Therefore the action of s on the idempotents f_i is trivial. In particular f_i, $i = 1, 2, 3$ are central idempotents of FG, therefore $f_i R$ are ring direct summands of R which should be decomposed into direct sum of blocks.

1) Consider first what the block $f_1 FK$ of FK, where $f_1 = e_1 = 2\sum_{g\in Q_8} g$.

Because the element $s^2 = a^2 = b^2$ belongs to the the center of G, it follows that $e_{11} = e_1(1-s)/2$ and $e_{12} = e_1(1+s)/2$ are orthogonal (noncentral) idempotents. Using calculations in GAP for $\mathbb{F}_3$ as a hint we see that bimodules $e_{11}Re_{12}$ and $e_{12}Re_{11}$ are 1-dimensional. Furthermore we can choose $0 \neq x_{12} \in e_{11}Re_{12}$ and $0 \neq x_{21} \in e_{12}Re_{11}$ such that $x_{11} = x_{12}x_{21} = \sum_{h\in H} h + 2\cdot \sum_{g\in G\setminus H} g$ and $x_{22} = x_{21}x_{12} = \sum_{g\in G} g$ are elements of the socle of R, in particular $x_{11}^2 = x_{22}^2 = 0$.

Also x_{11} generates the radical of the chain ring $e_{11}Re_{11}$, and x_{22} generates the radical of the ring $e_{22}Re_{22}$. Similarly x_{12} generates the radical of the module $e_{11}R$ and x_{21} generates the radical of the module $e_{12}R$. This shows that $e_{11}R$ is a chain module with composition series $e_{11}R \supset x_{12}R \supset x_{11}R \supset 0$; and $e_{12}R$ is a chain module with composition series $e_{12}R \supset x_{21}R \supset x_{22}R \supset 0$;

Thus the block $u_1 R$ is isomorphic to the ring B.

2) Now let us look at the block $f_2 FK = M_3(F)$ of FK.

Since $f_2 = e_2 + e_3 + e_4$ is a decomposition of f_2 into a sum of orthogonal isomorphic idempotents, it follows that the ring $f_2 R$ is isomorphic to the ring $M_3(e_2Re_2)$, where $e_2Re_2 \cong \mathrm{End}(e_2R)$. Calculating dimensions we see that this ring is 2-dimensional over F. Furthermore we will find the element $v = s(1+a)(1+a^2)d^2 = se_3d^2$ such that $v \in e_2Re_2$ and $e_2 = (e_2 - v) + (e_2 + v)$ is a decomposition of e_2 into a sum of orthogonal (non-isomorphic) idempotents. It easily follows that $e_2Re_2 \cong F \oplus F$, therefore $f_2R = M_3(F) \oplus M_3(F)$.

Thus it remains to consider what will happen with the third block $f_3FH = M_2(V)$ of FK.

3) Recall that $f_3 = e_5 = e_{51} + e_{52}$ for orthogonal isomorphic idempotents e_{51}, e_{52}. Using this we obtain that $f_3R \cong M_2(e_{51}Re_{51})$ where $e_{51}Re_{51} \cong \mathrm{End}(e_{51}R)$.

First we will clarify when the idempotent e_{51} decomposes.

Lemma 6. *The idempotent e_{51} is decomposable in R if and only if the equation $x^2 = 2$ is solvable in F (that is if F contains $\mathbb{F}_9$ as a subfield).*

Proof. Using GAP it is not difficult to choose a basis $v_1 = e_{51}, v_2, \ldots, v_6$ for $e_{51}Re_{51}$ with the following multiplication table:

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	v_3	v_4	v_5	v_6
v_2	v_2	$2v_1$	$2v_6$	v_5	$2v_4$	v_3
v_3	v_3	v_4	$2v_1$	$2v_2$	v_6	$2v_5$
v_4	v_4	$2v_3$	v_5	v_6	v_2	$2v_1$
v_5	v_5	v_6	$2v_4$	v_3	$2v_1$	$2v_2$
v_6	v_6	$2v_5$	v_2	$2v_1$	$2v_3$	$2v_4$

The idempotent e_{51} is decomposable if there exists an idempotent $f = \sum_{i=1}^{6} a_iv_i$ such that $f \neq 0, e_{51}$. The equation $f^2 = f$ is equivalent to the following system of equations:

$$\begin{cases} a_1 = a_1^2 + 2a_2^2 + 2a_3^2 + a_4a_6 + 2a_5^2, \\ a_2 = 2a_1a_2 + 2a_3a_4 + a_4a_5 + 2a_5a_6 + a_6a_3, \\ a_3 = 2a_1a_3 + a_2a_6 + 2a_4a_2 + a_5a_4 + 2a_6a_5, \\ a_4 = 2a_1a_4 + 2a_2a_5 + a_3a_2 + 2a_5a_3 + 2a_6^2, \\ a_5 = 2a_1a_5 + a_2a_4 + 2a_3a_6 + a_4a_3 + 2a_6a_2, \\ a_6 = 2a_1a_6 + 2a_2a_3 + a_3a_5 + a_4^2 + a_5a_2. \end{cases}$$

Calculating the Gröbner basis for this system in GAP we will find that the equations $(a_1 + 1)a_i = 0$, $i = 2, 3, 5, 6$, $a_4 + 2a_6 = 0$ and $a_1^2 + 2a_1 + 2a_2^2 + 2a_3^2 + 2a_5^2 + a_6^2 = 0$ are consequences of the above system.

If $a_1 \neq 2$ we conclude that $a_i = 0$ for $i = 2, \ldots, 6$ and $a_1(a_1 + 2) = 0$. Thus either $a_1 = 0$ (therefore $f = 0$) or $a_1 = 1$ (hence $f = e_{51}$).

Therefore we may assume $a_1 = 2$. Substituting and using the Gröbner basis again, we derive an equivalent system of equations:

$$\begin{cases} (a_2 + a_3 + 2a_5)^2 = 2 \\ \qquad\qquad a_4 = a_6 = a_2^2 + a_3^2 + a_5^2 + 1. \end{cases}$$

If this system has a solution, then the equation $x^2 = 2$ is solvable in F. Conversely, if the equation $x^2 = 2$ has a solution in F, then we can take $a_2 = x$ and $a_3 = a_5 = a_4 = a_6 = 0$, therefore $f = 2v_1 + xv_2$. □

Consider first the case when the idempotent e_{51} decomposes, say $e_{51} = u + u'$, where $P = uR$ and $P' = u'R$ are corresponding principal projectives. Calculating in MAGMA we see that these modules have the following composition series:

$$P : \begin{pmatrix} S \\ S' \\ S \end{pmatrix} \quad \text{and} \quad P' : \begin{pmatrix} S' \\ S \\ S' \end{pmatrix},$$

where S and S' are 2-dimensional simple non-isomorphic R-modules. Arguing as for the first block we obtain that $e_{51}Re_{51} \cong B$.

To complete the proof of Proposition 5 it suffices to consider the case when e_{51} remains indecomposable in R, say this is the case when $F = \mathbb{F}_3$. Then the ring $W = e_{51}Re_{51}$ is local, therefore is a chain ring (say, by Theorem 1). Calculating in MAGMA we see that $e_{51}R$ has (a unique) composition series of length 3 whose composition factors are isomorphic to the same 4-dimensional simple module. Furthermore the skew field $W/\mathrm{Jac}(W)$ is 2-dimensional over F. Since this factor is obtained by tensoring from the corresponding factor over $\mathbb{F}_3$, it is commutative.

However it is easily calculated that W is not commutative (for $\mathbb{F}_3$, hence for an arbitrary field of characteristic 3).

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