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Differential graded categories associated with the critical semi-definite quadratic forms

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Abstract. This work concerns with classification problem of differential graded categories with critical semi-definite quadratic form. We prove that such problem which satisfies some correctness conditions can be transformed to differential graded category with directed graded graph, which is a quiver of affine (extended) type.

Introduction

The reduction algorithm of linear categories and other structures is widely used in the representation theory. This approach allows to study representations inductively, reducing the corresponding categories step by step to representatively simpler ones. ([1]). On the other hand, the important characteristic of represented structure is the induced quadratic form whose roots under certain conditions correspond to the indecomposable representations. The theory of quadratic forms in application to representation theory is well known ([2], [3], [4]). We give the simultaneous reduction algorithm of transformation of the differential graded category with special properties and the underlined unit quadratic form to the canonical form.

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1. Differential graded category and directed graded graph

The $\Bbbk$-linear category $\mathcal{U}$ is called *graded* if $\mathcal{U}(\mathtt{i},\mathtt{j}) = \oplus_{q\in\mathbb{Z}}\mathcal{U}_q(\mathtt{i},\mathtt{j})$ is a sum of finite dimensional vector spaces $\mathcal{U}_q(\mathtt{i},\mathtt{j}) = \deg^{-1}(q)$, $\mathtt{i},\mathtt{j} \in \mathrm{Ob}\,\mathcal{U}$. The graded $\Bbbk$-category $\mathcal{U}$ is called the *differential graded category* or *dgc* if there is the differential $\mathtt{d} : \mathcal{U} \to \mathcal{U}$, $\mathtt{d} : \mathcal{U}_q(\mathtt{i},\mathtt{j}) \to \mathcal{U}_{q+1}(\mathtt{i},\mathtt{j})$, $q \in \mathbb{Z}$, $\mathtt{i},\mathtt{j} \in \mathrm{Ob}\,\mathcal{U}$, and the following properties hold: 1) $\mathtt{d}(1_{\mathtt{i}}) = 0$, $\mathtt{i} \in \mathrm{Ob}\,\mathcal{U}$; 2) Leibnitz rule: $\mathtt{d}(x_1 \dots x_i \dots x_k) = \sum\limits_{i=1}^{k} \hat{x}_1 \dots \hat{x}_{i-1}\mathtt{d}(x_i)x_{i+1}\dots x_k =$ $= \sum_{i=1}^{k}(-1)^{|x_1|}x_1 \dots (-1)^{|x_i|}x_i x_{i+1}\dots x_k$; 3) $\mathtt{d}^2 = 0$.

Let $\Gamma = (\Gamma_0, \Gamma_1, \mathtt{s}, \mathtt{t})$ be a *directed* graph with Γ_0 be a set of vertices and Γ_1 be a set of edges (arrows) equipped with two maps $\mathtt{s} : \Gamma_1 \to \Gamma_0$ and $\mathtt{t} : \Gamma_1 \to \Gamma_0$ that return starting and end (terminating) vertex of the edge correspondingly. Two vertices $\mathtt{i},\mathtt{j} \in \Gamma_0$ are called *incident* on Γ if $\Gamma_1(\mathtt{i},\mathtt{j}) \cup \Gamma_1(\mathtt{j},\mathtt{i}) \neq \varnothing$. The graph $\Gamma = (\Gamma_0, \Gamma_1, \mathtt{s}, \mathtt{t})$ is called *graded* (or $\mathbb{Z}$-graded) if there is the map $\deg : \Gamma_1 \to \mathbb{Z}$, such that $\Gamma_1 = \bigsqcup\limits_{q\in\mathbb{Z}} \Gamma_1^q$, $\Gamma_1^q = \bigsqcup\limits_{\mathtt{i},\mathtt{j}\in\Gamma_0} \Gamma_1^q(\mathtt{i},\mathtt{j}) = \deg^{-1}(q)$. We denote $|x| = \deg x$ and $\hat{x} = (-1)^{|x|}x$. The graph Γ is called 0-quiver or *quiver* if $\Gamma_1^q(\mathtt{i},\mathtt{j}) = \varnothing$ whenever $q \neq 0$.

Let $\Bbbk$ be an algebraically closed field. We consider $\Bbbk\Gamma$ the $\Bbbk$-linear path category of the graded graph Γ which is freely generated over $\Bbbk$ by all the pathes on Γ. We denote $\mathrm{coeff}_{x_1\dots x_k} x = \kappa$, $\kappa \in \Bbbk$ whenever $x = \kappa x_1 \dots x_k + \dots$ is a basis decomposition. Category $\Bbbk\Gamma$ inherits the graduation from Γ such that $\deg x_1 x_2 \dots x_k = \sum_{i=1}^{k} \deg x_i$.

Any edge $a \in \Gamma_1(\mathtt{i},\mathtt{j})$ is called *regular* if the differential of a does not have the summand of a type κx where $x \in \Gamma_1(\mathtt{i},\mathtt{j})$ and $\kappa \in \mathbb{Z}$, that is pathes with length $= 1$ are not summands of differential of a. The dgc $\mathcal{U}$ is called *regular* if all edges from Γ_1 are regular.

Given a dgc $\mathcal{U}$ with $|\mathrm{Ob}\,\mathcal{U}| < \infty$, define the underlined directed graded graph $\Gamma = \Gamma(\mathcal{U})$ such that $\Gamma_0 = \mathrm{Ob}\,\mathcal{U}$, and $\Gamma_1(\mathtt{i},\mathtt{j})$ is a basis of $(\mathcal{U}/\mathcal{U}^{\otimes 2})(\mathtt{i},\mathtt{j})$, $\mathtt{i},\mathtt{j} \in \Gamma_0$ with the induced graduation. The differential $\mathtt{d}$ induces the map $\mathtt{d} : \Gamma_1^q \to \Bbbk\Gamma_{q+1}(\mathtt{i},\mathtt{j})$, $\mathtt{i},\mathtt{j} \in \Gamma_0$, $q \in \mathbb{Z}$, which is extended on the whole $\Bbbk\Gamma$ by Leibnitz rule.

The graph Γ which is correspondent to the finite dimensional differential graded category is finite. The graph Γ is called *correctly defined* if it is directed cycle-free and it does not have parallel edges.

The full subgraph Γ_S, $S \subset \Gamma_0$, $|S| > 2$ is called *closed contour* if there is an ordering $S = \{\mathtt{i}_1, \dots, \mathtt{i}_k\}$ such that $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1}) \cup \Gamma_1(\mathtt{i}_{j+1}, \mathtt{i}_j)| > 0$, $j = 1, \dots, k-1$, and $|\Gamma_1(\mathtt{i}_1, \mathtt{i}_k) \cup \Gamma_1(\mathtt{i}_k, \mathtt{i}_1)| > 0$. The closed contour Γ_S, $S =$

$\{\mathtt{i}_1,\ldots,\mathtt{i}_k\}\subset\Gamma_0$ is *clear* if $\Gamma_1(\mathtt{i}_s,\mathtt{i}_t)\cup\Gamma_1(\mathtt{i}_t,\mathtt{i}_s)=\varnothing$, $|s-t|>1\,(\mathrm{mod}\ k)$ and $|\Gamma_1(\mathtt{i}_j,\mathtt{i}_{j+1})\cup\Gamma_1(\mathtt{i}_{j+1},\mathtt{i}_j)|=1$. The closed contour Γ_S is called *oriented cycle* if $|\Gamma_1(\mathtt{i}_j,\mathtt{i}_{j+1})|>0,\quad j=1,\ldots,k-1$, and $|\Gamma_1(\mathtt{i}_k,\mathtt{i}_1)|>0$. The closed contour Γ_S is called *detour contour* if $|\Gamma_1(\mathtt{i}_j,\mathtt{i}_{j+1})|>0,\quad j=1,\ldots,k-1$, and $|\Gamma_1(\mathtt{i}_1,\mathtt{i}_k)|>0$. Denote $x_{\mathtt{ij}}$ the edge starting in $\mathtt{i}$ and ending in $\mathtt{j}$. Detour contour Γ_S is called *active* (or contour of differential type) if $\kappa x_{\mathtt{i}_1\mathtt{i}_2}\ldots x_{\mathtt{i}_{k-1}\mathtt{i}_k}$ is a summand of differential of the edge $x_{\mathtt{i}_1\mathtt{i}_k}$. The edge $a\in\Gamma_1(\mathtt{i},\mathtt{j})$ is called *deep* if there are no other pathes on Γ from $\mathtt{i}$ to $\mathtt{j}$. $a\in\Gamma_1(\mathtt{i},\mathtt{j})$ is called *minimal* if $\mathtt{d}(a)=0$.

2. Quadratic form

We associate with correctly defined graded graph $\Gamma=(\Gamma_0,\Gamma_1,\mathtt{s},\mathtt{t})$ the undirected bigraph $\mathcal{B}=\mathcal{B}(\Gamma)=(\Gamma_0,\mathcal{B}_1)$ in the following way. We denote by $\mathcal{B}_1$ the set of pairs $\{\mathtt{i},\mathtt{j}\}$ of vertices from Γ_0 that are incident (have the common edge) in Γ. Graduation on $\mathcal{B}_1$ is correspondent to graduation on Γ: $\deg(\{\mathtt{i},\mathtt{j}\})=|\{\mathtt{i},\mathtt{j}\}|=\deg a \pmod 2$, $a\in\Gamma_1(\mathtt{i},\mathtt{j})\cup\Gamma_1(\mathtt{i},\mathtt{j})$, then $\mathcal{B}_1=\mathcal{B}_1^0\sqcup\mathcal{B}_1^1$. Denote by $\chi=\chi(\Gamma)$ the integral unit quadratic form $\chi:\mathbb{Z}^n\to\mathbb{Z}:\chi(x)=\sum_{\mathtt{i}\in\Gamma_0}x_{\mathtt{i}}^2-\sum_{\{\mathtt{i},\mathtt{j}\}\in\mathcal{B}_1}(-1)^{|\{\mathtt{i},\mathtt{j}\}|}x_{\mathtt{i}}x_{\mathtt{j}}$.

For the graph $\Gamma=(\Gamma_0,\Gamma_1,\mathtt{s},\mathtt{t})$ and $i,j\in\Gamma_0$ we denote by (i,j) - the edge of graph Γ with unknown or arbitrary direction. The edges with even (odd) degree are usually drown solid (dotted).

Let $n=|\Gamma_0|$. We say that χ is *positive* (*negative*) if $\chi(r)>0$ ($\chi(r)<0$) for all $r\neq 0$. A *semi-definite* quadratic form χ is defined as neither positive nor negative. An integer vector r of integer latice $\mathbb{Z}^n$ is called a *root* (real root)) if $\chi(r)=1$, and it is called an *image root* if $\chi(r)=0$. The canonical base vectors $\mathtt{e}^{\mathtt{i}}$ are called simple roots. The root $r=(r_{\mathtt{i}})_{\mathtt{i}\in\Gamma_0}$ is called *positive root* (resp., negative root) if in addition $r_{\mathtt{i}}\in\mathbb{Z}_+$ (resp., $r_{\mathtt{i}}\in\mathbb{Z}_-$) for any $\mathtt{i}\in\Gamma_0$ (we assume $0\in\mathbb{Z}_+\cap\mathbb{Z}_-$). The root r is called *sincere* if $r_{\mathtt{i}}\neq 0$ for all $\mathtt{i}\in\Gamma_0$.

The *kernel* of a symmetric bilinear form is the set of vectors $\ker\chi=\{x\in\mathbb{Q}^n\mid\chi(x,y)=0 \text{ for all } y\in\mathbb{Q}^n\}$. For semi-definite quadratic form, each image root belongs to kernel. The semi-definite quadratic form χ with bigraph $\mathcal{B}$ is called *critical* if each full sub form is positive. The critical forms have sincere one dimensional kernel.

For $\mathtt{i}\in\mathcal{B}_0$, we denote by $T_{\mathtt{i}}:\mathbb{Z}^n\to\mathbb{Z}^n$ the $\mathbb{Z}$-linear transformation:

$$T_{\mathtt{i}}(\mathtt{e}^t)=\begin{cases}\mathtt{e}^t, & \text{if } t\neq i;\\ -\mathtt{e}^i, & \text{if } t=i.\end{cases}\tag{1}$$

We call $T_{\mathtt{i}}$ a *sign change* for χ in the vertex $\mathtt{i}$. For $\{\mathtt{i},\mathtt{j}\} \in \mathcal{B}_1$, we denote by $T^{\varepsilon}_{\mathtt{ij}} : \mathbb{Z}^n \to \mathbb{Z}^n$ the $\mathbb{Z}$-linear transformation ([4], [5]):

$$T^{\varepsilon}_{\mathtt{ij}}(\mathtt{e}^t) = \begin{cases} \mathtt{e}^t, & \text{if } \ t \neq i; \\ \mathtt{e}^i + (-1)^{|\{\mathtt{i},\mathtt{j}\}|}\mathtt{e}^j, & \text{if } \ t = i. \end{cases} \tag{2}$$

with $\varepsilon = (-1)^{|\{\mathtt{i},\mathtt{j}\}|} \in \{+,-\}$. If a degree $|\{\mathtt{i},\mathtt{j}\}|$ is even then we call $T^{+}_{\mathtt{ij}}$ an *inflation* for χ, if $|\{\mathtt{i},\mathtt{j}\}|$ is odd, we call $T^{-}_{\mathtt{ij}}$ a *deflation* for χ.

We denote the corresponding transformations of quadratic form and an integral lattice $\mathbb{Z}^n$ by the same letter. So there are $T : \chi \to \chi' = \chi T$ for the quadratic form and $T : \mathtt{r} \to \mathtt{r}' = \mathtt{r}T$ for vector $r = \sum\limits_{\mathtt{j}\in\Gamma_0} r_{\mathtt{j}}\mathtt{e}_{\mathtt{j}}$, such that $\sum\limits_{\mathtt{j}\in\Gamma_0} r_{\mathtt{j}}\mathtt{e}_{\mathtt{j}} = \sum\limits_{\mathtt{j}\in\Gamma_0} r'_{\mathtt{j}}\mathtt{e}'_{\mathtt{j}}$ or $\chi(\mathtt{r}) = \chi'(\mathtt{r}')$.

Two integral forms $\chi, \chi' : \mathbb{Z}^n \to \mathbb{Z}$ are called *equivalent* (or $\mathbb{Z}$-*equivalent*) if they describe the same maps up to above changes of basis, that is, if there exists a linear $\mathbb{Z}$-invertible transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ which is a composition of admitted transformations such that $\chi' = \chi T$. The next simple lemma holds.

Lemma 1. *Let $T : \chi \to \chi T$ be an equivalence of the quadratic forms. If χ is an integral unit form, then χT is an integral unit form as well, and χT is positive (non negative, critical) if and only if χ is positive (non negative, critical). Besides, $T : \ker\chi \to \ker\chi'$.*

For bigraph $\mathcal{B}$ we will use notions of *chain*, *simple and closed chain*, *tree* and *forest* in common meaning. We say that tree $\mathcal{B}$ is 0-tree (0-forrest) if any edge has degree 0. Any point of tree which is incident with more than two edges is called *branch point.*

Proposition 1. *([4])Let χ be an integral positive (resp., semi-definite with* 1 *dimensional kernel) unit form, $\mathcal{B}$ its bigraph. Then there is a sequence of sign changes of type* (1) *and deflations of type* (2) *with the composition T such that the bigraph $\mathcal{B}T$ of the form χT is a* 0*-forrest of Dynkin (resp., affine) type. For the positive form, $\mathcal{B}T$ is a disjoint union of some of the following Dynkin diagrams: A_n $(n > 1)$, D_n $(n \geqslant 4)$, or E_n $(n = 6, 7, 8)$. For the case of affine form, $\mathcal{B}T$ is a disjoint union of some of the following affine diagrams: $\widetilde{A}_n$ $(n > 1)$, $\widetilde{D}_n$ $(n \geqslant 4)$, or $\widetilde{E}_n$ $(n = 6, 7, 8)$. If $\mathcal{B}$ is connected then $\mathcal{B}T$ is just a tree. The Dynkin (affine) type is uniquely defined by χ.*

The next two lemmas are simple consequences of this proposition.

Lemma 2. *Let* $(\chi, \mathcal{B})$ *be a critical quadratic form. Then* $\ker \chi = \mathbb{Z} \cdot \mathtt{r}$ *where* $\mathtt{r} = \sum_{i=1}^{n} r_i e^i \in \mathbb{Z}^n$ *is a sincere image root, besides, there is* $\mathtt{j} \in \Gamma_0$ *such that* $r_{\mathtt{j}} = 1$ *and* $\mathtt{r} = \sum_{i \neq j} r_i e^i \in \mathbb{Z}^{n-1}$ *is a sincere root on the positive defined full sub form* $\mathcal{B}|_{\Gamma_0 \setminus \{\mathtt{j}\}}$.

Lemma 3. *Let* $(\chi, \mathcal{B})$ *be a critical quadratic form and* $\mathtt{r}$ *be it's sincere root which has at least one entry* $= 1$. *Then there exists a sequence* T *of sign changes and deflations such that* $\mathtt{r}' = \mathtt{r}T$ *is a sincere positive root, there is* $\mathtt{j} \in \Gamma_0$ *such that* $r'_{\mathtt{j}} = 1$, *and* $r'_{\mathtt{i}} \leqslant 6$ *for any* $\mathtt{i} \in \Gamma_0$.

3. The main result

We consider the problems, that consist of the regular differential graded category (dgc) $\mathcal{U}$ together with it's underlined graded directed cycle-free graph Γ and undirected bigraph $\mathcal{B}$. We assume each clear contour of those dgc to be active, and underlined graph to be correctly defined. The class of such problems is denoted by Υ. The sub class of Υ consisting of the problems having positive quadratic form $\chi = \chi_{\mathcal{B}}$ will be denoted by Υ_+. The problem from Υ is considered as a triple $(\mathcal{U}, \Gamma, \mathcal{B})$.

The sub class to be considered in this paper consisting of the problems having the critical quadratic form $\chi = \chi_{\mathcal{B}}$, hence χ is a semi-definite quadratic form with sincere one parameter kernel. This class will be denoted by Υ_0. By Lemma 2 each problem from Υ_0 has a unique sincere image root $\mathtt{r} \in \ker \chi$ such that $r_{\mathtt{j}} = 1$ for some $\mathtt{j} \in \Gamma_0$. Therefore, the problem from Υ_0 is the quadruple $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}, \mathtt{r})$ where $(\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon$.

The connected problem $\mathfrak{A} \in \Upsilon_0$ is called an *affine problem*, and the correspondent graph Γ is called affine (extended) directed graded graph if $\mathcal{B}(\Gamma)$ is one of the affine diagrams ($\widetilde{A}_n$, $\widetilde{D}_n$ ($n \geqslant 4$), $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$). In this case $\mathtt{r}$ is a well known minimal positive image root having at least one entry 1. The sub class of affine problems is denoted by $\Upsilon_{\mathit{aff}} \subset \Upsilon_0$. We use the proposition from [5] which can be reformulated as follows.

Proposition 2. *For any* $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon_+$ *there exists a composition of admitted transformations* $\mathcal{R} : \mathfrak{A} \to \mathfrak{A}' = (\mathcal{U}', \Gamma', \mathcal{B}')$ *such that* $\mathfrak{A}' \in \Upsilon_+$ *is a tree, hence* $\mathcal{B}'$ *is a Dynkin diagram.*

In this parer we prove the following theorem

Theorem 1. *Let* $\mathcal{U}$ *be differential graded category having a correctly defined underlined graded graph* Γ *and critical semi-definite quadratic* *form* $(\chi, \mathcal{B})$. *We assume that any clear contour is of differential type. Then there exists a composition of transformations* $\mathcal{R} : \mathcal{U} \to \mathcal{U}'$ *such that* $\mathcal{U}'$ *is an affine problem.*

We reformulate the Theorem 1 in the way similar to Proposition 2:

Theorem 2. *For any* $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}, \mathtt{r}) \in \Upsilon_0$, *there exists a composition of admitted transformations* $\mathcal{R} : \mathfrak{A} \to \mathfrak{A}' = (\mathcal{U}', \Gamma', \mathcal{B}', \mathtt{r}')$ *such that* $\mathfrak{A}' \in \Upsilon_{aff}$, *hence* $\mathcal{B}'$ *is an affine diagram.*

4. Admitted transformations

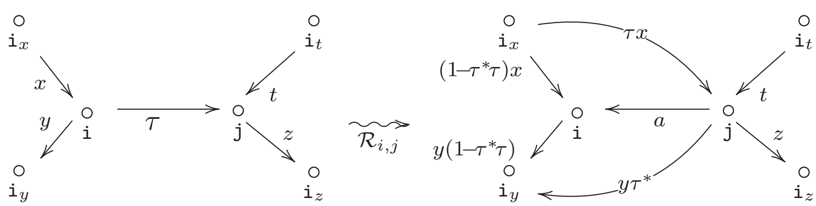
We consider a problem $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}, \mathtt{r}) \in \Upsilon$. The transformations on class Υ described in this section and their consequent combinations are called *admitted*. The problem obtained after using of such transformations again belongs to the class Υ. We repeat and extend the algorithm of reduction of the problem $(\mathcal{U}, \Gamma, \mathcal{B})$ shown in [5].

Denote by $\widehat{\mathcal{U}}$ the dg category, and by $\widehat{\Gamma}$ the correspondent graph, augmented by the set of loops $\Omega = \{\omega_{\mathtt{i}} \in \Gamma_1^1(\mathtt{i}, \mathtt{i}) \mid \mathtt{i} \in \Gamma_0\}$ with differential $\partial : \widehat{\mathcal{U}} \to \widehat{\mathcal{U}}$ such that $\partial(\omega_{\mathtt{i}}) = \omega_{\mathtt{i}}^2$ and $\partial(a) = a\omega_{\mathtt{i}} + (-1)^{|a|+1}\omega_{\mathtt{j}} a + \mathtt{d}(a)$, $a \in \Gamma_1(\mathtt{i}, \mathtt{j})$, $a \notin \Omega$. Then the condition $\partial^2 = 0$ and Leibnitz rule hold. The dgc $\widehat{\mathcal{U}}$ is called *augmented* for $\mathcal{U}$.

Here on the diagrams below we draw all edges as solid arrows but they can have different degrees, moreover, we depict the direction of the arrow, if it does not matter.

4.1. Reduction of a deep edge

Suppose that $\tau \in \Gamma_1(\mathtt{i}, \mathtt{j})$ is a deep minimal regular edge with degree $\deg \tau = |\tau|$. The general case is:



Define the reduction $\mathcal{R}_{\mathtt{ij}} : \Gamma \to \Gamma'$. We assume that there is $\tau^* : \mathtt{j} \to \mathtt{i}$ such that $\tau\tau^* = 1_{\mathtt{j}}$, and $1_{\mathtt{i}} = 1_{\mathtt{i}_1} + 1_{\mathtt{i}_2} = (1-\tau^*\tau) + \tau^*\tau$ is a decomposition on the sum of mutually commuting idempotents. Then in $\widehat{U}$ we obtain

$\omega_{\mathtt{i}} \quad \Longleftrightarrow \quad \begin{pmatrix} \omega_{\mathtt{i}_1} & \varphi_{21} \\ \varphi_{12} & \omega_{\mathtt{i}_2} \end{pmatrix} = \begin{pmatrix} (1-\tau^*\tau)\omega_{\mathtt{i}}(1-\tau^*\tau) & (1-\tau^*\tau)\omega_{\mathtt{i}}\tau^*\tau \\ \tau^*\tau\omega_{\mathtt{i}}(1-\tau^*\tau) & \tau^*\tau\omega_{\mathtt{i}}\tau^*\tau \end{pmatrix}$ and

$\tau \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & \tau \end{pmatrix}$.

Using that $\partial(\tau) = \tau\omega_{\mathtt{i}} + \omega_{\mathtt{j}}\tau$ on $\mathcal{U}$, we have $\varphi_{12} = 0$, and on $\widehat{U}$: $\partial(\tau) = \tau\omega_{\mathtt{i}_2} + \omega_{\mathtt{j}}\tau$, $\partial(\tau^*) = \omega_{\mathtt{i}_2}\tau^* + \tau^*\omega_{\mathtt{j}}$, $\partial(\varphi_{21}) = \varphi_{21}\omega_{\mathtt{i}_2} + \omega_{\mathtt{i}_1}\varphi_{21}$. Due to the construction, $1_{\mathtt{j}} = \tau\tau^*$ and $1_{\mathtt{i}_2} = \tau^*\tau$, hence the points $\mathtt{j}$, $\mathtt{i}_2$ are isomorphic. Denote by $\mathcal{R}_{\mathtt{ij}}\mathcal{U}$ and $\mathcal{R}_{\mathtt{ij}}\Gamma$ the dgc and graph which is obtained from the constructed above by factorization on the point $\mathtt{i}_2$. We denote $a = \varphi_{21}\tau^* : \mathtt{j} \to \mathtt{i}_1$, then $|a| = |\tau^*| + 1 = 1 - |\tau|$, and $\partial(a) = \partial(\varphi_{21}\tau^*) = \partial(\varphi_{21})\tau^* - \varphi_{21}\partial(\tau^*) = (\varphi_{21}\omega_{\mathtt{i}_2} + \omega_{\mathtt{i}_1}\varphi_{21})\tau^* - \varphi_{21}(\omega_{\mathtt{i}_2}\tau^* + \tau^*\omega_{\mathtt{j}}) = \omega_{\mathtt{i}_1}a - a\omega_{\mathtt{j}}$.

For any $x : \mathtt{i}_x \to \mathtt{i}$ we obtain the edges $(1-\tau^*\tau)x : \mathtt{i}_x \to \mathtt{i}$, $|(1-\tau^*\tau)x| = |x|$ and $\tau x : \mathtt{i}_x \to \mathtt{j}$, $|\tau x| = |x| + |\tau|$, besides, $\mathtt{d}'((1-\tau^*\tau)x) = a\tau x + (\mathtt{d}(x))'$. For any $y : \mathtt{i} \to \mathtt{i}_y$ there are: $y(1-\tau^*\tau) : \mathtt{i} \to \mathtt{i}_y$, $|y(1-\tau^*\tau)| = |y|$ and $y\tau^* : \mathtt{i} \to \mathtt{j}$, $|y\tau^*| = |y| - |\tau|$, and, $\mathtt{d}'(y\tau^*) = y(1-\tau^*\tau)a + (\mathtt{d}(y))'$.

The differential on $\mathcal{R}_{\mathtt{ij}}\mathcal{U}$ is obtained by substitution $1_{\mathtt{i}} = (1-\tau^*\tau) + \tau^*\tau$. Any path crossing on the point $\mathtt{i}$ is a combination of pathes:

$$y_1 \dots y_q \, yx \, x_p \dots x_1 \Longleftrightarrow y_1 \dots y_q \, \begin{pmatrix} y(1-\tau^*\tau) & y\tau^*\tau \end{pmatrix} \begin{pmatrix} (1-\tau^*\tau)x \\ \tau^*\tau x \end{pmatrix} x_p \dots x_1.$$

The reduction $\mathcal{R}_{\mathtt{ij}} : \mathcal{U} \to \mathcal{U}'$ is transferred to the reduction $\mathcal{R}_{\mathtt{ij}} : \Gamma \to \Gamma'$ and $\mathcal{R}_{\mathtt{ij}} : \mathcal{B} \to \mathcal{B}'$. For any $x \in \mathbb{Z}^n$ we obtain: $\mathcal{R}_{\mathtt{ij}} : \mathtt{x} \to \mathtt{x}'$ where $x'_i = x_i - (-1)^{|\tau|}x_j$ for $x'_k = x_k$ otherwise. Therefore the transformation $\mathcal{R}_{\mathtt{ij}} : \mathfrak{A} \to \mathfrak{A}'$ is defined. Sometimes we denote it by $\mathcal{R}^+_{\mathtt{ij}}$ if $|\tau|$ is even, and by $\mathcal{R}^-_{\mathtt{ij}}$ otherwise. Note that $\mathcal{R}^+_{\mathtt{ij}} : \mathcal{B} \to \mathcal{B}'$ is an inflation, and is an $\mathcal{R}^-_{\mathtt{ij}} : \mathcal{B} \to \mathcal{B}'$ deflation. If the points $\mathtt{i}$ and $\mathtt{j}$ are not incident on Γ then the reduction $\mathcal{R}_{i,j}$ is trivial, hence $\mathfrak{A}' = \mathfrak{A}$.

Note that $\mathcal{R}_{\mathtt{ij}}\mathcal{U}$ is not augmented dgc and it is directed cycle-free (as well as $\mathcal{U}$), but it is not necessarily regular. Namely, it is possible that $\Gamma_1(\mathtt{k}, \mathtt{j}) = \{x, y\}$ for some $\mathtt{k} \in \Gamma_0$. By the construction, in this case $\mathtt{d}(x) = \kappa y + l$ where $l \in \mathcal{P}^2$ and $|y| = |x| + 1$. Then we put: $x = 0$, $\mathtt{d}(x) = 0$, $y = -\kappa^{-1}l$, and obtain the new dgc $\mathcal{U}'$ with the graph Γ'. We say that $\mathcal{U}'$ is obtained from $\mathcal{U}$ by regularization on x, y. The quadratic form χ and the attached vector $\mathtt{r} \in \mathbb{Z}^n$ do not change after regularization operation. The case (or $|\Gamma_1(\mathtt{j}, \mathtt{k})| = 2$) is analogous. Given a reduced problem $\mathcal{R}_{\mathtt{ij}}\mathfrak{A}$, we can do some number of regularization procedure to obtain the regular problem. We call this transformation a *complete reduction* and denote it with the same letter $\mathcal{R}_{\mathtt{ij}}$.

The following lemma follows from the condition $\mathtt{d}^2 = 0$ and from the observation that for positive form the sum of degrees around the clear contour can not be even.

Lemma 4. *Let $\mathfrak{A} \in \Upsilon$, let $\tau \in \Gamma_1(\mathtt{i},\mathtt{j})$ be a minimal deep regular edge, and let $\mathcal{R}_{\mathtt{ij}} : \mathfrak{A} \to \mathfrak{A}'$ be a complete reduction. Then $\mathfrak{A}\mathcal{R}_{\mathtt{ij}} \in \Upsilon$ and $\mathfrak{A}\mathcal{R}_{\mathtt{ij}} \in \Upsilon_0$ whenever $\mathfrak{A} \in \Upsilon_0$.*

4.2. Turning arrows at a vertex

We assume the graph Γ to be directed cycle-free. Denote by Γ_0^+ (resp., Γ_0^-) the subset of vertices $\mathtt{i} \in \Gamma_0$ such that $\Gamma_1(\mathtt{i},\mathtt{j}) = \varnothing$ (resp., $\Gamma_1(\mathtt{j},\mathtt{i}) = \varnothing$) for any $\mathtt{j} \in \Gamma_0$. In this chapter we consider $\operatorname{add}\mathcal{P}$ the additive closure of the path category $\mathcal{P}$. Let $\mathtt{j} \in \Gamma_0^+$, $\mathtt{i}_k \in \Gamma_0$, and $a_k \in \Gamma_1(\mathtt{i}_k,\mathtt{j})$, $k \in \{1,\dots,p\}$ are all edges ending at a point $\mathtt{j}$. Consider the following mapping $\Theta_{\mathtt{j}} : \operatorname{add}\mathcal{P} \to \operatorname{add}\mathcal{P}$: $\Phi_{\mathtt{j}} : \mathtt{i}_1 \oplus \mathtt{i}_2 \oplus \ldots \oplus \mathtt{i}_p \to \mathtt{j}$, $\quad [\Phi_{\mathtt{j}}] = \big(-\hat{a}_k \big)_{k=1}^{p}$ where $\hat{a}_k = (-1)^{|a_k|} a_k$. There is the mapping

$$\Theta_{\mathtt{j}} : \mathtt{i}_1 \oplus \mathtt{i}_2 \oplus \ldots \oplus \mathtt{i}_p \to \mathtt{i}_1 \oplus \mathtt{i}_2 \oplus \ldots \oplus \mathtt{i}_p, \quad [\Theta_{\mathtt{j}}] = (\theta_{ij}), \quad \theta_{ij} \in \mathcal{P}(\mathtt{i}_i,\mathtt{i}_j)$$

such that the differentials are defined by:

$$[\Phi_{\mathtt{j}}\Theta_{\mathtt{j}}] = [\Phi_{\mathtt{j}}]\,[\Theta_{\mathtt{j}}] = [\mathtt{d}(a_i)] = (\mathtt{d}(a_1)\ldots\mathtt{d}(a_p)).$$

Now we turn over the arrows $a_1,\dots,a_p$, we obtain the arrows $b_k : \mathtt{j} \to \mathtt{i}_k$ and put $|b_k| = -|a_k|$. Denote by Γ' the new graded graph. It is directed cycle-free and the quadratic forms χ_Γ and $\chi_{\Gamma'}$ coincides.

Given any $\mathtt{r} \in \mathbb{Z}^n$, we define the vector $\mathtt{r}' \in \mathbb{Z}^n$ such that $r'_k = r_k$, if $k \neq j$, and r'_j is defined by the formulae: $r'_j = \sum\limits_{k=1}^{p} (-1)^{|\{\mathtt{i}_k,\mathtt{j}\}|} r_i - r_j$. If $\mathtt{r} \in \ker\chi$ then $\mathtt{r}'_{\mathtt{j}} = \mathtt{r}_{\mathtt{j}}$.

Lemma 5. *Let $\mathfrak{A} \in \Upsilon$, $\mathtt{j} \in \Gamma_0^+$. Define the dgc $\mathcal{U}'$ which has the arrows $b_1,\dots,b_p$ instead of $a_1,\dots,a_p$ and differential is given by the formulae:*

$$[\Theta_{\mathtt{j}}\Psi_{\mathtt{j}}] = [\Theta_{\mathtt{j}}]\,[\Psi_{\mathtt{j}}] = [\mathtt{d}(b_i)] = (\mathtt{d}(a_1)\ldots\mathtt{d}(b_p))^t$$

where $\Psi_{\mathtt{j}} : \mathtt{j} \to \mathtt{i}_1 \oplus \mathtt{i}_2 \oplus \ldots \oplus \mathtt{i}_p$, $\quad [\Psi_{\mathtt{j}}]^t = \big(b_k\big)_{k=1}^{p}$. Then the problem $\mathfrak{A}' = (\mathcal{U}',\Gamma',\mathcal{B}',\mathtt{r}')$ belongs to Υ, and, if any clear contour on Γ is active, then any clear contour on Γ' is active as well.

Proof. By definition of differential, we have: $\mathtt{d}(a_i) = \sum\limits_{k=1}^{p} (-1)^{|a_k|+1} a_k\theta_{ik}$ such that $\mathtt{d}^2(a_i) = 0$, $i = 1, \dots, p$, where $\theta_{ik} \in \mathcal{P}(\mathtt{i}, \mathtt{k})$ are pathes. Then

$$\mathtt{d}^2(a_i) = \sum_{k=1}^{p}\sum_{l=1}^{p} a_l(-1)^{|a_k|+|a_l|}\theta_{kl}\theta_{ik} + \sum_{l=1}^{p}(-1)^{|a_l|+|a_l|+1}a_l\mathtt{d}(\theta_{il}).$$

We obtain the condition: $\mathtt{d}(\theta_{il}) = \sum_{k=1}^{p}(-1)^{|a_k|+|a_l|}\theta_{kl}\theta_{ik} = 0$, $i, l = 1, \dots, p$ Besides, the following equalities hold: $|a_i| + 1 = |\theta_{ik}| + |a_k|$.

After turning we obtain the edges $b_l : \mathtt{j} \to \mathtt{i}_l$, $l = 1, \dots, p$. Denote the obtained graph by Γ'. Consider the differential $\mathtt{d} : \Gamma'^q_1(\mathtt{j}, \mathtt{i}_l) \to \mathcal{U}'_{q+1}(\mathtt{j}, \mathtt{i}_l)$ such that $\mathtt{d}(b_l) = \sum_i \theta_{il} b_i$. Here $|b_l| = -|a_l| = |\theta_{il}| - |a_i| - 1 = |\theta_{il}| + |b_i| - 1$. Prove that $\mathtt{d}^2(b_l) = 0$, $l = 1, \dots, p$. We have: $\mathtt{d}^2(b_l) = \sum_i \mathtt{d}(\theta_{il})b_i + \sum_k (-1)^{|\theta_{kl}|}\theta_{kl}\mathtt{d}(b_k) = \sum_i \mathtt{d}(\theta_{il})b_i + \sum_k(-1)^{|\theta_{kl}|}\theta_{kl}\sum_i \theta_{ik}b_i$, then $\mathtt{d}^2(b_l) = \sum_i \mathtt{d}(\theta_{il})b_i + \sum_i\sum_k(-1)^{|\theta_{kl}|}\theta_{kl}\theta_{ik}b_i$, and we obtain the required condition: $\mathtt{d}(\theta_{il}) = \sum_k(-1)^{|\theta_{kl}|+1}\theta_{kl}\theta_{ik}, \quad i, l = 1, \dots, p$. Because $|a_l| + |\theta_{kl}| = |a_k| + 1$, then it is the same condition as above. ☐

The case $\mathtt{j} \in \Gamma_0^-$ can be considered similarly.

4.3. Change of the degree

For any $k \in \mathbb{Z}$, $\mathfrak{A} \in \Upsilon$ and any $\mathtt{j} \in \Gamma_0$ we define the following transformation $\mathcal{D}_{\mathtt{j}}^{(k)} : \mathfrak{A} \to \mathfrak{A}'$. We assume that the category $\mathcal{U}'$ has the same objects and morphisms as $\mathcal{U}$. For any $\mathtt{i} \in \Gamma_0$, and any $a \in \Gamma_1$, we set $\deg_{\mathcal{U}'} a = \deg_{\mathcal{U}} a + k$ if $t(a) = \mathtt{j}$, and $\deg_{\mathcal{U}'} a = \deg_{\mathcal{U}} a - k$ if $s(a) = \mathtt{j}$. Then the differential $\mathtt{d}$ on $\mathcal{U}$ is correctly defined on $\mathcal{U}'$ too. Given any $\mathtt{r} \in \mathbb{Z}^n$, we define the vector $\mathtt{r}' \in \mathbb{Z}^n$ such that $r'_k = r_k$, if $k \neq j$, and r'_j is defined by the formulae: $r'_j = (-1)^k r_j$. Then $\mathfrak{A}' \in \Upsilon$. The transformation $\mathcal{D}_{\mathtt{j}}^{(k)}$ is called *change of the degree* for dgc $\mathcal{U}$.

5. Proof of the main result

We prove the theorem 2. We say that the tree-graph Γ is *well directed* if it does not have non trivial pathes of a length > 1. We prove in [5] that Dynkin graph Γ can be reduced to a well directed graded graph of the correspondent type. The following lema says that well directed graph can be transformated to well directed 0-tree.

Lemma 6. *Let $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon_+$, and let Γ be a well directed Dynkin tree. Then there exists the composition of change of degree transformations $\mathcal{R} : \mathfrak{A} \to \mathfrak{A}'$ such that the graph Γ' is well directed Dynkin 0-tree.*

Let $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}, \mathtt{r}) \in \Upsilon_0$, that is the quadratic form χ is critical semi-definite. Then Γ is a connected graph. By Lemma 3, for $\mathtt{r} = \sum\limits_{i=1}^{n} r_i e^i$, there is $\mathtt{j} \in \Gamma_0$ such that $r_{\mathtt{j}} = 1$. We denote by $\check{\mathtt{r}} = \sum_{i \neq j} r_i e^i \in \mathbb{Z}^{n-1}$ the sincere root of the positive full sub problem $\check{\mathfrak{A}} = \mathfrak{A}|_{\Gamma_0 \setminus \{\mathtt{j}\}}$.

Using the turning arrows at some vertices of $\mathfrak{A}$ if necessary, we obtain the condition $\mathtt{j} \in \Gamma_0^+$ (or analogously $\mathtt{j} \in \Gamma_0^-$). Then any edge $a \in \Gamma_1$ is deep on the subgraph $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ if and only if it is deep on the whole Γ. Therefore the point $\mathtt{j}$ still belongs to Γ_0^+ or Γ_0^- correspondently after reduction on the subproblem $\check{\mathfrak{A}}$.

By Theorem 2 concerning the positive forms, there exists admitted transformation without using of turning procedure $\mathcal{R}_1 : \mathfrak{A} \to \mathfrak{A}'$, $\mathcal{R}_1 : \mathtt{r} \mapsto \mathtt{r}'$, $r'_{\mathtt{j}} = r_{\mathtt{j}} = 1$ such that the subgraph $\check{\Gamma} = \Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ is reduced to the well directed tree of Dynkin type ([5]). By Lemma 6, there exists a reduction transformation $\mathcal{R}_2 : \mathfrak{A}' \to \mathfrak{A}''$, $\mathcal{R}_2 : \mathtt{r}' \mapsto \mathtt{r}''$ with $\mathcal{R}_2 : \check{\mathfrak{A}}' \to \check{\mathfrak{A}}''$ with $\check{\Gamma}''$ be a well directed Dynking 0-tree. Since $\mathtt{r}''_{\mathtt{j}} = \mathtt{r}'_{\mathtt{j}} = \mathtt{r}_{\mathtt{j}}$ then $\check{\mathtt{r}}'' = \sum\limits_{i \neq j} r''_i e^i$ is a sincere root on $\check{\mathfrak{A}}''$ by Lemma 2. The sincere root of Dynking 0-tree is either positive or negative, we assume it to be positive.

Denote $\mathtt{n} = |\Gamma_0|$, we can set the numeration on Γ_0 such that $\mathtt{j} = \mathtt{n}$. For the further proof, we can assume $\mathtt{n} \in \Gamma_0^+$, the case $\mathtt{n} \in \Gamma_0^-$ can be considered similarly. For the further proof, we assume that the problem already satisfies the condition: $\check{\Gamma} = \Gamma_{\Gamma_0 \setminus \{\mathtt{n}\}}$ is well directed Dynkin 0-tree.

Each clear contour on Γ is active triangle incident to $\mathtt{n}$ of a type

$$\begin{tikzcd} \circ \arrow[rr, "b"] \arrow[dr, "a_{\mathtt{k}}"'] & & \circ \arrow[dl, "a_{\mathtt{i}}"] \\ & \underset{\mathtt{n}}{\circ} & \end{tikzcd}$$

where $\mathtt{d}(a_{\mathtt{k}}) = \alpha b a_{\mathtt{i}} + \ldots$, $\alpha \in \Bbbk^*$. We write in this situation: $b a_{\mathtt{i}} \in \mathtt{d}(a_{\mathtt{k}})$. Then $\deg(a_{\mathtt{k}}) = \deg(a_{\mathtt{i}}) + \deg(b) - 1 = \deg(a_{\mathtt{i}}) - 1$ since $\deg(b) = 0$. If $a_{\mathtt{i}}, a_{\mathtt{k}}$ belongs to the same active triangle then one of this edges has even degree and another has odd degree.

Given $\mathtt{i}, \mathtt{i}' \in \Gamma_0 \setminus \{\mathtt{n}\}$, there is $k = k(\mathtt{i}, \mathtt{i}') \geqslant 2$ and there is a sequence of different vertices $\mathtt{i} = \mathtt{i}_1, \ldots, \mathtt{i}_k = \mathtt{i}' \in \Gamma_0 \setminus \{\mathtt{n}\}$ such that $\mathtt{i}_r, \mathtt{i}_{r+1}$ are incident for $r = 1, \ldots, k-1$. These vertices are called *intermediate* for $\mathtt{i}, \mathtt{i}'$ on $\check{\Gamma}$. The vertices $\mathtt{i}, \mathtt{i}'$ are *neighboring* on $\check{\Gamma}$ if $k(\mathtt{i}, \mathtt{i}') = 2$.

Lemma 7. *Let* $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}, \mathtt{r}) \in \Upsilon_0$, $\mathtt{n} \in \Gamma_0^+$ *(or* $\mathtt{n} \in \Gamma_0^-$*) and* $\Gamma_{\Gamma_0 \setminus \{\mathtt{n}\}}$ *be a well directed* 0*-tree then the next properties hold:*

1) *If* $\mathtt{i}_1, \mathtt{i}_2 \in \Gamma_0 \setminus \{\mathtt{n}\}$ *are neighboring and are both incident to* $\mathtt{n}$ *then the sub problem* $\Gamma|_{\{\mathtt{i}_1, \mathtt{i}_2, \mathtt{n}\}}$ *is an active triangle.*
2) *If* $\mathtt{i}_1, \mathtt{i}_2 \in \Gamma_0 \setminus \{\mathtt{n}\}$ *are both incident to* $\mathtt{n}$ *then all intermediate points are incident to* $\mathtt{n}$ *too.*

3) *Two triangles are incident either to common minimal or common maximal edge.*

4) *Any edge from* Γ_1 *is either minimal deep or maximal.*

The proof follows immediately from the construction, hence all clear contours are active and there are no pathes on $\check{\Gamma}$.

Lemma 8. *Let* $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}, \mathtt{r}) \in \Upsilon_0$, $\mathtt{n} \in \Gamma_0^{\pm}$ *and* $\check{\Gamma}$ *be a well directed* 0*-tree. Using the change of degree in the point* $\mathtt{n}$ *if necessary, and using the turning arrows at the point* $\mathtt{n}$ *if necessary, we can obtain the equivalent problem* $\mathfrak{A}'$ *such that* $\mathtt{n} \in \Gamma_0^{\pm}$, *all maximal edges on* Γ' *have the degree* 0, *and all minimal edges on* Γ' *incident to* $\mathtt{n}$, *have the degree* 1. *Besides,* $\mathtt{r}'$ *coincides with* $\mathtt{r}$.

Proof. We can raise degree in the point $\mathtt{n}$ on $2d$, $d \in \mathbb{Z}$ such that all minimal edges which are incident to $\mathtt{n}$, become the degree 0 or 1. For the second case, all maximal edges have the degree 0. For the first case, all edges have the degrees 0 and -1. We fulfil the turning arrows at the point $\mathtt{n}$ and obtain the demanded degrees 0 and 1. Note that $\mathtt{r}'_{\mathtt{n}} = \mathtt{r}_{\mathtt{n}} = 1$. $\square$

Lemma 9. *Let* $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}, \mathtt{r}) \in \Upsilon_0$, $\mathtt{n} \in \Gamma_0^{\pm}$ *and* $\check{\Gamma}$ *be a well directed* 0*-tree,* $\mathtt{r} \in \ker \chi$ *be a positive sincere vector with* $\mathtt{r}_{\mathtt{n}} = 1$. *Then there exists an equivalent problem* $\mathfrak{A}'$ *which satisfies the conditions of Lemma, and* $\sum\limits_{\mathtt{i} \in \Gamma_0} \mathtt{r}'_{\mathtt{i}} > \sum\limits_{\mathtt{i} \in \Gamma_0} \mathtt{r}_{\mathtt{i}}$.

Proof. We can immediately use Lemma 8 and assume that all maximal edges on Γ have the degree 0, and all minimal edges on Γ incident to $\mathtt{n}$, have the degree 1, besides there is at least one edge of degree 1. Now we prove that there is a sequence of reductions $\mathcal{R} : \mathfrak{A} \to \mathfrak{A}'$ such that $\mathfrak{A}'$ satisfies the conditions of Lemma, and $\sum\limits_{\mathtt{i} \in \Gamma_0} \mathtt{r}'_{\mathtt{i}} > \sum\limits_{\mathtt{i} \in \Gamma_0} \mathtt{r}_{\mathtt{i}}$.

We assume $\mathtt{n} \in \Gamma_0^{+}$. Denote by V_0 the set of vertices $\mathtt{i} \in \Gamma_0 \backslash \{\mathtt{n}\}$ such that $\Gamma_1(\mathtt{i}, \mathtt{n}) = \{a_{\mathtt{in}}\}$ and $\deg a_{\mathtt{in}} = 0$. Denote by V_1 the set of vertices $\mathtt{i} \in \Gamma_0 \backslash \{\mathtt{n}\}$ such that $\Gamma_1(\mathtt{i}, \mathtt{n}) = \{\varphi_{\mathtt{in}}\}$ and $\deg \varphi_{\mathtt{in}} = 1$. Furthermore, we denote by $N(\mathtt{i})$ the set of neighboring of $\mathtt{i}$ in $\Gamma_0 \backslash \{\mathtt{n}\}$. If $\mathtt{i}$ is incident to $\mathtt{n}$ then we can assume $N(\mathtt{i}) \neq \varnothing$ since otherwise there exists exactly one edge incident to $\mathtt{n}$ (remember that $\check{\Gamma}$ is a connected tree), hence Γ is already a tree. Let $\mathtt{i} \in V_0$. Then $N(\mathtt{i}) \cap V_0 = \varnothing$ and $1 \leqslant |N(\mathtt{i}) \cap V_1| \leqslant 2$. Indeed, if $\mathtt{i}' \in N(\mathtt{i}) \cap V_0$ then the restriction of a problem $\mathfrak{A}$ to the set $\{\mathtt{i}, \mathtt{i}', \mathtt{n}\}$ has a critical quadratic form, and the proof follows immediately. If $\mathtt{i}_1, \mathtt{i}_2, \mathtt{i}_3 \in N(\mathtt{i}) \cap V_1$ then the restriction of a problem to the set $\{\mathtt{i}, \mathtt{i}_1, \mathtt{i}_2, \mathtt{i}_3, \mathtt{n}\}$

has a critical quadratic form and the proof may be obtained directly in this case. Similarly, for $\mathtt{i} \in V_1$, we can assume that $N(\mathtt{i}) \cap V_1 = \varnothing$ and $1 \leqslant |N(\mathtt{i}) \cap V_0| \leqslant 2$.

Let $\mathcal{R}_1$ be a composition of reductions $\mathcal{R}_{\mathtt{in}}$ for all $\mathtt{i} \in V_1$. Denote $\mathfrak{A}' = \mathfrak{A}\mathcal{R}_1$. Then $\mathfrak{A}' \in \Upsilon_0$ and there hold: 1) $\Gamma|_{\Gamma_0\setminus\{\mathtt{n}\}}$ and $\Gamma'|_{\Gamma_0\setminus\{\mathtt{n}\}}$ coincides; 2) for any $\mathtt{i} \in V_1$, we have $\Gamma'_1(\mathtt{n},\mathtt{i}) = \{b_{\mathtt{ni}} = \varphi^*_{\mathtt{in}}\}$ with $\deg b_{\mathtt{ni}} = 0$; 3) for any $\mathtt{j} \in V_0$ such that $|N(\mathtt{j}) \cap V_1| = 1$, we have $\Gamma'_1(\mathtt{n},\mathtt{j}) = \Gamma'_1(\mathtt{j},\mathtt{n}) = \varnothing$; 4) for any $\mathtt{j} \in V_0$ such that $N(\mathtt{j}) \cap V_1 = \{\mathtt{i}_1, \mathtt{i}_2\}$, we have $\Gamma'_1(\mathtt{j},\mathtt{n}) = \{\psi_{\mathtt{jn}}\}$ with $\deg \psi_{\mathtt{nj}} = 1$ where $\psi_{\mathtt{jn}} = \varphi_{\mathtt{jn}} a_{\mathtt{i}_1\mathtt{j}} = \varphi_{\mathtt{jn}} a_{\mathtt{i}_2\mathtt{j}}$ ($a_{\mathtt{kj}} \in \Gamma_1(\mathtt{k},\mathtt{j})$); 5) for any $\mathtt{j} \notin V_0 \cup V_1$, $\mathtt{j} \neq \mathtt{n}$, we have $\Gamma'_1(\mathtt{j},\mathtt{n}) = \Gamma'_1(\mathtt{j},\mathtt{n}) = \varnothing$ if $N(\mathtt{j}) \cap V_1 = \varnothing$, and $\Gamma'_1(\mathtt{j},\mathtt{n}) = \{\psi_{\mathtt{jn}}\}$ with $\deg \psi_{\mathtt{nj}} = 1$ where $\psi_{\mathtt{jn}} = \varphi_{\mathtt{jn}} a_{\mathtt{ij}}$ if $N(\mathtt{j}) \cap V_1 = \{\mathtt{i}\}$. Here $\mathtt{r}'_{\mathtt{i}} = \mathtt{r}_{\mathtt{i}} + 1$ for any $\mathtt{i} \in V_1$, and $\mathtt{r}'_{\mathtt{j}} = \mathtt{r}_{\mathtt{j}}, i \neq j$.

Consider the problem $\mathfrak{A}'$. Let $V'_0 = \{\mathtt{i} \in \Gamma_0 \setminus \{\mathtt{n}\} \mid \exists\, b_{\mathtt{nj}}\}$. Define $V'_1 = \{\mathtt{j} \in \Gamma_0 \setminus \{\mathtt{n}\} \mid \exists\, \psi_{\mathtt{jn}}\}$ to be a set of all $\mathtt{j} \in V_0$ such that $|V_0(\mathtt{j})| = 2$. We denote by $\mathcal{R}_2$ a composition of complete reductions $\mathcal{R}_{\mathtt{jn}}$ for all $\mathtt{j} \in V'_1$. Denote $\mathfrak{A}'' = \mathfrak{A}'\mathcal{R}_2 = \mathfrak{A}\mathcal{R}_1\mathcal{R}_2$. Then Γ'' is correctly defined graph, $\Gamma|_{\Gamma_0\setminus\{\mathtt{n}\}}$ and $\Gamma''|_{\Gamma_0\setminus\{\mathtt{n}\}}$ coincides, and the following hold: 1) for any $\mathtt{j} \in V'_1$, we have $\Gamma''_1(\mathtt{n},\mathtt{j}) = \{c_{\mathtt{nj}} = \psi^*_{\mathtt{jn}}\}$ with $\deg c_{\mathtt{ni}} = 0$; 2) for any $\mathtt{i} \in V'_0$ such that $|N(\mathtt{i}) \cap V'_1| = 1$, we have $\Gamma''_1(\mathtt{n},\mathtt{i}) = \Gamma''_1(\mathtt{i},\mathtt{n}) = \varnothing$; 3) for any $\mathtt{i} \in V'_0$ such that $N(\mathtt{i}) \cap V'_1 = \{\mathtt{j}_1, \mathtt{j}_2\}$, we have $\Gamma''_1(\mathtt{i},\mathtt{n}) = \{\tau_{\mathtt{ni}}\}$ with $\deg \tau_{\mathtt{ni}} = -1$ where $\tau_{\mathtt{ni}} = \psi^*_{\mathtt{in}} a_{\mathtt{j}_1\mathtt{i}} = \psi^*_{\mathtt{in}} a_{\mathtt{j}_2\mathtt{i}}$ (here $a_{\mathtt{ki}} \in \Gamma_1(\mathtt{k},\mathtt{i})$); 4) for any $\mathtt{i} \notin V'_0 \cup V'_1$, $\mathtt{i} \neq \mathtt{n}$, we have $\Gamma''_1(\mathtt{i},\mathtt{n}) = \Gamma''_1(\mathtt{i},\mathtt{n}) = \varnothing$ if $N(\mathtt{i}) \cap V'_1 = \varnothing$, and $\Gamma''_1(\mathtt{i},\mathtt{n}) = \{\tau_{\mathtt{ni}}\}$, $\tau_{\mathtt{ni}} = \psi^*_{\mathtt{in}} a_{\mathtt{ji}}$ with $\deg \tau_{\mathtt{in}} = 1$ if $N(\mathtt{i}) \cap V'_1 = \{\mathtt{j}\}$. As before we have $\mathtt{r}''_{\mathtt{j}} = \mathtt{r}'_{\mathtt{j}} + 1$ for any $\mathtt{j} \in V'_1$, and $\mathtt{r}''_{\mathtt{i}} = \mathtt{r}'_{\mathtt{i}}$ otherwise.

It remains to turn arrows in the point $\mathtt{n}$ to obtain the problem $\mathfrak{A}'''$ of the same class but under the conditions: $\mathtt{r}''' > \mathtt{r}$ and $\mathtt{r}'''_{\mathtt{n}} = \mathtt{r}_{\mathtt{n}} = 1$. □

Of course, the analogous Lemma is valid for the case $\mathtt{n} \in \Gamma_0^-$.

To proof Theorem 2 one has to use the fact that the coordinates of the kernel vector having 1 can not be bigger 6 ([2], [4]).

Conclusion

This work concerns with classification problem of differential graded categories with critical semi-definite quadratic form. We prove that such problem which satisfies some correctness conditions can be transformed to differential graded category with directed graded graph, which is a quiver of affine (extended) type.

References

[1] Y.A. Drozd. *Reduction algorithm and representations of boxes and algebras*. Comtes Rendue Math. Acad. Sci. Canada, **23** (2001), 97–125.

[2] A.V. Roiter. *The roots of integral quadratic forms*. In: Quadratic forms and representations (1978), Kiev, 3–17.

[3] S.A. Ovsienko. *Integer weakly positive forms*. In: Shurian Matrix problems and quadratic forms (1978), Kiev, 3–17.

[4] M. Barot, J.A.de la Pena. *The Dynkin type of a non-negative unit form*. Expo. Math. **17**, (1999), 339–348.

[5] O. Gnatiuk, N. Golovashchuk. *Differential Graded Categories Associated with the Dynkin diagrams*. Scientific Journal of National Pedagogical Dragomanov University. Serie 1. Physics and Matematics Science, **14**, (2013).

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