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On the relation between completeness and H-closedness of pospaces without infinite antichains

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Abstract. We study the relation between completeness and H-closedness for topological partially ordered spaces. In general, a topological partially ordered space with an infinite antichain which is even directed complete and down-directed complete, is not H-closed. On the other hand, for a topological partially ordered space without infinite antichains, we give necessary and sufficient condition to be H-closed, using directed completeness and down-directed completeness. Indeed, we prove that a pospace X is H-closed if and only if each up-directed (resp. down-directed) subset has a supremum (resp. infimum) and, for each nonempty chain $L \subseteq X$, $\bigvee L \in \mathrm{cl}{\downarrow}L$ and $\bigwedge L \in \mathrm{cl}{\uparrow}L$. This extends a result of Gutik, Pagon, and Repovš [GPR].

In this paper, we study the relation between completeness and H-closedness for topological partially ordered spaces (or shortly pospaces). Though H-closedness is a generalization of compactness, H-closedness does not correspond with compactness for even chains and antichains (equipped with some pospace topologies). Indeed, since the pospaces which are antichains coincide with the Hausdorff topological spaces, we have that H-closed non-compact topological spaces are also such pospaces which are antichains. There is also another extremal example which is a

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countable linearly ordered H-closed non-compact pospace (e.g. Example 4.6 in [GPR]).

In [GPR], they have shown that a linearly ordered topological semilattice is H-closed if and only if it is H-closed as a topological pospace. They also have given the following characterization of H-closedness for a linearly ordered pospace to be H-closed (Corollary 3.5 in [GPR]): A linearly ordered pospace X is H-closed if and only if the following conditions hold:
(i) X is a complete set with respect to the partial order on X;
(ii) $x = \bigvee A$ for $A = \downarrow A \setminus \{x\}$ implies $x \in \mathrm{cl} A$, whenever $A \neq \emptyset \subseteq X$; and
(iii) $x = \bigwedge B$ for $B = \uparrow B \setminus \{x\}$ implies $x \in \mathrm{cl} B$, whenever $B \neq \emptyset \subseteq X$.

This result can rewrite the following statement: A linearly ordered pospace X is H-closed if and only if X is a complete lattice with $\bigvee L \in \mathrm{cl}{\downarrow} L$ and $\bigwedge L \in \mathrm{cl}{\uparrow} L$ for any nonempty chain $L \subseteq X$.

Naturally the following question arises: Is there a similar characterization of H-closedness for topological semilattices or pospaces?

It's easy to see that a discrete countable antichain is not H-closed as a pospace but directed complete and down-directed complete. This means that there is no similar characterization of H-closedness for pospaces. However, we give the necessary and sufficient condition for pospaces without infinite antichains to be H-closed.

In this paper, all topological spaces will be assumed Hausdorff. For a set X, denote by $X^{<\omega}$ the set of finite subsets of X. If A is a subset of a topological space X, then we denote the closure of the set A in X by $\mathrm{cl}_X A$ or $\mathrm{cl} A$. By a partial order on a set X we mean a reflexive, transitive and anti-symmetric binary relation $\leq$ on X. A set endowed with a partial order is called a partially ordered set (or poset).

Recall that a poset with a topology defined on it is called a topological partially ordered space (or pospace) if the partial order is a closed subset of $X \times X$. A partial order $\leq$ is said to be continuous or closed if $x \not\leq y$ in X implies that there are open neighborhoods U and V of x and y respectively such that ${\uparrow} U \cap V = \emptyset$ (equivalently $U \cap {\downarrow} V = \emptyset$). A partial order $\leq$ on a topological space X is continuous if and only if $(X, \leq)$ is a pospace [W]. In any pospace, ${\downarrow} x$ and ${\uparrow} x$ are both closed for any element x of it.

A Hausdorff pospace X is said to be an H-closed pospace if X is a closed subspace of every Hausdorff pospace in which it is contained. Obviously that the notion of H-closedness is a generalization of compactness.

For an element x of a poset X, ${\uparrow} x := \{y \in X \mid x \leq y\}$ (resp. ${\downarrow} x := \{y \in X \mid y \leq x\}$) is called the upset (resp. the downset) of x. Define ${\updownarrow} x := {\uparrow} x \cup {\downarrow} x$. For a subset $Y \subseteq X$, ${\uparrow} Y := \bigcup_{y \in Y} {\uparrow} y$ (resp.

$\downarrow Y := \bigcup_{y \in Y} \downarrow y$) is called the upset (resp. the downset) of Y. Define $\updownarrow Y := \uparrow Y \cup \downarrow Y$. For a subset S of a poset X, denote by $S^{\uparrow}$ (resp. $S^{\downarrow}$) the set of upper (resp. lower) bounds of S. In other words, $S^{\downarrow} = \{x \in X \mid x \leq s \text{ for all } s \in S\}$ and $S^{\uparrow} = \{x \in X \mid x \geq s \text{ for all } s \in S\}$. Define $\max S := \{s \in S \mid S \cap \uparrow s = \{s\}\}$ and $\min S := \{s \in S \mid S \cap \downarrow s = \{s\}\}$.

For elements x, y of a poset, $x \| y$ means that x and y are incomparable. For a subset A of a poset, A is said to be a chain if A is linearly ordered, and is said to be an antichain if any distinct elements are incomparable. A maximal chain (resp. antichain) is a chain (resp. antichain) which is properly contained in no other chain (resp. antichain). The Axiom of Choice implies the existence of maximal chains in any poset. A subset D of a poset X is (up-)directed (resp. down-directed) if every finite subset of D has an upper (resp. lower) bound in D. A poset X is said to be down-directed complete (resp. (up-)directed complete) if each down-directed (resp. up-directed) set S of X has $\bigwedge S$ (resp. $\bigvee S$). An up-directed complete poset is also called a directed complete poset or a dcpo. It is well-known that a poset X is directed complete if and only if each chain L of X has $\bigvee L$. Now we state the main result.

Theorem 1. *Let X be a pospace without infinite antichains. Then X is an H-closed pospace if and only if X is directed complete and down-directed complete such that $\bigvee L \in \mathrm{cl}\downarrow L$ and $\bigwedge L \in \mathrm{cl}\uparrow L$ for any nonempty chain $L \subseteq X$.*

Note that $\updownarrow F = X$ for a maximal antichain F of a poset X. Moreover notice that that if X has no infinite antichains, then all subposet and all extensions of X by adding finite points have no infinite antichain neither. First, we show the *only if* part in the main theorem.

Lemma 1. *Let X be a pospace without infinite antichains. Suppose that X is directed complete and down-directed complete such that $\bigvee L \in \mathrm{cl}\downarrow L$ and $\bigwedge L \in \mathrm{cl}\uparrow L$ for any nonempty chain $L \subseteq X$. Then X is an H-closed pospace.*

Proof. Suppose there is a non-H-closed pospace X without infinite antichains such that X is directed complete and down-directed complete, and $\bigvee L \in \mathrm{cl}\downarrow L$ and $\bigwedge L \in \mathrm{cl}\uparrow L$ for any nonempty chain $L \subseteq X$. Then there is an embedding from X to a pospace $\widetilde{X}$ such that X is a dense proper subspace. Fix any element $x \in \widetilde{X} \setminus X$. Since X is directed complete and down-directed complete, we have $X = \downarrow_X \max X = \uparrow_X \min X$. Since X has no infinite antichains, we have that $\max X$ and $\min X$ are finite subsets. If $x \notin {\downarrow}\max X$, then the fact that ${\downarrow}\max X$ is a closed subset of $\widetilde{X}$ implies that $x \notin \mathrm{cl}X$, which contradicts that X is a dense subset of $\widetilde{X}$. Thus there is an element $\omega \in X$ such that $x < \omega$. By the symmetry of pospace, there is an element $\alpha \in X$ such that $\alpha < x$. Since X is embedded into $\widetilde{X}$, we have that $A' := {\downarrow}x \cap X$ and $B' := {\uparrow}x \cap X$ are closed subsets of X. Let $A := \{\bigvee_X L \mid L \neq \emptyset \subseteq A'$ is a chain$\}$ and $B := \{\bigwedge_X L \mid L \neq \emptyset \subseteq B'$ is a chain$\}$ be subsets of X. Since X is directed complete and down-directed complete, we obtain $\max A$ and $\min B$ are nonempty such that ${\downarrow}_X \max A \supseteq A$ and ${\uparrow}_X \min B \supseteq B$. Next we show that $x \notin {\downarrow}\max A$. Indeed, suppose that there is an element $y \in \max A$ such that $x < y$. By the definition of A', we have $y \notin A'$. Since $y \in \max A$, there is a chain $L \subseteq A'$ such that $y = \bigvee_X L \notin L$. By the assumption, $y = \bigvee_X L \in \mathrm{cl}_X{\downarrow}L \subseteq \mathrm{cl}_X A' = A'$, which is impossible.

By the symmetry of pospace, we have $x \notin {\uparrow}\min B$. Let $I := \{y \in X \mid x \| y\}$. Since X has no infinite antichains, there is a finite subset F of I such that $I \subseteq {\updownarrow}F$. Since $X = A' \sqcup B' \sqcup I$, the facts that ${\downarrow}_X \max A \supseteq A'$ and ${\uparrow}_X \min B \supseteq B'$, imply that $X \subseteq {\downarrow}\max A \cup {\uparrow}\min B \cup {\updownarrow}F$ and so $\mathrm{cl}X \subseteq {\downarrow}\max A \cup {\uparrow}\min B \cup {\updownarrow}F$. Since $x \notin {\downarrow}\max A \cup {\uparrow}\min B \cup {\updownarrow}F$, we obtain that $x \notin \mathrm{cl}X$, which contradicts that X is a dense subset of $\widetilde{X}$. Therefore X is H-closed. □

Now we show the another direction. Later we assume that X is an H-closed pospace without infinite antichains. Note that the following three statements are equivalent for subsets U, V of a poset X: ${\uparrow}U \cap V = \emptyset$. $U \cap {\downarrow}V = \emptyset$. ${\uparrow}U \cap {\downarrow}V = \emptyset$.

Lemma 2. *Let X be a poset and x a point of X. Suppose that there is a subset F of X such that the disjoint union $F \sqcup \{x\}$ is a maximal antichain in X. Let $U := X \setminus {\updownarrow}F$. Then ${\updownarrow}U \subseteq {\updownarrow}x$.*

Proof. Put $X_- := {\downarrow}(F \sqcup \{x\})$ and $X_+ := {\uparrow}(F \sqcup \{x\})$. Then $X = X_- \cup X_+$. By symmetry, it suffices to show that ${\uparrow}U \subseteq {\updownarrow}x$. For any $y \in U$, if $y \in {\uparrow}x$, then obviously ${\uparrow}y \in {\uparrow}x$. Otherwise $y \in {\downarrow}x$. Since ${\uparrow}U \cap {\downarrow}F = \emptyset$ and $y \in U$, we have ${\uparrow}y \cap {\downarrow}F = \emptyset$. Then ${\uparrow}y \cap X_- \subseteq {\downarrow}x$ and ${\uparrow}y \cap X_+ \subseteq {\uparrow}x$. Thus ${\uparrow}y \subseteq {\downarrow}x \cup {\uparrow}x$ and so ${\uparrow}U \subseteq {\updownarrow}x$. □

Lemma 3. *Any maximal chain of X is complete.*

Proof. Suppose that there is a maximal chain L which is not complete. Then there is a subset S of L such that either $\bigvee_L S$ or $\bigwedge_L S$ does not exist. We may assume that $\bigvee_L S$ does not exist. If $S^{\uparrow} = \emptyset$, then let $\widetilde{X} := X \sqcup \{\infty\}$ be the extension with the greatest element ∞. Define a topology τ on $\widetilde{X}$ by an open subbase $\tau_X \cup \{\widetilde{X} \setminus {\downarrow}F \mid F \in X^{<\omega}\}$. Since any neighborhood of ∞ meets S, we have $\infty \in \mathrm{cl}X \setminus X$. Then X is a non-closed subset of $\widetilde{X}$. We will show that $\widetilde{X}$ is a pospace. For any element x of X, if $x \notin \max X$, then the fact that $\bigvee_L S$ does not exist implies that there are an element $y > x$ of X and a maximal antichain F of X containing y. Then $U := \widetilde{X} \setminus {\uparrow}F$ is an open neighborhood of x and $V := \widetilde{X} \setminus {\downarrow}F$ is an open neighborhood of ∞ such that ${\downarrow}U \cap V = \emptyset$. Otherwise $x \in \max X$. Since $S^{\uparrow} = \emptyset$, there is an element $y \in S$ such that $y \| x$. Then there is a maximal antichain F of X containing x and y. Let $U = \widetilde{X} \setminus {\updownarrow}(F \setminus \{x\})$ be an open neighborhood of x and $V = \widetilde{X} \setminus {\downarrow}F$ an open neighborhood of ∞. Since $x \in \max X$, we have ${\uparrow}x = \{x, \infty\}$. Since $F \setminus \{x\} \neq \emptyset$, we obtain $U \subseteq {\downarrow}x$. Then ${\downarrow}U \cap V = \emptyset$. Thus $\widetilde{X}$ is a pospace and so X is not H-closed, which contradicts the hypothesis on X.

Thus we may assume that $S^{\uparrow} \neq \emptyset$. Then $\bigwedge_L (L \setminus {\downarrow}S)$ does not exist. Let $A := L \cap {\downarrow}S$ and $B := L \setminus {\downarrow}S$. Extend X to $\widetilde{X} := X \sqcup \{\alpha\}$ by ${\uparrow}\alpha := \{\alpha\} \sqcup A^{\uparrow}$ and ${\downarrow}\alpha := \{\alpha\} \sqcup {\downarrow}A$. Define a topology τ by an open subbase $\tau_X \cup \{\widetilde{X} \setminus ({\uparrow}E \cup {\downarrow}F) \mid F, E \in X^{<\omega}\}$. Since ${\uparrow}\alpha = \{\alpha\} \sqcup A^{\uparrow}$ and since $\bigvee_L S = \bigvee_L A$ does not exist, we have $\bigvee A = \alpha$. Since A is a chain with $\max A = \emptyset$ and ${\uparrow}\alpha \supseteq A^{\uparrow}$, we obtain that any neighborhood of α meets A. Thus X is a non-closed subset of $\widetilde{X}$. Therefore it suffices to show that $\widetilde{X}$ is a pospace, which implies that X is not an H-closed pospace. Indeed, let $x \in X$ be any element. Then either $x < \alpha$, $x \| \alpha$, or $x > \alpha$. If $x < \alpha$, then there is an element $y \in X$ such that $x < y < \alpha$. Let E be a maximal antichain containing y, $U = \widetilde{X} \setminus {\downarrow}E$ an open neighborhood of α, and $V = \widetilde{X} \setminus {\uparrow}E$ an open neighborhood of x. Then $U \cap {\downarrow}V = \emptyset$. If $x \| \alpha$, then the fact that ${\uparrow}\alpha = \{\alpha\} \sqcup A^{\uparrow}$ implies that $x \notin A^{\uparrow}$. Since $x \notin {\downarrow}A \subseteq {\downarrow}\alpha$, we have that either $x \geq y$ or $x \| y$ for any $y \in A$. Since $x \notin A^{\uparrow} \subseteq {\uparrow}\alpha$, there is an element $y \in A$ such that $x \| y$. Let F be a maximal antichain containing x and y, $U = \widetilde{X} \setminus {\updownarrow}x$ and $V = \widetilde{X} \setminus {\updownarrow}(F \setminus \{x\}) \subseteq {\updownarrow}x$. Then $U \cap {\updownarrow}V = \emptyset$ and V is an open neighborhood of x. Since $x \| \alpha$, we have U is an open neighborhood of α. Otherwise $x > \alpha$. Then $x \in A^{\uparrow}$. If $x \notin \min A^{\uparrow}$, then there is an element $y \in A^{\uparrow}$ such that $\alpha < y < x$. Let F be a maximal antichain containing y, $U = \widetilde{X} \setminus {\downarrow}F$, and $V = \widetilde{X} \setminus {\uparrow}F$. Then U is an open neighborhood of x and V is an open neighborhood of α such that $U \cap {\downarrow}V = \emptyset$. Otherwise $x \in \min A^{\uparrow}$. Since L is a maximal chain and $\bigwedge_L B$ does not exist, there is an element $y \in B$ such that $y \| x$. Let F be a maximal antichain containing x and y, $U = \widetilde{X} \setminus {\updownarrow}(F \setminus \{x\})$, E a maximal antichain containing α, and $V = \widetilde{X} \setminus ({\updownarrow}(E \setminus \{\alpha\}) \cup {\uparrow}F) \subseteq ({\updownarrow}\alpha \setminus \{x\}) \cap {\downarrow}F$. Obviously U is an open neighborhood of x. Since $y > \alpha$, we have V is an open neighborhood of α. Since $x \in \min A^{\uparrow}$ and ${\uparrow}\alpha = \{\alpha\} \sqcup A^{\uparrow}$, we have

$\uparrow\alpha \cap \downarrow x = \{\alpha, x\}$. Since F is a maximal antichain containing y and since $\alpha < y$, we have $(\uparrow\alpha \setminus \{x\}) \cap \downarrow F \subseteq \downarrow(F \setminus \{x\})$. Since $\downarrow\alpha \subset \downarrow y \subseteq \downarrow(F \setminus \{x\})$, we have $V \subseteq \downarrow(F \setminus \{x\})$. By the definition of U, we have $\downarrow V \cap U = \emptyset$. This show that $\widetilde{X}$ is a pospace. □

Lemma 4. *X is directed complete.*

Proof. Let L be any infinite chain of X. Then $\min(L^{\uparrow}) = \min((\downarrow L)^{\uparrow})$. By the Axiom of choice, there is a maximal chain L' containing L. By Lemma 3, there is the element $x \in L' \cap \min(L^{\uparrow})$ and so $\min(L^{\uparrow}) \neq \emptyset$ and $\uparrow \min(L^{\uparrow}) = L^{\uparrow}$. Since X has no infinite antichains, we have $\min(L^{\uparrow})$ is a nonempty finite subset. Put $\{x_1, \ldots, x_n\} = \min(L^{\uparrow})$. Since X has no infinite antichains and since $\min L$ is an antichain, there is a maximal antichain K containing $\{x_1, \ldots, x_n\}$. Suppose that $\min(L^{\uparrow})$ is not a singleton. Then $x_i \| x_j$ for any distinct pair $i \neq j$. The facts that X is a pospace and $x_i \| x_j$ for any $i \neq j$, imply that for any $i = 1, \ldots, n$, there is an open neighborhood $U_i \subseteq \updownarrow x_i \setminus \updownarrow(K \setminus \{x_i\})$ of x_i such that $U_i \cap \downarrow L = \emptyset$ and $U_i \cap \updownarrow U_j = \emptyset$ for any distinct j. Extend X to $\widetilde{X} := X \sqcup \{\alpha\}$ by $\uparrow\alpha := \uparrow_X\{x_1, \ldots, x_n\} \sqcup \{\alpha\} = L^{\uparrow} \sqcup \{\alpha\}$ and $\downarrow\alpha := \downarrow_X L \sqcup \{\alpha\}$. Define a topology τ on $\widetilde{X}$ by an open subbase $\tau_X \cup \{\widetilde{X} \setminus (\uparrow F \cup \downarrow E) \mid F, E \in X^{<\omega}\}$, where τ_X is the topology of X. Since $\max L = \emptyset$, $\bigvee_{\widetilde{X}} L = \alpha$, and $\uparrow\alpha \supseteq L^{\uparrow}$, we have that any neighborhood of α meets L and so X is an embedded subset of $\widetilde{X}$ which is not closed. Next we show that $\widetilde{X}$ is a pospace. Indeed, let x be any element of X. Then either $x \in \uparrow_X\{x_1, \ldots, x_n\}$, $x \leq \alpha$, or $x \| \alpha$. Let G be a finite subset of X such that $G \sqcup \{\alpha\}$ is a maximal antichain in $\widetilde{X}$. If $x = x_i$ for some i, then let $U := \widetilde{X} \setminus (\uparrow K \cup \updownarrow G)$ be an open neighborhood of α. Since $\uparrow\alpha \setminus \{\alpha\} \subseteq \uparrow K$, we have $U \subseteq \downarrow\alpha$. Since $\downarrow\alpha = \downarrow L \sqcup \{\alpha\}$ and $\downarrow L \cap U_i = \emptyset$, we have $\downarrow U \cap U_i = \emptyset$. If $x \in \uparrow\alpha \setminus \{x_1, \ldots, x_n\}$, then let $U := \widetilde{X} \setminus \uparrow K$ be an open neighborhood of α and $V := \widetilde{X} \setminus \downarrow K \subseteq \uparrow K$ an open neighborhood of α. Now $\downarrow U \cap V = \emptyset$. If $x \in \downarrow\alpha$, then there is an element $y \in L$ such that $x < y < \alpha$. Now there is a maximal antichain F of X containing y. Then $U := \widetilde{X} \setminus \downarrow F$ is an open neighborhood of α and $V := \widetilde{X} \setminus \uparrow F$ is an open neighborhood of x such that $U \cap \downarrow V = \emptyset$. Otherwise $x \| \alpha$. Since $x \notin \downarrow L \subseteq \downarrow\alpha$, we obtain that either $x \geq y$ or $x \| y$ for any $y \in L$. Since $x \notin L^{\uparrow} \subseteq \uparrow\alpha$, there is an element $y \in L$ such that $y \| x$. Then there is a maximal antichain F of X containing x and y. Thus $U := \widetilde{X} \setminus \updownarrow x$ is an open neighborhood of α and $V := \widetilde{X} \setminus \updownarrow(F \setminus \{x\}) \subseteq \updownarrow x$ is an open neighborhood of x. Since F is a maximal antichain, by Lemma 2, we have $\updownarrow V \subseteq \updownarrow x$ and so $\updownarrow V \cap U = \emptyset$. This show that $\widetilde{X}$ is a pospace. Therefore X is not H-closed, which contradicts the hypothesis of X. Thus $\min(L^{\uparrow})$ is a singleton and so $\bigvee L$ exists. Hence X is directed complete. □

Notice that the symmetry of pospace implies that the dual statement of Lemma 4 holds (i.e. An H-closed pospace without infinite antichains is down-directed complete).

Lemma 5. *For any nonempty chain $L \subseteq X$, we have $\bigvee L \in \mathrm{cl}{\downarrow}L$.*

Proof. Suppose that there is a chain $L \subseteq X$ such that $\bigvee L \notin \mathrm{cl}{\downarrow}L$. Put $a := \bigvee L$. Let $\widetilde{X} := X \sqcup \{\alpha\}$ be an extension of X by ${\uparrow}\alpha = \{\alpha\} \sqcup {\uparrow}a$ and ${\downarrow}\alpha = \{\alpha\} \sqcup {\downarrow}L$. Define a topology τ on $\widetilde{X}$ by an open subbase $\tau_X \cup \{\widetilde{X} \setminus ({\uparrow}F \cup {\downarrow}E) \mid F, E \in X^{<\omega}\}$, where τ_X is the topology of X. Since any neighborhood of α meets L, we have $\alpha \in \mathrm{cl}{\downarrow}L \setminus {\downarrow}L$. Thus X is a subset of $\widetilde{X}$ which is not closed. Next we show that $\widetilde{X}$ is a pospace. Indeed, let x be an element of X. Then either $x > \alpha$, $x < \alpha$, or $x \| \alpha$. If $x \in {\uparrow}\alpha$, then there is a finite subset F of X such that $F \sqcup \{a\}$ is a maximal antichain. Since ${\updownarrow}\alpha \subseteq {\updownarrow}a$, we obtain $F \sqcup \{\alpha\}$ is an antichain. If $x = a$, then $U = \widetilde{X} \setminus ({\uparrow}a \cup {\updownarrow}F) \subseteq {\downarrow}a$ is an open neighborhood of α. Since $a \notin \mathrm{cl}_X{\downarrow}L$, there is an open neighborhood $W \subset X$ of a such that $W \cap {\downarrow}L = \emptyset$. Then $V := W \setminus {\updownarrow}F \subseteq {\updownarrow}a$ is an open neighborhood of a. Since $\alpha \notin V \subseteq W$, we have ${\downarrow}U \cap V \subseteq {\downarrow}a \setminus ({\downarrow}L \sqcup \{\alpha, a\}) = {\downarrow}a \setminus ({\downarrow}\alpha \sqcup \{a\})$. We may assume that ${\downarrow}U \cap V \neq \emptyset$. Let E be a maximal antichain of ${\downarrow}a \setminus ({\downarrow}\alpha \sqcup \{a\})$. Note that $E \cap {\updownarrow}\alpha = \emptyset$. Since $\alpha \notin {\updownarrow}E$, we have $U' := U \setminus {\updownarrow}E$ is an open neighborhood of α. Since ${\downarrow}a \setminus {\downarrow}\alpha \subseteq {\updownarrow}E$, we have $U' \subseteq {\downarrow}\alpha = \{\alpha\} \sqcup {\downarrow}L$. Since $W \cap {\downarrow}L = \emptyset$ and $\alpha \notin W$, we have ${\downarrow}U' \cap V = \emptyset$. Otherwise $x > a$. Then $U = \widetilde{X} \setminus {\uparrow}(F \cup \{a\})$ is an open neighborhood of α and $V := \widetilde{X} \setminus {\downarrow}(F \cup \{a\})$ is an open neighborhood of x with ${\downarrow}U \cap V = \emptyset$. If $x < \alpha$, then there are an element $y \in L$ and a maximal antichain $F \in X^{<\omega}$ containing y such that $x < y < \alpha$. Then $U := \widetilde{X} \setminus {\uparrow}F$ is an open neighborhood of x and $V := \widetilde{X} \setminus {\downarrow}F$ is an open neighborhood of α. Then ${\downarrow}U \cap V = \emptyset$. Otherwise $x \| \alpha$. If $x \| a$, then there is a finite subset E of X such that $E \sqcup \{x, a\}$ is a maximal antichain. Thus $U := \widetilde{X} \setminus {\updownarrow}x$ is an open neighborhood of α and $V := \widetilde{X} \setminus {\updownarrow}(E \sqcup \{a\})$ is an open neighborhood of x. Since $E \sqcup \{x, a\}$ is a maximal antichain, we have ${\updownarrow}V \subseteq {\updownarrow}x$. Since $U \subseteq {\downarrow}\widetilde{X} \setminus {\updownarrow}x$, we obtain ${\updownarrow}V \cap U = \emptyset$. Otherwise $x < a$. Since $x \| \alpha$ and $a = \bigvee L \notin L$, there is an element $y \in L$ such that $y \| x$. Let E be a maximal antichain of X containing x and y. Then $U := \widetilde{X} \setminus {\updownarrow}x$ is an open neighborhood of α and $V := \widetilde{X} \setminus {\updownarrow}(E \setminus \{x\})$ is an open neighborhood of x. Since E is a maximal antichain, we have ${\updownarrow}V \subseteq {\updownarrow}x$. Since $U \subseteq \widetilde{X} \setminus {\updownarrow}x$, we obtain ${\updownarrow}V \cap U = \emptyset$. This shows that $\widetilde{X}$ is a pospace. Therefore X is not an H-closed pospace. □

Notice that the symmetry of pospace implies that the dual statement of Lemma 5 holds (i.e. For any nonempty chain $L \subseteq X$, we have

$\bigwedge L \in \mathrm{cl}{\uparrow}L$.). Therefore Theorem 1 is induced by Lemma 4, 5, their dual statements, and Lemma 1.

Note that all H-closed pospaces which the author knows are directed complete. Naturally the following question arises:

Question. Is there an H-closed pospace which is not directed complete?

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