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The symmetries of McCullough-Miller space

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Abstract. We prove that if W is the free product of at least four groups of order 2, then the automorphism group of the McCullough-Miller space corresponding to W is isomorphic to group of outer automorphisms of W. We also prove that, for each integer $n \geq 3$, the automorphism group of the hypertree complex of rank n is isomorphic to the symmetric group of rank n.

1. Introduction

A simplicial complex K is a *geometric model* for a group G if there exists a homomorphism $m : G \to \mathrm{Aut}(\mathrm{K})$, where Aut(K) denotes the group of simplicial automorphisms of K—in other language, we would say that K is *equipped with a G-action*. The smaller the kernel of m, the less the model simplifies G; in the best case, m is injective, and the model represents G precisely as a subgroup $m(G)$ of Aut(K). The larger $m(G)$ in Aut(K), the greater the expectation that Aut(K) in its entirety, rather than the subgroup $m(G)$, can offer insights into G; in particular, it is natural to believe that a model is better if $m(G)$ is large (say, of finite index) in Aut(K), and best if $m(G) = \mathrm{Aut}(\mathrm{K})$. Following [2], we say that

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K is an *accurate geometric model* of G if there exists an isomorphism $m\colon G \to \mathrm{Aut}(\mathrm{K})$.

For each positive integer n, we write W_n for the universal Coxeter group of rank n; that is, W_n is the free product of n groups of order two, as presented $\langle a_1, \ldots, a_n \mid a_1^2, \ldots, a_n^2 \rangle$. We write $\mathrm{Out}(W_n)$ for the group of outer automorphisms of W_n.

For each $n \geq 3$, $\mathrm{Out}(W_n)$ is the outer automorphism group of the most simple free product with n factors. The group $\mathrm{Out}(W_n)$ is related to, but much more simple in structure than, $\mathrm{Out}(F_{n-1})$ (see, for example, [3]), where F_m denotes the free group of rank m. Even so, there are a number of questions one may ask about a group which have been answered for $\mathrm{Out}(F_n)$, but not for $\mathrm{Out}(W_n)$. In particular, one may wish to identify an accurate geometric model for a group of interest. In [2], Bridson and Vogtmann showed that if $n \geq 3$, then the spine of the appropriate outer space is an accurate geometric model for $\mathrm{Out}(F_n)$. In the present article we prove that, provided $n \geq 4$, a well-known geometric model of $\mathrm{Out}(W_n)$ is in fact an accurate geometric model.

Given a group G and a fixed free-product decomposition of G, the corresponding *McCullough-Miller space* $\mathrm{K}(G)$ is a contractible simplicial complex which is a geometric model for the group of symmetric outer automorphisms of G [6]—those outer automorphisms of G which map each free factor in the fixed decomposition of G to a conjugate of a free-factor. Further, it is known that the modeling homomorphism is injective. In the present article we consider the case that $G = W_n$, equipped with the canonical decomposition; we write K_n for the corresponding McCullough-Miller space. In this case, each outer automorphism of W_n is a symmetric outer automorphism, so K_n is a geometric model for $\mathrm{Out}(W_n)$. We show that the modeling homomorphism is surjective, and hence prove the following.

Theorem 1.1. For each integer $n \geq 4$, the McCullough-Miller space K_n is an accurate geometric model of $\mathrm{Out}(W_n)$.

Remark 1.2. The hypothesis $n \geq 4$ is necessary because: $\mathrm{Out}(W_3)$ is finitely generated, and hence it is countably infinite; K_3 is the barycentric subdivision of the regular trivalent tree, and hence $\mathrm{Aut}(\mathrm{K}_3)$ is uncountably infinite.

In general, a McCullough-Miller space is constructed by gluing together copies of a finite complex HT_n, called *the hypertree complex of rank n*, which is the simplicial realization of a poset $(\mathcal{HT}_n, \leq)$, called *the hypertree* *poset of rank n* (see Remark 1.5 below). The poset $(\mathcal{HT}_n, \leq)$ is the set of hypertrees on n labeled vertices, partially-ordered by the operation of folding. It is well-known, and easily seen, that HT_n is a geometric model for Σ_n, the symmetric group of rank n. As part of our proof of Theorem 1.1, and for its independent interest, we prove the following.

Theorem 1.3. For each integer $n \geq 3$, the hypertree complex HT_n is an accurate geometric model of the symmetric group Σ_n.

Remark 1.4. The hypothesis $n \geq 3$ is necessary because there is only one hypertree on 2 labeled vertices, so $\mathrm{Aut}(\mathrm{HT}_2)$ is the trivial group, and it is not isomorphic to Σ_2.

Remark 1.5. In McCullough and Miller's original account [6] of the construction now named for them, the hypertree complex is not explicitly used. In its place is used a complex called the Whitehead complex, which is the simplicial realization of a poset called the Whitehead poset. As explained in [5], the Whitehead poset is isomorphic to the hypertree poset, and thus the corresponding simplicial realizations are interchangeable in the construction of McCullough-Miller space.

We now describe the structure of the paper, and proofs. In Section 2 we describe the hypertree poset, and a number of its subsets. In Section 3 we prove Theorem 1.3. Our argument proceeds by: identifying a subset of $\mathcal{HT}_n$, the set of "star trees", on which Σ_n acts as the full permutation group; observing that the corresponding subset of vertices in the hypertree complex is geometrically distinguishable, and hence must be fixed setwise by an arbitrary simplicial automorphism; and showing that every other vertex in the hypertree complex is uniquely identified either by its relative proximities to vertices corresponding to star trees, or to vertices which can be so identified.

In Section 4 we describe the construction of K_n and prove Theorem 1.1. To do so we consider an arbitrary simplicial automorphism f of K_n. We argue that: since $\mathrm{Out}(W_n)$ acts transitively on the copies of HT_n from which K_n is built, and Σ_n is the full automorphism group of HT_n, there exist elements $\sigma, \phi \in \mathrm{Out}(W_n)$ such that the actions of $\sigma\phi$ and f agree pointwise on one of the copies of HT_n. We then establish that overlapping copies of HT_n are sufficiently intertwined that if one copy is fixed pointwise by a simplicial automorphism, overlapping copies are fixed pointwise too.

2. The hypertree complex

In this section we introduce the hypertree complex. The interested reader may find an alternative account of the hypertree complex in [5].

2.1. Hypertrees

A *hypergraph* Γ is an ordered pair (V_Γ, E_Γ) consisting of a set of *vertices* V_Γ, and a collection (often a set) E_Γ of *hyperedges*, each of which is a subset of V_Γ containing at least two elements. When we want to emphasize the vertex set of Γ, we say Γ is a *hypergraph on* V_Γ. A *graph (without loops)* is a hypergraph in which each hyperedge contains exactly two vertices.

Hypergraphs $\Gamma = (V_\Gamma, E_\Gamma)$ and $\Gamma' = (V'_\Gamma, E'_\Gamma)$ are *isomorphic as unlabeled hypergraphs* if there exists a bijection $f : V_\Gamma \to V'_\Gamma$ such that for each subset $S \subset V_\Gamma$, $f(S) \in E'_\Gamma$ if and only if $S \in E_\Gamma$; in this case f is called a *hypergraph isomorphism*. Hypergraphs $\Gamma = (V_\Gamma, E_\Gamma)$ and $\Gamma' = (V_{\Gamma'}, E_{\Gamma'})$ are *isomorphic as labeled hypergraphs* if $V_\Gamma = V_{\Gamma'}$, and the identity map $V_\Gamma \to V_\Gamma$ is a hypergraph isomorphism $\Gamma \to \Gamma'$. We shall usually consider hypergraphs up to labeled-hypergraph isomorphism.

Let Γ be a hypergraph. Distinct vertices $v, v' \in V_\Gamma$ are said to be *adjacent in* Γ if $\{v, v'\} \subset e$ for some hyperedge $e \in E_\Gamma$. The *valence in* Γ of a vertex $v \in V_\Gamma$ is the number of hyperedges containing v. A vertex with valence one is called a *leaf*. The *degree in* Γ of a hyperedge $e \in E_\Gamma$ is $\#e$, the number of vertices it contains. In general, we shall write $\#S$ for the cardinality of a set S.

Given a hypergraph Γ, and vertices $v, v' \in V_\Gamma$, a *walk* in Γ from v to v' is a diagram

$$v = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots \xrightarrow{e_p} v_p = v'$$

with: $p \geq 0$; each v_k a vertex; each e_k an edge; and $v_{i-1} \neq v_i$ and $\{v_{i-1}, v_i\} \subset e_i$ for each $i = 1, \dots, p$. Such a walk is said: to *visit* the vertices $v_0, \dots, v_p$; to *join* the vertices v and v'; and to *cross* the hyperegdes $e_1, \dots, e_p$. Such a walk is *simple* if the vertices $v_0, \dots, v_{p-1}$ are distinct, and the edges $e_1, \dots, e_p$ are distinct.

We say a hypergraph Θ is: *connected* if for each pair of vertices $v, v' \in V_\Theta$, there is at least one simple walk from v to v'; and a *hypertree* if for each pair of vertices $v, v' \in V_\Theta$, there is exactly one simple walk from v to v'. It follows immediately that if Θ is a hypertree, then: Θ is connected; the intersection of two or more distinct hyperedges contains

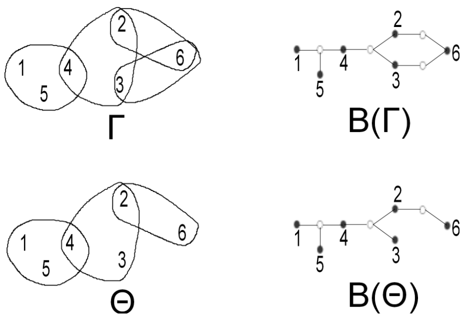

Figure 1. Γ is a hypergraph but not a hypertree, Θ is a hypertree.

at most one vertex; and the collection E_Θ of edges is a set. A hypertree Θ is a *tree* if it is a graph.

Remark 2.1. A hypergraph Γ may be represented as a labeled bipartite graph $B(\Gamma)$ as follows: each vertex in V_Γ is a labeled vertex of $B(\Gamma)$; each hyperedge in E_Γ is an unlabeled vertex of $B(\Gamma)$; an unlabeled vertex u is adjacent to a labeled vertex ℓ if $\ell \in u$. Then Γ is a hypertree if and only if $B(\Gamma)$ is a tree. Thus there is a bijective correpondence between the set of hypertrees on a set S, and the set of labeled bipartite trees with labeled vertices in bijective correspondence with S. Two hypergraphs, one of which is a hypertree, and the corresponding labeled bipartite graphs are shown in Figure 1

2.2. The Hypertree Complex

For each positive integer n we write: $[n]$ for the set $\{1, \ldots, n\}$; and $\mathcal{HT}_n$ for the set of hypertrees on $[n]$, considered up to labeled-hypergraph isomorphism.

Remark 2.2. A general formula for $\#\mathcal{HT}_n$, the number of hypertrees on $[n]$, was calculated in [4] and [7]. The sequence $(\#\mathcal{HT}_n)$ begins:

$$1, 1, 4, 29, 311, 4447, 79745, \ldots$$

More terms of the sequence, and the general formula for $\#\mathcal{HT}_n$, can be found in the On-Line Encyclopedia of Integer Sequences [1, Sequence A030019].

Example 2.3. The four hypertrees of $\mathcal{HT}_3$ are depicted in Figure 2. The 29 hypertrees of $\mathcal{HT}_4$ are depicted in Figure 4.

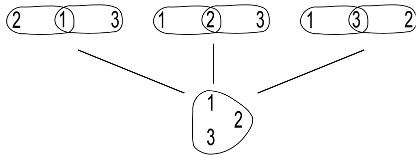

FIGURE 2. $(\mathcal{HT}_3, \leq)$.

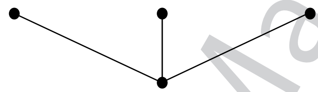

FIGURE 3. HT_3.

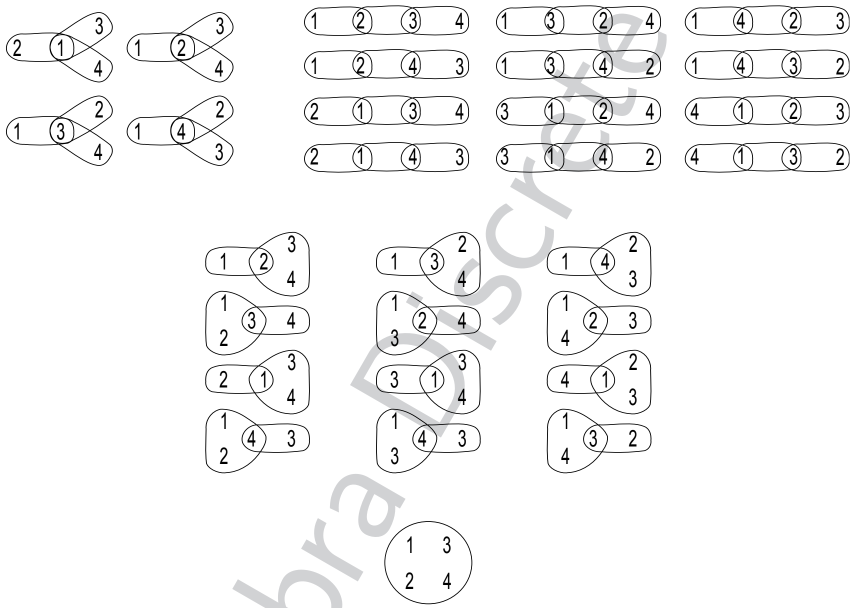

FIGURE 4. The 29 hypertrees in $\mathcal{HT}_4$.

For the remainder of this section we fix an integer $n \geq 3$.

There is a partial order $\leq$ on $\mathcal{HT}_n$, determined by an operation called *folding*. Given hypertrees $\Theta, \Theta' \in \mathcal{HT}_n$, we say Θ' is obtained from Θ by *a single fold* if there exist distinct hyperedges $e, e' \in E_\Theta$ such that $e \cap e' \neq \emptyset$ and

$$E_{\Theta'} = (E_\Theta \setminus \{e, e'\}) \cup \{e \cup e'\};$$

that is, $E_{\Theta'}$ is the result of replacing e and e' by their union. The requirement that $e \cap e' \neq \emptyset$ ensures that, in such a case, Θ' is also a hypertree on $[n]$. For each pair $\Theta, \Lambda \in \mathcal{HT}_n$, we write $\Theta \leq \Lambda$, and we say that Θ is a result of *folding* Λ, if Θ may be obtained from Λ by a (possibly empty) sequence of folds. Then $(\mathcal{HT}_n, \leq)$ is a partially ordered set called the *hypertree poset of rank n*.

Example 2.4. The Hasse diagram of $(\mathcal{HT}_3, \leq)$ is shown in Figure 2. The simplicial complex HT_3 is shown in Figure 3.

A hypertree $\Theta \in \mathcal{HT}_n$ must have at least one hyperedge, and may have as many as $n-1$ hyperedges. For each $h \in \{0, \ldots, n-2\}$, we write $\mathcal{HT}_n^h$ for the set of hypertrees on $[n]$ with $h+1$ hyperedges; a hypertree in $\mathcal{HT}_n^h$ is said to have *height h*. The unique hypertree of height 0 is denoted Θ_n^0. The hypertrees in $\mathcal{HT}_n^{n-2}$ are precisely the trees on $[n]$; it follows that there are n^{n-2} hypertrees in $\mathcal{HT}_n^{n-2}$.

Remark 2.5 (An alternative, but equivalent, definition of a hypertree)**.** Since the maximal elements in $(\mathcal{HT}_n, \leq)$ are exactly the trees on $[n]$, and $\mathcal{HT}_n$ is closed under folding, $\mathcal{HT}_n$ is exactly the set of hypergraphs obtained by folding trees on $[n]$. More generally, the set of hypertrees on a vertex set V is the set of hypergraphs obtained by folding trees on V.

The *hypertree complex of rank n*, denoted HT_n, is the simplicial realization of $(\mathcal{HT}_n, \leq)$. Recall that this means: there exists a bijection $\mathcal{V}_n$ from $\mathcal{HT}_n$ to the vertex set of HT_n; for distinct hypertrees $\Theta_1, \ldots, \Theta_k \in \mathcal{HT}_n$, the vertices $\mathcal{V}_n(\Theta_1), \ldots, \mathcal{V}_n(\Theta_k)$ span a $(k-1)$-simplex if and only if there exists a maximal chain in $(\mathcal{HT}_n, \leq)$ which contains $\Theta_1, \ldots, \Theta_k$.

It is immediate that each single fold reduces the number of hyperedges by one, and a hypertree can be folded provided it has more than one hyperedge. It follows that each maximal chain in $(\mathcal{HT}_n, \leq)$ contains exactly $n-1$ hypertrees, the minimal element of which is Θ_n^0, and the maximal element of which is a tree. Thus the simplicial complex HT_n has dimension $n-2$.

We shall often consider the hypertree poset without its minimal element. We write $\mathcal{HT}_n^+ := \mathcal{HT}_n \setminus \{\Theta_n^0\}$, and we write HT_n^+ for simplicial realization of $(\mathcal{HT}_n^+, \leq)$. Equivalently, we may consider HT_n^+ to be the subcomplex of HT_n spanned by $\mathcal{V}_n(\mathcal{HT}_n^+)$, or the link in HT_n of $\mathcal{V}_n(\Theta_n^0)$. We write $\mathcal{V}_n^+(\Theta)$ for the vertex in HT_n^+ corresponding to Θ.

Example 2.6. The complex HT_4^+ is shown in Figure 5; this figure is an adaptation of [6, Figure 8]. Some vertices are represented as stars, some as filled circles and some as unfilled circles, for reasons described in Section 3.

We shall make use of a metric on HT_n^+ which reflects the geometry of the 1-skeleton of HT_n^+.

Definition 2.7 ($d_n^+(\cdot,\cdot)$)**.** For hypertrees $\Theta, \Lambda \in \mathcal{HT}_n^+$, we write $d_n^+(\Theta, \Lambda)$ for the combinatorial length of the minimal length paths in the 1-skeleton of HT_n^+ between the vertices $\mathcal{V}_n^+(\Theta)$ and $\mathcal{V}_n^+(\Lambda)$.

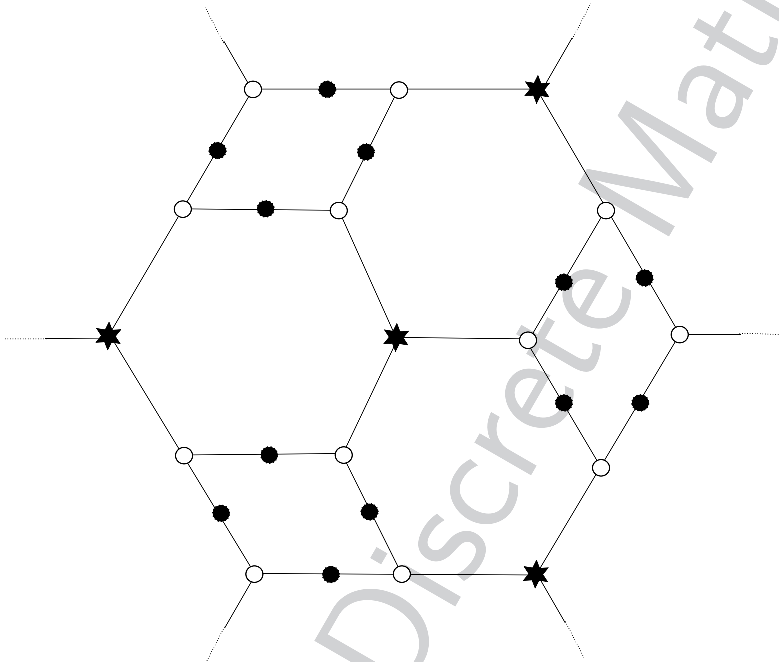

FIGURE 5. The endpoints of antipodal dashed edges should be identified to create HT_4^+, the link in HT_4 of Θ_4^0.

2.3. Some subsets of $\mathcal{HT}_n$

In the arguments which follow, a number of subsets of $\mathcal{HT}_n$ prove important. We gather the definitions here for the convenience of the reader. We have also included a table of notation at the end of the paper.

Definition 2.8 (Star trees)**.** For each $j \in [n]$, we write Ξ_n^j for the hypertree on $[n]$ with $(n-1)$ hyperedges, each of which contains j; we say that Ξ_n^j is the *star tree of rank n and common vertex j*. We write $\mathcal{S}_n$ for the set of star trees of rank n.

It is immediate that the elements of $\mathcal{S}_n$ are isomorphic as unlabeled hypergraphs; that is, any two elements of $\mathcal{S}_n$ differ by a permutation of the vertices. It is also immediate that each star tree is a tree. The hypothesis that $n \geq 3$ ensures that there are n star trees in $\mathcal{HT}_n$.

Definition 2.9 (Line trees)**.** A hypertree in which exactly two vertices are leaves is called a *line tree*; we write $\mathcal{L}_n$ for the set of line trees.

It is immediate that the elements of $\mathcal{L}_n$ are isomorphic as unlabeled hypergraphs. It is also immediate that each line tree is a tree, and that each vertex in a line tree has valence one or valence two. There are $n!/2$ line trees in $\mathcal{HT}_n$.

Definition 2.10. For each $h \in \{1, \ldots, n-2\}$, we write $\mathcal{M}_n^h$ for those hypertrees in $\mathcal{HT}_n^h$ that contain a vertex of valence $h+1$, and a hyperedge of degree $n-h$.

It is immediate that $\mathcal{M}_n^{n-2} = \mathcal{S}_n$, and, for each h, the elements of $\mathcal{M}_n^h$ are isomorphic as unlabeled hypergraphs. Examples are shown in Figure 6. The notation, a script M, was chosen because, amongst the vertices in $\mathcal{V}_n^+(\mathcal{HT}_n^h)$, the vertices in $\mathcal{V}_n^+(\mathcal{M}_n^h)$ prove to have maximal valence in $\mathcal{HT}_n^+$, provided $n \geq 5$ (Lemma 3.12 below).

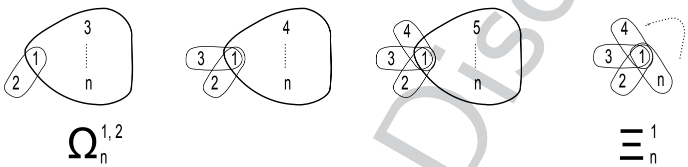

Figure 6. From left to right, a hypertree in $\mathcal{M}_n^1$, $\mathcal{M}_n^2$, $\mathcal{M}_n^3$, and $\mathcal{M}_n^{n-2}$, with $n > 4$.

The set $\mathcal{M}_n^1$ is particularly important as, provided $n \geq 5$, the vertices in $\mathcal{V}_n^+(\mathcal{M}_n^1)$ prove to be the vertices in HT_n^+ of maximal valence. It is convenient to define notation for the elements of $\mathcal{M}_n^1$. For each pair $j, k \in [n]$ with $j \neq k$, we write $\Omega_n^{j,k}$ for the hypertree on $[n]$ with hyperedges $\{j, k\}$ and $[n] \setminus \{k\}$. It follows that $\mathcal{M}_n^1 = \{\Omega_n^{j,k} \mid j, k \in [n], j \neq k\}$.

3. Automorphisms of HT_n

In this section we prove Theorem 1.3.

Fix an integer $n \geq 3$. Recall that we write Σ_n for the symmetric group of rank n, which we identify with the group of bijections $[n] \to [n]$. For a bijection $\sigma \in \Sigma_n$, and a subset $S \subset [n]$, we write $\sigma(S)$ for the set $\{\sigma(s) \mid s \in S\}$. For a bijection $\sigma \in \Sigma_n$, and a hypertree $\Theta \in \mathcal{HT}_n$, we write $\sigma(\Theta)$ for the hypertree on $[n]$ such that $E_{\sigma(\Theta)} = \{\sigma(e) \mid e \in E_\Theta\}$; that is, $E_{\sigma(\Theta)}$ is obtained by replacing each hyperedge $e \in E_\Theta$ with $\sigma(e)$.

Evidently, the map $\Theta \mapsto \sigma(\Theta)$ preserves the partial order $\leq$, and hence determines an automorphism of $(\mathcal{HT}_n, \leq)$; thus we have a homomorphism $\Sigma_n \to \mathrm{Aut}(\mathcal{HT}_n, \leq)$. Since $\mathrm{Aut}(\mathcal{HT}_n, \leq)$ embeds in $\mathrm{Aut}(\mathrm{HT}_n)$, we also have a homomorphism $\iota_n : \Sigma_n \to \mathrm{Aut}(\mathrm{HT}_n)$. Theorem 1.3 is proved if we show that ι_n is bijective. We can simplify this task a little using the following lemma. The lemma follows immediately from the observation that for each hypertree except Θ_n^0, there is another hypertree with the same number of hyperedges.

Lemma 3.1. For each integer $n \geq 3$, $\mathcal{V}_n(\Theta_n^0)$ is the unique vertex in HT_n of maximal valence.

It follows from the lemma that for each simplicial automorphism $f \in \mathrm{Aut}(\mathrm{HT}_n)$, f fixes $\mathcal{V}_n(\Theta_n^0)$, and restricts to a simplicial automorphism $f^+ \in \mathrm{Aut}(\mathrm{HT}_n^+)$. The restriction $f \mapsto f^+$ is an isomorphism $\mathrm{Aut}(\mathrm{HT}_n) \to \mathrm{Aut}(\mathrm{HT}_n^+)$. Pre-composing this isomorphism with ι_n gives a homomorphism $\iota_n^+ : \Sigma_n \to \mathrm{Aut}(\mathrm{HT}_n^+)$. To prove Theorem 1.3 it suffices to show that ι_n^+ is bijective.

That ι_n^+ is injective follows immediately from an analysis of how Σ_n acts on $\mathcal{S}_n = \{\Xi_n^1, \dots, \Xi_n^n\}$, the set of star trees.

Lemma 3.2. For each integer $n \geq 3$, the homomorphism $\iota_n^+ : \Sigma_n \to \mathrm{Aut}(\mathrm{HT}_n^+)$ is injective.

Proof. Since $n \geq 3$, there are n distinct star trees, and Σ_n acts on $\mathcal{S}_n$ by permuting superscripts. It follows that Σ_n acts as the full permutation group on $\mathcal{V}_n^+(\mathcal{S}_n)$, and distinct elements of Σ_n act distinctly on $\mathcal{V}_n^+(\mathcal{S}_n)$. Thus distinct elements of Σ_n act distinctly on HT_n^+. □

Notation 3.3. Empowered by the lemma, we shall not distinguish between an element of Σ_n and the corresponding automorphisms of HT_n, HT_n^+ and $(\mathcal{HT}_n, \leq)$.

It remains to show that ι_n^+ is surjective. The cases $n = 3$ and $n = 4$ are easily dispatched by inspection of HT_3^+ and HT_4^+ respectively.

Lemma 3.4. The homomorphism ι_3^+ is surjective.

Proof. The complex HT_3^+ consists of three disjoint vertices. The vertices are the elements of $\mathcal{V}_3^+(\mathcal{S}_3)$. Evidently, each automorphism of HT_3^+ permutes these vertices. The result follows. □

Lemma 3.5. The homomorphism ι_4^+ is surjective.

Proof. Let $f^+ \in \mathrm{Aut}(\mathrm{HT}_4^+)$. It suffices to show that there exists $\sigma \in \Sigma_4$ such that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_4^+(\mathcal{HT}_4^+)$, because then $\sigma^{-1} f^+$ is the identity automorphism of HT_4^+, and $f^+ = \sigma \in \Sigma_4$.

Consider the complex HT_4^+, as shown in Figure 5. The vertices in $\mathcal{V}_4^+(\mathcal{S}_4)$ are shown as stars, the vertices in $\mathcal{V}_4^+(\mathcal{L}_4)$ are shown as filled circles, and the vertices in $\mathcal{V}_4^+(\mathcal{HT}_4^1)$ are shown as unfilled circles. Note that $\mathcal{HT}_4^+ = \mathcal{S}_4 \cup \mathcal{L}_4 \cup \mathcal{HT}_4^1$. Inspection of Figure 5 shows that, for each $\Theta \in \mathcal{HT}_4^+$, $\Theta \in \mathcal{S}_4$ if and only $\mathcal{V}_4^+(\Theta)$ has valence three, and $\mathcal{V}_4^+(\Theta)$ is not adjacent to a vertex of valence two. Since the elements of $\mathcal{V}_4^+(\mathcal{S}_4)$ can be identified geometrically, f^+ fixes setwise $\mathcal{V}_4^+(\mathcal{S}_4)$. Since Σ_4 acts as the full permutation group on $\mathcal{V}_4^+(\mathcal{S}_4)$, there exists $\sigma \in \Sigma_4$ such that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_4^+(\mathcal{S}_4)$.

Inspection also shows that, for each $\Theta \in \mathcal{HT}_4^+$, $\Theta \in \mathcal{HT}_4^1$ if and only if $\mathcal{V}_4^+(\Theta)$ has valence three, and $\mathcal{V}_4^+(\Theta)$ is adjacent to a vertex of valence two. Since the elements of $\mathcal{V}_4^+(\mathcal{HT}_4^1)$ can be identified geometrically, $\sigma^{-1} f^+$ fixes setwise $\mathcal{V}_4^+(\mathcal{HT}_4^1)$, and hence also fixes setwise $\mathcal{V}_4^+(\mathcal{L}_4)$.

Further inspection shows that distinct elements of $\mathcal{V}_4^+(\mathcal{HT}_4^1)$ (that is, distinct unfilled circles) can be distinguished by their relative proximities in HT_4^+ to the elements of $\mathcal{V}_4^+(\mathcal{S}_4)$; that is, if Θ and Λ are distinct elements of $\mathcal{HT}_4^1$, then there exists $\Upsilon \in \mathcal{S}_4$ such that $d_4^+(\Theta, \Upsilon) \neq d_4^+(\Lambda, \Upsilon)$. It follows that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_4^+(\mathcal{HT}_4^1)$.

Finally, further inspection shows that distinct elements in $\mathcal{V}_4^+(\mathcal{L}_4)$ can be distinguished by their relative proximities in HT_4^+ to the elements of $\mathcal{V}_4^+(\mathcal{HT}_4^1)$. It follows that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_4^+(\mathcal{L}_4)$.

We have that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_4^+(\mathcal{HT}_4^+)$, as required. $\square$

To prove that ι_n^+ is surjective for $n \geq 5$, we adapt the argument for $n = 4$, replacing each use of inspection by a general argument. Although our general argument still works with the subsets $\mathcal{L}_n$, $\mathcal{S}_n$ and $\mathcal{HT}_n^1$, it must recognize that these sets no longer combine to give the entirety of HT_n^+. In particular, it takes more effort to show that an arbitrary simplicial automorphism of HT_n^+ fixes setwise $\mathcal{V}_n^+(\mathcal{HT}_n^1)$.

To ensure the structure of our general argument is easily visible, we describe it assuming a series of technical claims, to be proved immediately after. For each $n \geq 5$, we claim the following:

(A) The vertices in $\mathcal{V}_n^+(\mathcal{M}_n^1)$ are exactly the vertices in HT_n^+ of maximal valence.

(B) The vertices in $\mathcal{V}_n^+(\mathcal{S}_n)$ are exactly the vertices in HT_n^+ which are adjacent to $n-1$ vertices in $\mathcal{V}_n^+(\mathcal{M}_n^1)$.

(C) The vertices in $\mathcal{V}_n^+(\mathcal{L}_n)$ are exactly the vertices in HT_n^+ which, although not adjacent to any vertex in $\mathcal{V}_n^+(\mathcal{S}_n)$, are distance exactly two from $n-2$ vertices in $\mathcal{V}_n^+(\mathcal{S}_n)$.

(D) The vertices in $\mathcal{V}_n^+(\mathcal{HT}_n^1)$ are exactly the vertices in HT_n^+ which are adjacent to some vertex in $\mathcal{V}_n^+(\mathcal{S}_n)$, and adjacent to some vertex in $\mathcal{V}_n^+(\mathcal{L}_n)$.

(E) For all $\Theta, \Delta \in \mathcal{M}_n^1$, $\Theta = \Delta$ if and only if $d_n^+(\Theta, \Upsilon) = d_n^+(\Delta, \Upsilon)$ for all $\Upsilon \in \mathcal{S}_n$.

(F) For all $\Theta, \Delta \in \mathcal{HT}_n^1$, $\Theta = \Delta$ if and only if $d_n^+(\Theta, \Upsilon) = d_n^+(\Delta, \Upsilon)$ for all $\Upsilon \in \mathcal{M}_n^1$.

(G) For all $\Theta, \Delta \in \mathcal{HT}_n^+$, $\Theta = \Delta$ if and only if $d_n^+(\Theta, \Upsilon) = d_n^+(\Delta, \Upsilon)$ for all $\Upsilon \in \mathcal{HT}_n^1$.

Proposition 3.6. For each integer $n \geq 5$, ι_n^+ is surjective.

Proof, assuming claims (A) through (G). Considered in order, Claims (A) through (D) combine to give that an arbitrary automorphism of HT_n^+ fixes setwise $\mathcal{V}_n^+(\mathcal{M}_n^1)$, $\mathcal{V}_n^+(\mathcal{S}_n)$, $\mathcal{V}_n^+(\mathcal{L}_n)$ and $\mathcal{V}_n^+(\mathcal{HT}_n^1)$.

Let $f^+ \in \mathrm{Aut}(\mathrm{HT}_n^+)$. As in the case that $n = 4$, it suffices to show that there exists $\sigma \in \Sigma_n$ such that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{HT}_n^+)$.

Since f^+ fixes setwise $\mathcal{V}_n^+(\mathcal{S}_n)$, and Σ_n acts as the full permutation group of $\mathcal{V}_n^+(\mathcal{S}_n)$, there exists $\sigma \in \Sigma_n$ such that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{S}_n)$. Since $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{S}_n)$, and fixes setwise $\mathcal{V}_n^+(\mathcal{M}_n^1)$, (E) implies that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{M}_n^1)$. Since $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{M}_n^1)$, and fixes setwise $\mathcal{V}_n^+(\mathcal{HT}_n^1)$, (F) implies that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{HT}_n^1)$. Finally, since $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{HT}_n^1)$ and fixes setwise $\mathcal{V}_n^+(\mathcal{HT}_n^+)$, (G) implies that $\sigma^{-1} f^+$ fixes pointwise $\mathcal{V}_n^+(\mathcal{HT}_n^+)$, as required. $\square$

Remark 3.7. Claim (A) fails in the case $n = 4$ because, as is evident in Figure 5, the vertices in $\mathcal{V}_4^+(\mathcal{HT}_4^1)$ (shown as unfilled circles) are not the only vertices of maximal valence.

3.1. Proving Claims (A) through (G)

Throughout this subsection we assume $n \geq 5$.

Claim (A) requires that we understand the valences of vertices in HT_n^+. For each $\Theta \in \mathcal{HT}_n^+$: we write $A_n^+(\Theta)$ for the set of hypertrees in $\mathcal{HT}_n^+$, distinct from Θ, which fold to Θ; and we write $B_n^+(\Theta)$ for the set of hypertrees in $\mathcal{HT}_n^+$, distinct from Θ, which can be obtained from Θ by folding. Thus the valence in HT_n^+ of $\mathcal{V}_n^+(\Theta)$ is $\#A_n^+(\Theta) + \#B_n^+(\Theta)$.

A convenient formula for $\#A_n^+(\Theta)$ follows from the observation that the operation of folding has a natural inverse. Let $\Theta, \Lambda \in \mathrm{HT}_n^+$, let $e \in E_\Theta$, and let Δ be a hypertree on e (that is, Δ is a hypertree with vertex set e). We say Λ is obtained from Θ by *unfolding e to Δ*, or just *unfolding e*, if $E_\Lambda = (E_\Theta \setminus \{e\}) \cup E_\Delta$; that is, if E_Λ is the result of replacing $e \in E_\Theta$ by the elements of E_Δ. If, in the above, Δ has only one hyperedge, then $\Theta = \Lambda$ and we say the unfolding is *trivial*; otherwise the unfolding is *nontrivial*, and Λ has more hyperedges than Θ. Evidently, Λ is obtained from Θ by unfolding if and only if Θ is obtained from Λ by folding.

Lemma 3.8 (c.f Lemma 2.5(1) [5])**.** For each $\Theta \in \mathrm{HT}_n^+$,

$$\#A_n^+(\Theta) = -1 + \prod_{e \in E_\Theta} \#(\mathcal{HT}_{\#e}).$$

Proof. Let $\Theta \in \mathrm{HT}_n^+$. For each hyperedge $e \in E_\Theta$, there are $\#(\mathcal{HT}_{\#e})$ distinct hypertrees on e, and hence $\#(\mathcal{HT}_{\#e})$ distinct hypertrees, including Θ itself, which may be obtained from Θ by unfolding e. The result follows. $\square$

Let $h \in \{1, \ldots, n-2\}$. If $\Theta \in \mathcal{HT}_n^h$, then a hyperedge in Θ has degree at most $n-h$. Our next result records that those hypertrees in $\mathcal{HT}_n^h$ that contain a hyperedge of degree $n-h$ are precisely the hypertrees in $\mathcal{HT}_n^h$ for which $\#A_n^+(\Theta)$ is maximal. We note that if $\Theta \in \mathcal{HT}_n^h$ has a hyperedge of degree $n-h$, then the other hyperedges in Θ have degree exactly two.

Lemma 3.9. Let $h \in \{1, \ldots, n-2\}$, let $\Theta \in \mathcal{HT}_n^h$, and let $\Lambda \in \mathcal{HT}_n^h$ be such that Λ has a hyperedge of degree $n-h$. If Θ also has a hyperedge of degree $n-h$, then $\#A_n^+(\Theta) = \#A_n^+(\Lambda)$; otherwise $\#A_n^+(\Theta) < \#A_n^+(\Lambda)$.

The lemma follows inductively from Lemma 3.8, and the following result.

Lemma 3.10. For all integers $p, q \geq 3$, $\#\mathcal{HT}_p \,.\, \#\mathcal{HT}_q < \#\mathcal{HT}_{p+q-2}$.

Proof. Let Λ be the hypertree on $[p+q-1]$ with hyperedges $\{1, \ldots, p-1\} \cup \{p+q-1\}$ and $\{p, p+1, \ldots, p+q-1\}$. By Lemma 3.8 we have $\#A_n^+(\Lambda) = -1 + \#\mathcal{HT}_p \,.\, \#\mathcal{HT}_q$; recall that $\#A_n^+(\Theta^0_{p+q-2}) = -1 + \#\mathcal{HT}_{p+q-2}$. Thus to prove the lemma it suffices to exhibit an injective, but not surjective, map $A_n^+(\Lambda) \to A_n^+(\Theta^0_{p+q-2})$.

Let Υ be an arbitrary hypertree in $A_n^+(\Lambda)$. Note that if $e \in E_\Upsilon$, and $p+q-1 \in e$, then either $e \setminus \{p+q-1\} \subset \{1, \ldots, p-1\}$ or $e \setminus \{p+q-1\} \subset \{p, \ldots, p+q-2\}$. Let $a \in \{1, \ldots, p-1\}$ be maximal such that a is adjacent in Υ to $p+q-1$, and let $e_a \in E_\Upsilon$ be the hyperedge such that $\{a, p+q-1\} \in e_a$; let $b \in \{p, \ldots, p+q-2\}$ be minimal such that b is adjacent in Υ to $p+q-1$, and let $e_b \in E_\Upsilon$ be the hyperedge such that $\{b, p+q-1\} \in e_b$. We construct a hypertree $\Upsilon' \in \mathcal{HT}_{p+q-2}$ as follows: $(e_a \cup e_b) \setminus \{p+q-1\}$ is a hyperedge in Υ'; for each hyperedge $e \in E_\Upsilon$ with $p+q-1 \in e$ and $e \setminus \{p+q-1\} \subset \{1, \ldots, p-1\}$ and $e \neq e_a$, we replace e by $\{b\} \cup e \setminus \{p+q-1\}$; for each hyperedge $e \in E_\Upsilon$ with $p+q-1 \in e$ and $e \setminus \{p+q-1\} \subset \{p, \ldots, p+q-2\}$ and $e \neq e_b$, we replace e by $\{a\} \cup e \setminus \{p+q-1\}$.

To show that the map $\Upsilon \mapsto \Upsilon'$ is injective, we now explain how, given Υ', we can reconstruct Υ. Suppose Υ' is as described above. We can identify a because it is maximal amongst the elements of $\{1, \ldots, p-1\}$ which are adjacent in Υ' to at least one element of $\{p, \ldots, p+q-2\}$; similarly, we can identify b. Hence we can identify the hyperedge $(e_a \cup e_b) \setminus \{p+q-1\}$, and we can recover e_a and e_b. Having done this, it is now evident that we can recover the other hyperedges of Υ.

In this paragraph we show that the map $\Upsilon \mapsto \Upsilon'$ is not surjective. It follows from the construction that any Υ' must have the property that, amongst the elements of $\{1, \ldots, p-1\}$ which are adjacent to an element of $\{p, \ldots, p+q-2\}$, only the maximal element may be adjacent to more than one element of $\{p, \ldots, p+q-2\}$. Hence the hypertree with hyperedges

$$\{1, p+q-2\}, \{1\} \cup \{p, \ldots, p+q-3\}, \{2, p+q-2\}, \{2, 3, \ldots, p-1\}$$

is contained in $A_n^+(\Theta^0_{p+q-2})$, but it is not the image of any hypertree $\Upsilon \in A_n^+(\Lambda)$ (it is here we use the hypothesis that $p, q \geq 3$). $\square$

Next we look to understand $\#B_n^+(\Theta)$ for an arbitrary hypertree $\Theta \in \mathcal{HT}_n^+$. We begin by observing that each hypertree $\Lambda \in B_n^+(\Theta)$ corresponds to an equivalence relation on E_Θ, but, in general, only certain equivalence relations on E_Θ correspond to hypertrees in $\mathcal{B}_n^+(\Theta)$. Given $\Theta \in \mathcal{HT}_n$ and $\Lambda \in B_n^+(\Theta)$, we define a relation $\sim_\Lambda$ on E_Θ as follows: if $e, e' \in E_\Theta$, then $e \sim_\Lambda e'$ if there exists $u \in E_\Lambda$ such that $e \cup e' \subset u$; that is, hyperedges in Θ are related if and only if they are eventually folded in Λ. We leave the reader to verify that $\sim_\Lambda$ is an equivalence relation. The restriction, that two hyperedges must intersect nontrivially if they are to be merged in a single fold, implies that each equivalence class must satisfy the following condition:

(∗) if $e \in E_\Theta$ and $i, j \in \cup_{e \sim e'} e'$, then the unique simple walk in Λ from i to j visits only vertices in $\cup_{e \sim e'} e'$.

It is easily verified that an equivalence relation $\sim$ on E_Θ corresponds to a hypertree $\Theta_\sim$ if and only if the equivalence relation satisfies $(*)$; in which case, the hypertree $\Theta_\sim$ is in $B_n^+(\Theta)$ unless there is only one equivalence class of hyperedeges (because then $\Theta_\sim = \Theta_n^0$), or all equivalence classes are singletons (because then $\Theta_\sim = \Theta$).

If $\Theta \in \mathcal{HT}_n^h$ has a vertex j of valence $h+1$, then j is contained in every hyperedge of Θ, and every vertex but j is a leaf (otherwise Θ would fail to be a hypertree). We now show that the hypertrees in $\mathcal{HT}_n^h$ that contain a vertex of valence $h+1$ are precisely the hypertrees in $\mathcal{HT}_n^h$ for which $\#B_n^+(\Theta)$ is maximal.

Lemma 3.11. Let $h \in \{1, \ldots, n-2\}$, let $\Theta \in \mathcal{HT}_n^h$, and let $\Lambda \in \mathcal{HT}_n^h$ be such that Λ has a vertex of valence $h+1$. If Θ also has a vertex of valence $h+1$, then $\#B_n^+(\Lambda) = \#B_n^+(\Theta)$; otherwise, $\#B_n^+(\Theta) < \#B_n^+(\Lambda)$.

Proof. Let $\Upsilon \in \mathcal{HT}_n^h$. If some vertex j has valence $h+1$, then j is contained in every hyperedge of Υ, and every partition of E_Υ satisfies $(*)$; otherwise, there exist partitions of E_Υ which do not satisfy $(*)$. The result follows. □

Lemmas 3.9 and 3.11 combine with the definition of $\mathcal{M}_n^h$ (Definition 2.10) to give the following.

Lemma 3.12. Let $h \in \{1, \ldots, n-2\}$. If $\Theta \in \mathcal{HT}_n^h \setminus \mathcal{M}_n^h$, and $\Lambda \in \mathcal{M}_n^h$, then the valence in HT_n^+ of $\mathcal{V}_n^+(\Lambda)$ strictly exceeds that of $\mathcal{V}_n^+(\Theta)$.

We now compare the valences of vertices in $\mathcal{M}_n^h$ for different values of h, and so establish Claim (A). Recall that for distinct integers $j, k \in [n]$, we write $\Omega_n^{j,k}$ for the hypertree with hyperedges $\{j, k\}$ and $[n] \setminus \{k\}$; and we write $\mathcal{M}_n^1 := \{\Omega_n^{j,k} \mid j, k \in [n], j \neq k\}$. We begin by observing that there is a simple way to characterize those vertices in $\mathcal{V}_n^+(\mathcal{HT}_n)$ which are adjacent to $\mathcal{V}_n^+(\Omega_n^{j,k})$.

Definition 3.13 ((j,k)-tag)**.** Given $\Theta \in \mathcal{HT}_n^+$, and distinct integers $j, k \in [n]$ with $j \neq k$, we say Θ *has a* (j,k)*-tag* if $\{j, k\}$ is a hyperedge in Θ, and no other hyperedge in Θ contains k.

Lemma 3.14. For each pair $j, k \in [n]$ with $j \neq k$, and for each hypertree $\Lambda \in \mathcal{HT}_n^+$, $\mathcal{V}_n^+(\Lambda)$ and $\mathcal{V}_n^+(\Omega_n^{j,k})$ are adjacent in HT_n^+ if and only if Λ has at least three hyperedges, and Λ has a (j,k)-tag.

Proof. Since $\Omega_n^{j,k}$ is minimal in $(\mathcal{HT}_n^+, \leq)$, $\mathcal{V}_n^+(\Lambda)$ and $\mathcal{V}_n^+(\Omega_n^{j,k})$ are adjacent in HT_n^+ if and only if $\Omega_n^{j,k} \in B_n^+(\Lambda)$.

If Λ has only two hyperedges, then $B_n^+(\Lambda) = \emptyset$.

If Λ has at least three hyperedges and Λ has a (j,k)-tag, then we obtain $\Omega_n^{j,k}$ by folding, in some allowable order, all of the hyperedges in Λ except $\{j,k\}$; hence $\Omega_n^{j,k} \in B_n^+(\Lambda)$.

Suppose that Λ does not have a (j,k)-tag. Then there exists some vertex $\ell \neq j$ such that k and ℓ are adjacent in Λ. It follows from the definition of folding that k and ℓ are adjacent in each element of $B_n^+(\Lambda)$; thus $\Omega_n^{j,k} \notin B_n^+(\Lambda)$. $\square$

Proposition 3.15. [Claim (A)] The vertices in $\mathcal{V}_n^+(\mathcal{M}_n^1)$ are exactly the vertices in HT_n^+ of maximal valence.

Proof. Let $\Omega \in \mathcal{M}_n^1$, and let Θ be a hypertree in $\mathcal{M}_n^h$ for some $h \in \{2,\dots,n\}$. In light of Lemma 3.12, it suffices to show that the valence in HT_n^+ of $\mathcal{V}_n^+(\Omega)$ exceeds that of $\mathcal{V}_n^+(\Theta)$. To do so we will exhibit an injective, but not surjective, map $p\colon A_n^+(\Theta) \cup B_n^+(\Theta) \to A_n^+(\Omega)$.

Since the hypertrees in $\mathcal{M}_n^1$ are isomorphic as unlabeled hypertrees, the corresponding vertices have the same valences in HT_n^+, and we may assume $\Omega = \Omega_n^{1,2}$. Similarly, we may assume

$$E_\Theta = \{\{1,2\},\{1,3\},\dots,\{1,h{+}1\},\{1,h{+}2,h{+}3,\dots,n\}\},$$

as shown in Figure 7.

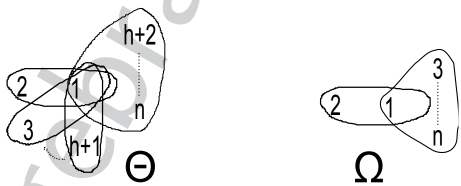

FIGURE 7. Θ and Ω as in the proof of Proposition 3.15.

We define $p(\Omega) = \Theta$, and $p(\Upsilon) = \Upsilon$ for each $\Upsilon \in (A_n^+(\Theta) \cup B_n^+(\Theta)) \cap A_n^+(\Omega)$. Since Θ has a $(1,2)$-tag, $\Theta \in A_n^+(\Omega)$; it follows that $A_n^+(\Theta)$ is a subset of $(A_n^+(\Theta) \cup B_n^+(\Theta)) \cap A_n^+(\Omega)$. Thus it remains only to define $p(\Upsilon)$ for $\Upsilon \in B_n^+(\Theta) \setminus (A_n^+(\Omega) \cup \{\Omega\})$. Consider a hypertree $\Upsilon \in B_n^+(\Theta) \setminus (A_n^+(\Omega) \cup \{\Omega\})$. Since $\Upsilon \in B_n^+(\Theta)$, 1 is the only non-leaf vertex; since $\Upsilon \notin A_n^+(\Omega) \cup \{\Omega\}$, the unique hyperedge $e \in E_\Upsilon$ such that e contains 2 is such that $e \neq \{1,2\}$. Let j be minimal such that $j \geq 3$ and $j \in e$. We let $E_{\Upsilon'}$ be the set of hyperedges obtained from E_Υ by removing 2 from e; swapping each occurrence of 1 with j, and vice-versa; and then adding the hyperedge $\{1,2\}$. An example of this process is shown in Figure 8. Since Υ' has at least three hyperedges, and it has a $(1,2)$-tag, $\Upsilon' \in A_n^+(\Omega)$.

Since j is a non-leaf in Υ', $\Upsilon' \notin B_n^+(\Theta)$. We define $p(\Upsilon) = \Upsilon'$. Since j and 1 are the only non-leaves in Υ', we can recognize j in Υ', and hence we can recover Υ from Υ'. It follows that p is injective.

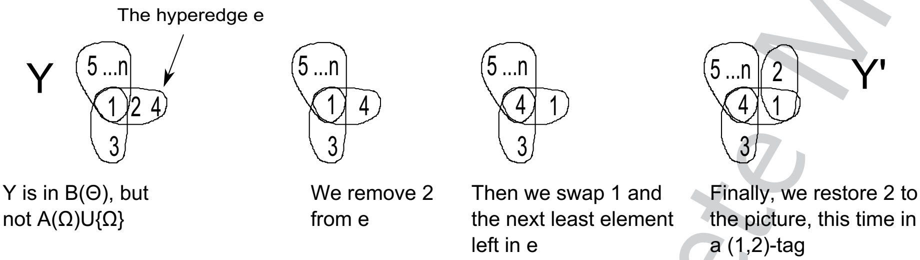

Figure 8. An example computing Υ' for $\Upsilon \in B_n^+(\Theta) \setminus (A_n^+(\Omega) \cup \{\Omega, \Theta_n^0\})$, as in the proof of Proposition 3.15.

It remains only to show that p is not surjective. To do so, consider $\Delta \in \mathcal{HT}_n$ such that

$$E_\Delta = \{\{1,2\},\{1,3\},\{3,4\},\{4,5,\ldots,n\}\}$$

(recall that, by hypothesis, $n \geq 5$). Since Δ has a $(1,2)$-tag, $\Delta \in A_n^+(\Omega)$; since 1 is not the only non-leaf in Δ, $\Delta \notin B_n^+(\Theta)$; evidently, $\Delta \neq \Theta$; since Δ has 3 non-leaves, and any Υ' constructed as above has only two non-leaves, $\Delta \neq \Upsilon'$. Since $\Delta \in A_n^+(\Omega)$, and Δ is not in the image of p, p is not surjective. $\square$

We now turn our attention to claims (B) through (G).

Lemma 3.16 (Claim (B))**.** The vertices in $\mathcal{V}_n^+(\mathcal{S}_n)$ are exactly the vertices in HT_n^+ which are adjacent to $n-1$ vertices in $\mathcal{V}_n^+(\mathcal{M}_n^1)$.

Proof. The lemma follows immediately from Lemma 3.14, and the observation that the star trees are exactly the hypertrees with $n-1$ distinct tags. $\square$

Lemma 3.17. Let $j \in [n]$ and let $\Theta \in \mathcal{HT}_n^+$. Then

1) $d_n^+(\Theta, \Xi_n^j) \leq 1$ if and only if j is the only vertex in Θ that is not a leaf;
2) $d_n^+(\Theta, \Xi_n^j) = 2$ if and only if j is not a leaf in Θ, and there is at least one other vertex in Θ which is not a leaf.

Proof. Since Ξ_n^j is maximal in $(\mathcal{HT}_n^+, \leq)$, $d_n^+(\Theta, \Xi_n^j) = 1$ if and only if $\Theta \in B_n^+(\Xi_n^j)$. Evidently, j is the only non-leaf vertex in Ξ_n^j, folding cannot make a leaf into a non-leaf, and no element of $\mathcal{HT}_n^+$ has n-leaves. Thus each element of $B_n^+(\Xi_n^j)$ has exactly $n-1$ leaves. Property (1) follows.

Property (2) follows immediately from the observations that: $d_n^+(\Theta, \Xi_n^j) = 2$ if and only if $\Theta \notin B_n^+(\Xi_n^j) \cup \{\Xi_n^j\}$, but $B_n^+(\Theta) \cap B_n^+(\Xi_n^j) \neq \emptyset$; if j is a non-leaf in a hypertree Λ, then we can arrange that j is the only non-leaf by repeatedly folding two hyperedges which both contain some $i \in [n] \setminus \{j\}$. □

Corollary 3.18 (Claim (C))**.** The vertices in $\mathcal{V}_n^+(\mathcal{L}_n)$ are exactly the vertices in HT_n^+ which, although not adjacent to any vertex in $\mathcal{V}_n^+(\mathcal{S}_n)$, are distance exactly two from $n-2$ of the n vertices in $\mathcal{V}_n^+(\mathcal{S}_n)$.

Proof. Line trees and star trees have the same height, and so cannot be adjacent in $\mathcal{HT}_n^+$. By definition, the lines trees are exactly the hypertrees with exactly two leaves. Equivalently, the line trees are exactly the hypertrees in which there are $n-2$ vertices which are not leaves. The result now follows from Lemma 3.17. □

Lemma 3.19 (Claim (D))**.** The vertices in $\mathcal{V}_n^+(\mathcal{HT}_n^1)$ are exactly the vertices in HT_n^+ which are adjacent to some vertex in $\mathcal{V}_n^+(\mathcal{S}_n)$, and adjacent to some vertex in $\mathcal{V}_n^+(\mathcal{L}_n)$.

Proof. Suppose $\mathcal{V}_n^+(\Theta)$ is adjacent to both $\mathcal{V}_n^+(\Xi)$ and $\mathcal{V}_n^+(\Lambda)$ for some star-tree Ξ, and some line-tree Λ. Then, since Ξ and Λ are maximal in $(\mathcal{HT}_n^+, \leq)$, $\Theta \leq \Xi$ and $\Theta \leq \Lambda$. Since $\Theta \leq \Xi$, Θ has only one non-leaf vertex. Since distinct hyperedges in a hypertree can contain at most one common vertex, a single fold can turn at most one non-leaf into a leaf. Since Λ has n–2 non-leaves, it will take at least n–3 single folds to obtain from Λ a hypertree that has only one non-leaf. Since $\Lambda \in \mathcal{HT}_n^{n-2}$, and Θ is obtained by at least $n-3$ single folds, $\Theta \in \mathcal{HT}_n^1$.

Conversely, suppose that $\Theta \in \mathcal{HT}_n^1$. Then $E_\Theta = \{A \cup \{j\}, B \cup \{j\}\}$ for some $j \in [n]$ and some non-trivial partition $\{A, B\}$ of $[n] \setminus \{j\}$. Since j is the only non-leaf of Θ, $\Theta \in B_n^+(\Xi_n^j)$. Let $a \in A$ and let Λ_A be a line tree on A in which a is a leaf; let $b \in B$ and let Λ_B be a line tree on B in which b is a leaf. Let Λ be the hypertree on $[n]$ such that $E_\Lambda = E_{\Lambda_A} \cup E_{\Lambda_B} \cup \{\{a, j\}, \{b, j\})\}$. Then Λ is a line tree on $[n]$ and $\Theta \in B_n^+(\Lambda)$. □

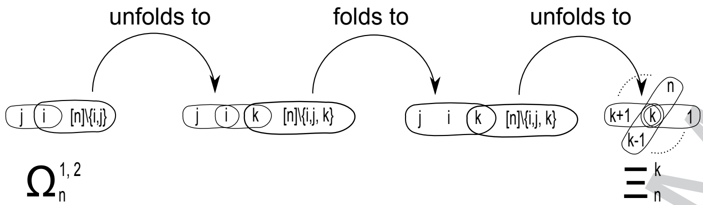

Figure 9. A 3-step unfolding and folding sequence.

The author thanks Andy Eisenberg for observing an error in an earlier proof of the following lemma, and suggesting the proof which appears below.

Lemma 3.20 (Claim (E))**.** For all $\Omega, \Omega' \in \mathcal{M}_n^1$, $\Omega = \Omega'$ if and only if $d_n^+(\Omega, \Xi) = d_n^+(\Omega', \Xi)$ for all $\Xi \in \mathcal{S}_n$.

Proof. Let i, j, k be distinct elements of $[n]$. It is clear that $d_n^+(\Omega_n^{i,j}, \Xi_n^i) = 1$.

We claim that $d_n^+(\Omega_n^{i,j}, \Xi_n^k) = 3$. The unfolding and folding sequence in Figure 9 shows that $d_n^+(\Omega_n^{i,j}, \Xi_n^k) \leq 3$. Since k is a leaf in $\Omega_n^{i,j}$, Lemma 3.17 implies that $d_n^+(\Omega_n^{i,j}, \Xi_n^k) > 3$.

Finally, we claim that $d_n^+(\Omega_n^{i,j}, \Xi_n^j) > 3$. Because it is minimal in $(\mathcal{HT}_n^+, \leq)$, we cannot fold $\Omega_n^{i,j}$ and stay in $\mathcal{HT}_n^+$. Suppose that Λ' is obtained by unfolding $\Omega_n^{i,j}$. Then Λ' has a (i, j)-tag. Since j is a leaf in Λ', Lemma 3.17 implies that $d_n^+(\Lambda', \Xi_n^j) > 2$. Hence $d_n^+(\Omega_n^{i,j}, \Xi_n^j) > 3$ as claimed. The result follows. $\square$

Lemma 3.21 (Claim (F))**.** For all $\Theta, \Delta \in \mathcal{HT}_n^1$, $\Theta = \Delta$ if and only if $d_n^+(\Theta, \Upsilon) = d_n^+(\Delta, \Upsilon)$ for all $\Upsilon \in \mathcal{M}_n^1$.

Proof. Let $\Theta \in \mathcal{HT}_n^1 \setminus \mathcal{M}_n^1$. Then $E_\Theta = \{A \cup \{j\}, B \cup \{j\}\}$ for some $j \in [n]$ and some disjoint nonempty subsets $A, B \subset [n] \setminus \{j\}$ such that $A \cup B = [n] \setminus \{j\}$, and $\#A, \#B \geq 3$.

Let $\Upsilon \in \mathcal{M}_n^1$. Then $\Upsilon = \Omega_n^{k,\ell}$ for some $k, \ell \in [n]$. Since Θ and $\Omega^{k,\ell}$ are distinct elements of the same height, $d_n^+(\Theta, \Omega^{k,\ell}) \geq 2$.

If k and ℓ are adjacent in Θ, and ℓ is a leaf, then Θ can be unfolded to a hypertree with a (k, ℓ)-tag, so $d_n^+(\Theta, \Omega_n^{k,\ell}) = 2$. If k and ℓ are not adjacent in Θ, or ℓ is not a leaf, then Θ cannot be unfolded to a hypertree with a (k, ℓ)-tag, so $d_n^+(\Theta, \Omega_n^{k,\ell}) \geq 3$.

It follows that the function $\Xi \mapsto d_n^+(\Theta, \Xi)$, for $\Xi \in \mathcal{S}_n$, can be used to identify which pairs of vertices in Θ are adjacent (and which vertex is a non-leaf). Since the hyperedges in Θ are precisely the maximal subsets of $[n]$ with the property that elements are pairwise adjacent in Θ, the result follows. $\square$

Lemma 3.22 (Claim (G))**.** For all $\Theta, \Delta \in \mathcal{HT}_n^+$, $\Theta = \Delta$ if and only if $d_n^+(\Theta, \Upsilon) = d_n^+(\Delta, \Upsilon)$ for all $\Upsilon \in \mathcal{HT}_n^1$.

Proof. Let $\Lambda \in \mathcal{HT}_n^1$. Then $E_\Lambda = \{A \cup \{j\}, B \cup \{j\}\}$ for some vertex j, and some disjoint non-empty subsets $A, B \subset [n] \setminus \{j\}$ such that $A \cup B = [n] \setminus \{j\}$. For all $\Theta \in \mathcal{HT}_n^+$, $\Theta \in A_n^+(\Lambda)$ if and only if, for all $a \in A$ and $b \in B$, a and b are not adjacent in Θ. Since there is an element of $\mathcal{HT}_n^1$ for each choice $j \in [n]$, and subsequent choices of A and B, the function $\Upsilon \mapsto d_n^+(\Theta, \Upsilon)$, for $\Upsilon \in \mathcal{HT}_n^1$, contains sufficient information to establish exactly which vertices are not adjacent in Θ, and hence exactly which vertices are adjacent in Θ. It follows from the definition of a hypertree that the hyperedges in Θ are precisely the maximal subsets of $[n]$ with the property that elements are pairwise adjacent in Θ. Thus we can use the knowledge of which vertices are adjacent in Θ to reconstruct E_Θ. The result follows. $\square$

4. The symmetries of McCullough-Miller space

4.1. Automorphisms of W_n

Fix a positive integer n. There are exactly n conjugacy classes of involutions in W_n, each represented by a generator. Each permutation of the generators induces an automorphism of W_n which permutes these conjugacy classes; we write Σ_n for the group of these automorphisms. It follows that $\mathrm{Aut}(W_n)$ acts transitively on the set of conjugacy classes of involutions; we write $\mathrm{Aut}^0(W_n)$ for the kernel of this action. It is easily verified that $\mathrm{Aut}(W_n) = \mathrm{Aut}^0(W_n) \rtimes \Sigma_n$. It follows that, writing $\mathrm{Out}^0(W_n)$ for quotient $\mathrm{Aut}^0(W_n)/\mathrm{Inn}(W_n)$, we have $\mathrm{Out}(W_n) \cong \mathrm{Out}^0(W_n) \rtimes \Sigma_n$. Thus for each $\alpha \in \mathrm{Out}(W_n)$, there exist unique automorphisms $\phi \in \mathrm{Out}^0(W_n)$ and $\sigma \in \Sigma_n$ such that $\alpha = \phi\sigma$. For $n \geq 3$, $\mathrm{Out}^0(W_n)$ is an infinite group.

Definition 4.1 (Partial conjugation)**.** For an integer $i \in [n]$, and a proper subset $D \subset [n] \setminus \{i\}$, we write x_{iD} for the outer automorphism of W_n determined by the map:

$$a_j \mapsto \begin{cases} a_i a_j a_i & \text{if } j \in D, \\ a_j & \text{if } j \in [n] \setminus D; \end{cases}$$

we say that x_{iD} is the *partial conjugation* with *acting letter* i and *domain* D.

If x_{iD} is a partial conjugation, then x_{iD} is an involution (if D were not a proper subset of $[n] \setminus \{i\}$, then x_{iD} would be the identity outer automorphism). If $i \in [n]$ and D, D' are disjoint proper subsets of $[n] \setminus \{i\}$ such that $D \cup D' = [n] \setminus \{i\}$, then $x_{iD} = x_{iD'}$. We adopt the convention that whenever we write x_{iD}, it is assumed that either $i = 1$ and $2 \notin D$, or $i \neq 1$ and $1 \notin D$.

The partial conjugations generate $\mathrm{Out}^0(W_n)$ (see, for example [3]). The following definition and lemma, due to McCullough and Miller, together establish a relationship between the hypertree poset and the automorphisms of W_n.

Definition 4.2 (Carried by)**.** Given a partial conjugation x_{iD}, and a hypertree $\Theta \in \mathcal{HT}_n$, we say that x_{iD} *is carried by* Θ if: for all $d \in D$ and for all $j \in [n] \setminus D$, the simple walk in Θ from j to d visits i. Given an automorphism $\alpha \in \mathrm{Out}^0(W_n)$ and a hypertree $\Theta \in \mathcal{HT}_n$, we say that α *is carried by* Θ if $\alpha = x_{i_pD_p} \ldots x_{i_1D_1}$ for some partial conjugations $x_{i_pD_p}, \ldots, x_{i_1D_1}$, each of which is carried by Θ.

Remark 4.3. It follows that x_{iD} is carried by Θ if and only if D is a union of connected components of $\Theta \setminus \{i\}$.

Lemma 4.4. Let $x_{i_1D_1}, \ldots, x_{i_pD_p}$ be partial conjugations and let $\Theta \in \mathcal{HT}_n$. If Θ carries the product $x_{i_pD_p} \ldots x_{i_1D_1}$, then the partial conjugations $x_{i_1D_1}, \ldots, x_{i_pD_p}$ pairwise commute.

Proof. Suppose that Θ carries $x_{i_jD_j}$ for each $j \in \{1, \ldots, k\}$. Let $p, q \in \{1, \ldots, k\}$. It follows from [6, p.14, second paragraph] that $x_{i_pD_p}$ commutes with $x_{i_qD_q}$ if $i_p \neq i_q$. Because the factors in the free product decomposition of W_n are abelian, $x_{i_pD_q}$ commutes with $x_{i_qD_q}$ if $i_p = i_q$ (this is not necessarily true in the more general setting considered by McCullough and Miller). $\square$

Given a hypertree $\Theta \in \mathcal{HT}_n$, and an integer $j \in [n]$, it is easy to count the partial conjugations x_{jD} carried by Θ: there is one for each collection of connected components of $\Theta \setminus \{j\}$, provided the collection excludes the connected component containing the least vertex of $[n] \setminus \{j\}$. The next lemma follows.

Lemma 4.5. For each $h \in \{0, \ldots, n–2\}$, and each hypertree $\Theta \in \mathcal{HT}_n^h$, Θ carries exactly 2^h automorphisms, including the identity automorphism.

4.2. McCullough-Miller space K_n

We define a relation $\sim$ on $\mathrm{Out}^0(W_n) \times \mathcal{HT}_n$ as follows: $(\alpha, \Theta) \sim (\beta, \Lambda)$ if and only if $\Theta = \Lambda$, and $\alpha^{-1}\beta$ is carried by Θ. It is easily verified that $\sim$ is an equivalence relation. We write $[\alpha, \Theta]$ for the $\sim$-equivalence class of (α, Θ), and we write $\mathcal{K}_n$ for the set of $\sim$-equivalence classes.

We define a partial order $\leq$ on $\mathcal{K}_n$ as follows: $[\alpha, \Theta] \leq [\beta, \Lambda]$ if and only if $\alpha^{-1}\beta$ is carried by Λ, and Λ folds to Θ. Equivalently, $[\alpha, \Theta] \leq [\beta, \Lambda]$ if and only if Λ folds to Θ and $[\beta, \Lambda] = [\alpha, \Lambda]$.

McCullough-Miller space K_n is the simplicial realization of $(\mathcal{K}_n, \leq)$. We write $\mathcal{V}_n(\alpha, \Theta)$ for the vertex in K_n corresponding to $[\alpha, \Theta]$.

Remark 4.6. Since Θ_n^0 carries only the identity in $\mathrm{Out}^0(W_n)$, equivalence classes of the form $[\alpha, \Theta_n^0]$ are singletons, and $[\alpha, \Theta_n^0] \leq [\beta, \Lambda]$ if and only of $\beta = \alpha$. Thus K_n consists of copies of HT_n, one copy for each element of $\mathrm{Out}^0(W_n)$, glued appropriately. Vertices of the form $\mathcal{V}_n(\alpha, \Theta_n^0)$ are called *nuclear vertices.*

4.3. A map $\mathrm{Out}(W_n) \to \mathrm{Aut}(\mathrm{K}_n)$

The set $\mathrm{Out}^0(W_n) \times \mathcal{HT}_n$ is naturally equipped with a left $\mathrm{Out}(W_n)$-action: for all $\phi \in \mathrm{Out}^0(W_n)$, $\sigma \in \Sigma_n$, and $(\alpha, \Theta) \in \mathrm{Out}^0(W_n) \times \mathcal{HT}_n$, we define $\phi\sigma.(\alpha, \Theta) = (\phi\sigma\alpha\sigma^{-1}, \sigma\Theta)$. It is easily verified that this action preserves the equivalence relation $\sim$, and that the induced action on $\mathcal{K}_n$ preserves the partial order $\leq$. Thus we have a homomorphism $\mathrm{Out}(W_n) \to \mathrm{Aut}(\mathcal{K}_n, \leq)$, which induces a homomorphism $\chi_n \colon \mathrm{Out}(W_n) \to \mathrm{Aut}(\mathrm{K}_n)$.

Lemma 4.7. For each integer $n \geq 3$, the homomorphism $\chi_n \colon \mathrm{Out}(W_n) \to \mathrm{Aut}(\mathrm{K}_n)$ is injective.

Proof. Recall that each element of $\mathrm{Out}(W_n)$ has the form $\phi\sigma$ for some $\phi \in \mathrm{Out}^0(W_n)$ and some $\sigma \in \Sigma_n$. If ϕ is non-trivial and ι denotes the identity in $\mathrm{Out}(W_n)$, then

$$\chi_n(\phi\sigma)\mathcal{V}_n(\iota, \Theta_n^0) = \mathcal{V}_n(\phi\sigma\iota\sigma^{-1}, \sigma\Theta_n^0) = \mathcal{V}_n(\phi, \Theta_n^0) \neq \mathcal{V}_n(\iota, \Theta_n^0).$$

If ϕ is trivial, but σ is not, then $\sigma\Theta \neq \Theta$ for some $\Theta \in \mathcal{HT}_n$, hence

$$\begin{aligned}\chi_n(\phi\sigma)\mathcal{V}_n(\iota, \Theta) &= \chi_n(\sigma)\mathcal{V}_n(\iota, \Theta) = \\ &= \mathcal{V}_n(\sigma\iota\sigma^{-1}, \sigma\Theta) = \mathcal{V}_n(\iota, \sigma\Theta) \neq \mathcal{V}_n(\iota, \Theta). \quad \square\end{aligned}$$

To prove Theorem 1.1 it suffices to show that χ_n is surjective, which is achieved in the next proposition. To ensure the structure of the argument is most clear, we describe it assuming the following technical claims, to be proved immediately after. We claim the following:

(H) For each integer $n \geq 3$, the nuclear vertices are exactly the vertices of maximal valence in K_n.

(I) Let $n \geq 3$, let $x_{iD} \in \mathrm{Out}^0(W_n)$ be a partial conjugation, and let $\beta \in \mathrm{Out}^0(W_n)$ be an automorphism such that, for each $\Theta \in \mathcal{HT}_n$, Θ carries β if and only if Θ carries x_{iD}. Then $\beta = x_{iD}$.

(J) Let $n \geq 4$, let $g \in \mathrm{Aut}(\mathrm{K}_n)$, let $\alpha \in \mathrm{Out}^0(W_n)$, and let x_{iD} be a non-trivial partial conjugation. If g fixes $\mathcal{V}_n(\alpha x_{iD}, \Theta_0)$, and fixes pointwise the star of $\mathcal{V}_n(\alpha, \Theta_0)$, then g fixes pointwise the star of $\mathcal{V}_n(\alpha x_{iD}, \Theta_0)$.

Remark 4.8. Claim (J) fails in the case that $n = 3$ because, in that case, K_n is the barycentric subdivision of the regular trivalent tree, and an automorphism of the tree may fix pointwise the star of a valence-three vertex v without fixing pointwise the star of those valence-three vertex distance two from v.

Proof that χ_n is surjective, for $n \geq 4$, assuming Claims (H),(I) and (J). Consider an arbitrary simplicial automorphism $f \in \mathrm{Aut}(\mathrm{K}_n)$. It follows from Claim (H) that f maps nuclear vertices to nuclear vertices; that is, f maps $\mathcal{V}_n(\iota, \Theta_n^0)$ to $\mathcal{V}_n(\alpha, \Theta_n^0)$ for some $\alpha \in \mathrm{Out}^0(W_n)$. It follows that $\chi_n(\alpha^{-1})f$ fixes $\mathcal{V}_n(\iota, \Theta_n^0)$, and fixes setwise the star of $\mathcal{V}_n(\iota, \Theta_n^0)$. By Theorem 1.3, there exists $\sigma \in \Sigma_n$ such that $\chi_n(\alpha^{-1})f\mathcal{V}_n(\iota, \Theta) = \mathcal{V}_n(\iota, \sigma\Theta)$ for all $\Theta \in \mathcal{HT}_n$; hence $\chi_n(\sigma^{-1}\alpha^{-1})f$ fixes pointwise the star of $\mathcal{V}_n(\iota, \Theta_n^0)$.

Now suppose that $\chi_n(\sigma^{-1}\alpha^{-1})f$ fixes pointwise the star of $\mathcal{V}_n(\alpha, \Theta_n^0)$, for some $\alpha \in \mathrm{Out}^0(W_n)$. Let x_{iD} be a partial conjugation. By Claims (H) and (I), amongst the vertices in K_n of maximal valence, $\mathcal{V}_n(\alpha x_{iD}, \Theta_n^0)$ is distinguished by the set of vertices in the star of $\mathcal{V}_n(\alpha, \Theta_n^0)$ to which it is adjacent. It follows that $\chi_n(\sigma^{-1}\alpha^{-1})f$ fixes $\mathcal{V}_n(\alpha x_{iD}, \Theta_n^0)$. Claim (J) then gives that $\chi_n(\sigma^{-1}\alpha^{-1})f$ fixes pointwise the star of $\mathcal{V}_n(x_{iD}\alpha, \Theta_n^0)$.

By induction we have that $\chi_n(\sigma^{-1}\alpha^{-1})f$ fixes pointwise the star of $\mathcal{V}_n(\alpha, \Theta_n^0)$ whenever α can be written as a product of partial conjugations. Since the partial conjugations generate $\mathrm{Out}^0(W_n)$, $\chi_n(\sigma^{-1}\alpha^{-1})f$ fixes pointwise the star of every nuclear vertex, and hence the entire space K_n. The result follows. $\square$

It remains only to prove Claims (H), (I) and (J). Before addressing Claim (H), it is convenient to define some notation. For each hypertree $\Theta \in \mathcal{HT}_n$, we write $A(\Theta)$ for the set of hypertrees in $\mathcal{HT}_n$, distinct from Θ, which fold to Θ; and $B(\Theta)$ for the set of hypertrees in $\mathcal{HT}_n$, distinct from Θ, which can be obtained from Θ by folding.

Proposition 4.9 (Claim (H))**.** For each integer $n \geq 3$, the nuclear vertices are exactly the vertices of maximal valence in K_n.

Proof of Proposition 4.9. Consider first the case that $n = 3$. Nuclear vertices have valence three, and since each hypertree $\theta \in \mathcal{HT}_n$ carries exactly one partial conjugation, and folds only to the nuclear hypertree, each non-nuclear vertex in K_3 is adjacent to two nuclear vertices, and no non-nuclear vertices. Thus the result holds.

We therefore assume that $n \geq 4$. It is clear that each nuclear vertex has valence $\#A(\Theta_n^0)$.

Consider an arbitrary element $[\alpha, \Theta] \in \mathcal{K}_n$, with $\Theta \in \mathcal{HT}_n^h$ for some $h \in \{1, \ldots, n-2\}$. The definitions immediately give that, for any other element $[\beta, \Lambda] \in \mathcal{K}_n$:

1) $[\alpha, \Theta] < [\beta, \Lambda]$ if and only if $\beta^{-1}\alpha$ is carried by Θ, and $\Theta < \Lambda$ (from which it follows that $[\alpha, \Lambda] = [\beta, \Lambda]$);
2) $[\beta, \Lambda] < [\alpha, \Theta]$ if and only if $\beta^{-1}\alpha$ is carried by Θ, and $\Lambda < \Theta$ (which does not necessarily imply that $[\beta, \Lambda] = [\alpha, \Lambda]$).

It follows that the valence in K_n of $\mathcal{V}_n(\alpha, \Theta)$ is at most

$$\#A(\Theta) + \#(\text{automorphisms carried by } \Theta)\,.\,\#B(\Theta).$$

By Proposition 3.15, $\#A(\Theta) \leq \#A(\Omega_n^{1,2})$. By Lemma 4.5, there are 2^h automorphisms carried by Θ. It follows from Lemma 3.11 that $\#B(\Theta) \leq \#B(\Xi_n^1) = -1 + B_{n-1}$, where B_{n-1} is the number of partitions of $[n-1]$ (B_{n-1} is the $(n-1)$-th Bell number). Thus we have that the valence in K_n of $\mathcal{V}_n(\alpha, \Theta)$ is at most

$$\#A(\Omega_n^{1,2}) + 2^{n-2}(-1 + B_{n-1}),$$

and the proposition is proved if we show that

$$\#A(\Omega_n^{1,2}) + 2^{n-2}(-1 + B_{n-1}) < \#A(\Theta_n^0).$$

We make the following choices, in order:

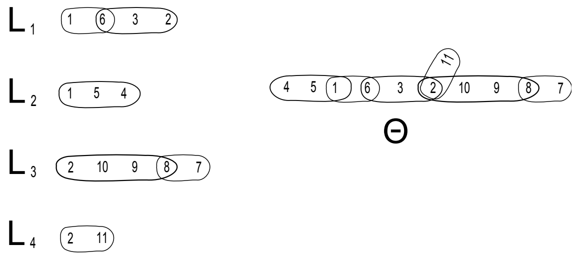

FIGURE 10. The construction of Θ, as in the proof of Proposition 4.9.

- we choose a partition $\mathbb{P} = \{\mathbb{P}_1, \dots \mathbb{P}_p\}$ of $\{2, \dots, n\}$ subject only to the restriction that $p > 1$ (there are $(-1 + B_{n-1})$ such partitions);
- we then choose a function $m\colon \{1, \dots, p\} \to \{1, 2\}$ subject only to the restriction that $m(j) = 1$ if $2 \in \mathbb{P}_j$ (there are 2^{p-1} such functions);
- for each $j \in \{1, \dots, p\}$, we choose a composition $C_j := (c_j^1, \dots, c_j^q)$ of $\#\mathbb{P}_j$ (so $c_j^1, \dots, c_j^q$ are positive integers which sum to $\#\mathbb{P}_j$; there are $2^{-1+\#\mathbb{P}_j}$ such compositions).

In making these choices, we have chosen one combination of data from a possible

$$(-1 + B_{n-1})2^{p-1} \prod_{j=1}^{p} 2^{-1+\#\mathbb{P}_j} = (-1 + B_{n-1})2^{n-2}$$

combinations.

Now for each $j \in \{1, \dots, p\}$, we construct a set Λ_j of hyperedges as follows: if $\mathbb{P}_j = \{s_1, \dots, s_q\}$ with $s_1 > \cdots > s_q$, then we define

$$\Lambda_j := \Big\{ \{m(j), s_1, \dots, s_{c_j^1}\}, \{s_{c_j^1}, \dots, s_{c_j^1 + c_j^2}\}, \dots, \{s_{-c_j^q + \#\mathbb{P}_j}, \dots, s_q\} \Big\}.$$

Finally, we define Θ to be the hypertree on $[n]$ such that $E_\Theta = \bigcup_{j=1}^{p} \Lambda_j$.

An example construction is shown in Figure 10, using the data: $n = 11$; $p = 4$; $\mathbb{P}_1 = \{2, 3, 6\}$, $m(1) = 1$, $C_1 = (1, 2)$; $\mathbb{P}_2 = \{4, 5\}$, $m(2) = 1$, $C_1 = (2)$; $\mathbb{P}_3 = \{7, 8, 9, 10\}$, $m(3) = 2$, $C_3 = (3, 1)$; $\mathbb{P}_4 = \{11\}$, $m(4) = 2$, $C_4 = (1)$.

Let $\mathcal{H}$ denote the set of hypertrees constructed in the manner described above. Given $\Theta \in \mathcal{H}$: the corresponding partition $\mathbb{P}$, and the function m, can be recovered from Θ by considering the connected components of

$\Theta\backslash\{1\}$ and $\Theta\backslash\{2\}$; the compositions C_j can be recovered from considering the subhypertrees of Θ corresponding to each partition set. It follows that distinct choices of input data determine distinct hypertrees. The requirement that $p > 1$ ensures that $\Theta \neq \Theta_n^0$. Thus we have exhibited $(-1+B_{n-1})2^{n-2}$ distinct hypertrees in $\mathcal{H} \subset \mathcal{HT}_n$.

We wish to identify at least $1 + \#A(\Omega_n^{1,2})$ more hypertrees in $\mathcal{HT}_n$. First we consider the elements in $A(\Omega_n^{1,2})$. Recall that these are precisely the hypertrees with a $(1,2)$-tag and at least three hyperdges. Because some elements of $\mathcal{H}$ have $(1,2)$-tags, this falls short of the extra hypertrees required by $1 + \#A(\Omega_n^{1,2}) \cap \mathcal{H}$.

Suppose $\Theta \in A(\Omega_n^{1,2}) \cap \mathcal{H}$. Since $\Theta \in \mathcal{H}$: the valence of j in Θ is at most two for each $j \in \{3, \dots, n\}$. Since $\Theta \in A(\Omega_n^{1,2})$, it has a $(1,2)$ tag. The only way this can happen is if $\{2\}$ is a partition set, and $m(j) = 1$ for each $j \in \{2, \dots, p\}$. It follows that 1 has valence in Θ at least three, and no other vertex in Θ has valence exceeding two. We write Θ' for the hypertree obtained from Θ by swapping the vertices 1 and 3. Then Θ' is not contained in $\mathcal{H}$ (because 3 has valence in Θ' at least three), and it does not have a $(1,2)$-tag (because 1 and 2 are not adjacent in Θ').

Evidently, distinct choices of $\Theta \in \mathcal{H} \cap A(\Omega_n^{1,2})$ give distinct hypertrees Θ'. Hence the set

$$\mathcal{H} \cup A(\Omega_n^{1,2}) \cup \{\Theta' \mid \Theta \in \mathcal{H} \cap A(\Omega_n^{1,2})\}$$

contains exactly

$$(-1+B_{n-1})2^{n-2} + \#A(\Omega_n^{1,2})$$

hypertrees.

It remains only to find one more hypertree in $\mathcal{HT}_n$. The star tree Ξ_n^4 suffices because: it does not have a $(1,2)$-tag, and hence is not contained in $A(\Omega_n^{1,2})$; the valence in Ξ_n^4 of 4 exceeds two, and hence Ξ_n^4 cannot be an element of $\mathcal{H} \cup \{\Theta' \mid \Theta \in \mathcal{H} \cap A(\Omega_n^{1,2})\}$. $\square$

Lemma 4.10 (Claim (I))**.** Let $n \geq 3$, let $x_{iD} \in \mathrm{Out}^0(W_n)$ be a partial conjugation, and let $\beta \in \mathrm{Out}^0(W_n)$ be an automorphism such that, for each $\Theta \in \mathcal{HT}_n$, Θ carries β if and only if Θ carries x_{iD}. Then $\beta = x_{iD}$.

Proof. There exists at least one hypertree $\Theta \in \mathcal{HT}_n$ which carries x_{i_D}. It follows, by Definition 4.2 and hypothesis, $\beta = x_{i_pD_p} \dots x_{i_1D_1}$ for some partial conjugations $x_{i_1D_1}, \dots, x_{i_pD_p}$, each of which is carried by Θ. Thus it suffices to show that if x_{jF} is a partial conjugation and $x_{jF} \neq x_{iD}$, then there exists a hypertree $\Lambda \in \mathcal{HT}_n$ such that Λ carries x_{iD} but Λ does not carry x_{jF}. Equivalently, it suffices to show that if x_{jF} is a partial conjugation and $x_{jF} \neq x_{iD}$, then there exists a hypertree $\Lambda \in \mathcal{HT}_n$ such that D is a union of connected components of $\Lambda \setminus \{i\}$, but F is not a union of connected components of $\Lambda \setminus \{j\}$. We leave the reader to verify this statement. □

Lemma 4.11 (Claim (J))**.** Let $n \geq 4$, let $g \in \mathrm{Aut}(\mathrm{K}_n)$, let $\alpha \in \mathrm{Out}^0(W_n)$, and let x_{iD} be a non-trivial partial conjugation. If g fixes $\mathcal{V}_n(\alpha x_{iD}, \Theta_0)$, and fixes pointwise the star of $\mathcal{V}_n(\alpha, \Theta_0)$, then g fixes pointwise the star of $\mathcal{V}_n(\alpha x_{iD}, \Theta_0)$.

Proof. Suppose g fixes $\mathcal{V}_n(\alpha x_{iD}, \Theta_0)$, and fixes pointwise the star of $\mathcal{V}_n(\alpha, \Theta_0)$.

Recall that a line tree is a hypertree which has exactly two leaves. It is immediate from the definitions that x_{iD} is carried by exactly $(\#D)!(n - \#D - 1)!$ line trees; let X denote this set of line trees. It follows that the star of $\mathcal{V}_n(\alpha, \Theta_0)$ shares at least $(\#D)!(n-\#D-1)!$ vertices with the star of $\mathcal{V}_n(x_{iD}\alpha, \Theta_0)$. Since $n \geq 4$ and $1 \leq \#D \leq n-2$, $(\#D)!(n-\#D-1)! \geq 2$. Since g fixes $\mathcal{V}_n(x_{iD}\alpha, \Theta_0)$, it fixes setwise the star of $\mathcal{V}_n(x_{iD}\alpha, \Theta_0)$. Hence g fixes setwise the set of vertices common to the stars of $\mathcal{V}_n(x_{iD}\alpha, \Theta_0)$ and $\mathcal{V}_n(x_{iD}\alpha, \Theta_0)$, and this set contains at least two vertices corresponding to line trees.

Now each line tree which carries x_{iD} is fixed by exactly one nontrivial element of Σ_n, and no two line trees are fixed by the same nontrivial element of Σ_n. It follows that the pointwise stabilizer in $\mathrm{Aut}(\mathrm{HT}_n)$ of X is the trivial subgroup of Σ_n. By Theorem 1.3, g acts as an element of Σ_n on the star of $\mathcal{V}_n(x_{iD}\alpha, \Theta_0)$. But since g is contained in the pointwise stabiliser of the vertices shared with the star of $\mathcal{V}_n(\alpha, \Theta_0)$, g acts as the identity on the star of $\mathcal{V}_n(\alpha, \Theta_0)$. That is, g fixes pointwise the star of $\mathcal{V}_n(\alpha x_{iD}, \Theta_0)$, as required. □

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Appendix. Table of notation

$[n]$	the set $\{1, \ldots, n\}$
Θ_n^0	the hypertree on $[n]$ with exactly one hyperedge
Ξ_n^j	the hypertree on $[n]$ with exactly $n-1$ hyperedges, each of which contains j
$\Omega_n^{j,k}$	the hypertree on $[n]$ with exactly two hyperedges, $\{j, k\}$ and $[n] \setminus \{j\}$
$\mathcal{HT}_n$	the set of hypertrees on $[n]$
$\mathcal{HT}_n^+$	the set of hypertrees on $[n]$ that have at least two hyperedges
$\mathcal{HT}_n^h$	the set of hypertrees on $[n]$ that have exactly h+1 hyperedges
$\mathcal{S}_n$	the set $\{\Xi_n^j \mid j \in [n]\}$; elements of $\mathcal{S}_n$ are called star trees
$\mathcal{L}_n$	the set of hypertrees on $[n]$ that have exactly two leaves; elements of $\mathcal{L}_n$ are called line trees
$\mathcal{M}_n^h$	the set of hypertrees on $[n]$ which have exactly h+1 hyperedges, a vertex of valence $h+1$, and a hyperedge of degree $n-h$ (note: $\mathcal{M}_n^1 = \{\Omega_n^{j,k} \mid j, k \in [n], j \neq k\}$ and $\mathcal{M}_n^{n-2} = \mathcal{S}_n = \{\Xi_n^j \mid j \in [n]\}$)
$A_n^+(\Theta)$	the set of hypertrees on $[n]$, distinct from Θ, which fold to Θ
$B_n^+(\Theta)$	the set of hypertrees on $[n]$, distinct from Θ, which can be obtained by folding Θ
HT_n	the simplicial realization of $(\mathcal{HT}_n, \leq)$, called the hypertree complex of rank n
HT_n^+	the simplicial realization of $(\mathcal{HT}_n^+, \leq)$

TABLE 1. Notation relating to hypertrees

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