Algebra and Discrete Mathematics
Volume 14 **(2012)**. Number 2. pp. 230 – 235

RESEARCH ARTICLE

Reduction of matrices over Bezout domains of stable range 1 with Dubrovin's condition in which maximal nonprincipal ideals are two-sides

Tetyana Kysil, Bogdan Zabavskiy, Olga Domsha

Communicated by V. V. Kirichenko

Abstract. It is proved that each matrix over Bezout domain of stable range 1 with Dubrovin's condition, in which every maximal nonprincipal ideals are tho-sides ideals, is equivalent to diagonal one with right total division of diagonal elements

Introduction

I.Kaplansky began systematic study of elementary divisors rings [1]. Many articles about commutative elementary divisors rings are already written and many are still under research. But as to non-commutative elementary divisors rings not much is known. It's classical that principal ideals domain is elementary divisors domain [2].

It's necessary to mention the result of P.Cohn [3] who proved, that the right principal Bezout domain is the domain, over which every matrix is equivalent to the diagonal matrix with the condition of the right total division of the diagonal elements. It's also important to consider the work of Dubrovin, who proved, that the semi-local semi-prime Bezout ring is the elementary divisors ring if and only if for any element $a \in R$ there exists the element $b \in R$ R such that $RaR = bR = Rb$ (today

Key words and phrases: Bezout domain, domain of stable range 1, Dubrovin's condition, maximal nonprincipal ideal, right total division.

this condition is called Dubrovin's condition). In [5] it is shown, that the distributive Bezout domain is the elementary divisors ring if and only if it's duo-domain (as a distributive Bezout domain we understand the Bezout domain, where every maximal one-sided ideal is two-sided).

During last year K-theory was actively applied for the research of elementary divisors rings. There particularly important is the use of such invariant as the stable range of rings [8, 7]. Since the semi-local ring is the ring of the stable range 1, then taking into consideration the above-mentioned the next step in the research is Bezout domains of the stable range 1 with Dubrovin's condition. Generally, Bezout domains differ from the principal ideal domains by existence of non-principal two-sided ideals. Therefore study of the influence of the structure of non-principal right ideals on possibility of the diagonal reduction is actual.

1. Preliminaries

For the future as a ring we shall understand the associative ring with $1 \neq 0$.

Definition 1. A ring is called the right (left) Bezout ring if every right (left) finitely generated ideal is principal. If ring is right and left Bezout ring at once, it is called Bezout ring

In this work the Bezout domain of the stable range 1 is researched through the study of influence of the structure of non-principal right ideals on the diagonal reductions of matrices. It is understandable that such research narrows the class of the rings which are studied. Further the Bezout domains that aren't the rings of the principle right ideals will be in consideration.

Definition 2. A right ideal of ring, which is maximal in the set of non-principal right ideals concerning inclusion of right ideals is called maximal non-principal right ideal. Existing of these right ideals is proved in [9].

Definition 3. A ring R is called a ring with stable range 1, if for any $a, b \in R$, such that $aR + bR = R$, exist the element $t \in R$, such that $a + bt$ is invertible in R [2].

Definition 4. An element a is called right total divisor of element $b \in R$, (in denotation $a_r||b$,) if there exists duo-element $c \in R$, that $bR \subset Rc \subset aR$. An element $c \in R$ is called duo-element of ring R if $cR = Rc$[3].

Definition 5. A matrix A over ring R is equivalent to matrix B, (in denotation $A \sim B$), if there exist invertible matrices P and Q corresponding sides such that $PAQ = B$ [5].

Definition 6. Let say that in ring R Dubrovin's condition is true if for any element $a \in R$ there exist duo-element $c \in R$, such that $RaR = cR = Rc$ [8].

2. Main Results

Proposition 1. *Let R is a Bezout domain with Dubrovin's condition. If every maximal non- principal right ideal of R is two-sided and element a doesn't belong to any one maximal non- principal, right ideal, then every 2×2 - matrix A, which element is a, is equivalent to diagonal $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha_r || \beta$*

Proof. According to [9] and limitations imposed on the ring R, we can state that the matrix A looks like:

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

Let's denote as X a set of divisors x of element a, such that $(x, 0)$ is the first row of matrices equivalent to A. Since x is a divisor (right or left) of element a, than any element of X doesn't belong to any maximal non-principal right ideal[1]. Let a_1 be a minimal element relatively right division that's if $a_1 = p_2 a_2 q_2$ for some $a_2 \in X$, then element $q_2 \in U(R)$. We need to prove that such element exists.

Let

$$a = p_1 a_1 q_1 = p_1 p_2 a_2 q_2 q_1 = \dots$$

and

$$aR \subset p_1 a_1 R \subset p_1 p_2 a_2 R \subset \dots \tag{1}$$

Since the element a doesn't belong to any maximal non-principal right ideal, then right ideal

$$\bigcup_{i=1}^{\infty} (\prod_{j=1}^{i} p_j) a_i R$$

is principal. Let $\bigcup\limits_{i=1}^{\infty} (\prod_{j=1}^{i} p_j) a_i R = yR$.

As $y \in (\prod_{j=1}^{i} p_j)a_iR$, then $yR = (\prod_{j=1}^{i} p_j)a_iR$. Thus chain (1) finishes after finitely step.

So, $A \sim \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}$, where a_1 doesn't belong to any maximal non-principal right ideal and $a_1 \in X$ is minimal relatively the right division.

Let $Ra_1 + Rb_1 = Ra_2$. According to [7], since R - Bezout domain, there exists such invertible matrix $P_1 \in GL_2(R)$, that

$$P_1 \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix},$$

where $a_2 = pa_1$ for some $p \in R$. As any maximal non-principal right ideal is two-sided and the element a doesn't belong to any maximal non-principal right ideal, a_2 also.

Analogically,

$$\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} Q_1 = \begin{pmatrix} a_3 & 0 \\ b_3 & c_3 \end{pmatrix}$$

for some invertible 2×2 matrix $Q_1 \in GL_2(R)$ and $a_2R + b_2R = a_3R$, $a_2 = a_3q$, $q \in R$. Then $a_1 = pa_3q$. As a_1 is the minimal element, $q \in U(R)$, that's why $b_2R \in a_2R$. In other words, $b_2 = a_2z$ for some $z \in R$. So,

$$\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix}$$

.

Let t belongs to R. To the first row of the matrix $\begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix}$ we shall add the second, which is multiplied by t. Thinking in the same way we shall get $a_2R + tc_2R = a_1R$, whence $tc_2R \in a_1R$. On the strength of accident of the element t we can consider that $Rc_2 \subset a_2R$. So, $c_2R \subseteq Rc_2R \subseteq a_2R$. Since the Dubrovin's condition is true in ring R, then exists such duo-element $c \in R$, that $Rc_2R = cR = Rc$, what proves the given part of proposition.

Now we shall consider event, when the element a is located on casual location of the matrix A. Let's move it on the location (1,1) by transposition of the rows and columns of the matrix. So, we can consider that the matrix A looks like:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $aR + bR = a'R$. According to the definition of the element a and since a' the left division of a, follows that a' doesn't belong to any maximal non-principal right ideal neither. Since R is the Bezout domain, the matrix A is equivalent to $\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix}$ [7]. Considering what is proved above the matrix $\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix}$ is equivalent to the diagonal matrix with the condition of the right total division of diagonal elements. □

Theorem 1. *Let R be the Bezout domain of the stable range 1 with Dubrovin's condition. If every maximal non-principal right ideal in R is two-sided, then any matrix over R is equivalent to the diagonal matrix with the condition of the right total division of diagonal elements.*

Proof. According to [1, 4] for proof of the theorem it is enough to show that any matrix $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $RaR + RbR + RcR = R$, is equivalent to the diagonal matrix with the condition of the right total division of diagonal elements.

Since R is the Bezout domain of the stable range 1, considering [6] for any $a, b \in R$ exist such elements $x, d \in R$, that $xa + b = d$ and $Ra + Rb = Rd$. Therefore

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} xa + b & c \\ a & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ a & 0 \end{pmatrix},$$

where $a = a_0 d$ for some element $a_0 \in R$.

Analogically, let $cR + dR = zR$ and $cy + d = z$ for some element y from R. Then

$$\begin{pmatrix} d & c \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} d + cy & c \\ a & 0 \end{pmatrix} = \begin{pmatrix} z & c \\ a & 0 \end{pmatrix},$$

where $d = xt$, $c = zc_0$ for some elements $t, c_0 \in R$. Thereby we proved the matrix $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ is equivalent to the matrix $\begin{pmatrix} z & c \\ a & 0 \end{pmatrix}$, where $a = a_0 zt$ and $c = zc_0$. Since $a = a_0 z$, $c = zc_0$ and $RaR + RbR + RcR = RzR$, then $RaR + RbR + RcR = R$, and so $RzR = R$.

The element z cannot belong to any maximal non-principal right ideal, therefore according to proposition 1 the matrix $\begin{pmatrix} z & c \\ a & 0 \end{pmatrix}$, as well as the matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ is equivalent to the diagonal matrix with the condition of the right total division of diagonal elements. □

References

[1] *Kaplansky I.* Elementary divisors and modules // Trans. Amer. Math. Soc. – 1949. – **66**. – P. 464 – 491.

[2] *Jacobson N.* Ring Theory.- M.: Publishers of foreign literature, 1947.

[3] *Cohn P.M.* Right principal Bezout domains // J. London Math. Soc., 35, N2, (1987), 251-162.

[4] *Zabavskiy B.V., Komarnitskiy N.Ya.* Distributive domains jf elementary divisors // Ukr. Math.J.-1990.-42.-N7,-1000-1004.

[5] *Vaserstein L.N.* Bass's first stable range condition // J. of Pure and Appl. Alg.,1984,-34,319-330.

[6] *Zabavskiy B.V.* Reduction of matrices over Bezout rings of stable range at most 2 // Ukr. Math.J.,-2003.-55.-N4.-550-554.

[7] *Zabavskiy B. V.* On noncommutative rings with elementary divisors // Ukr.Math.J. -1990. -42. -N6. -847-850.

[8] *Dubrovin N.I.* On noncommutative rings with elementary divisors // Reports of institutes of higher education. Mathematics,-1986.-N11,-14-20.

[9] *Zabavskiy B. V.* Factorial analogue of distributive Bezout domains // Ukr.Math.J.-2001.-53.-N11,-1564-1567.

[10] *Amitsur S.A.* Remarks of principal ideal rings // Osaka Math.Journ. -1963, -15, -59-69.

Contact information

T. Kysil Khmelnitsky national university, The faculty of Applied Mathematics and Computer Technologies, Applied Mathematics and Social Informatics Department
E-Mail: kysil_tanya@mail.ru

B. Zabavskiy Lviv national university named after I. Franko, The faculty of Mechanics and Mathematics, The chair of Algebra and Logic
E-Mail: b_zabava@franko.lviv.ua

O. Domsha Lviv national university named after I. Franko, The faculty of Mechanics and Mathematics, The chair of Algebra and Logic
E-Mail: olya.domsha@i.ua

Received by the editors: 21.04.2012
and in final form 19.05.2012.