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RESEARCH ARTICLE

On inverse operations in the lattices of submodules

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Dedicated to Prof. V.V. Kirichenko
on the occasion of his seventieth birthday

Abstract. In the lattice $\boldsymbol{L}({}_RM)$ of submodules of an arbitrary left R-module ${}_RM$ four operation were introduced and investigated in the paper [3]. In the present work the approximations of inverse operations for two of these operations (for α-product and ω-coproduct) are defined and studied. Some properties of *left quotient* with respect to α-product and *right quotient* with respect to ω-coproduct are shown, as well as their relations with the lattice operations in $\boldsymbol{L}({}_RM)$ (sum and intersection of submodules). The particular case ${}_RM = {}_RR$ of the lattice $\boldsymbol{L}({}_RR)$ of left ideals of the ring R is specified.

1. Preliminaries

Let R be an associative ring with unity and R-Mod be the category of unitary left R-modules. We denote by $\boldsymbol{L}({}_RM)$ the lattice of submodules of an arbitrary left R-module ${}_RM$, and by $\boldsymbol{L}^{ch}({}_RM)$ the lattice of *characteristic* (fully invariant) submodules of ${}_RM$ (i.e. submodules $N \in \boldsymbol{L}({}_RM)$ such that $f(N) \subseteq N$ for every $f : {}_RM \to {}_RM$).

We remind that a *preradical* r in the category R-Mod is a subfunctor of identity functor of R-Mod, i.e. $r(M) \subseteq M$ and $f\,(r(M)) \subseteq r(M')$ for every

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$f : {}_RM \to {}_RM'$ ([4], [5], [6]). Every pair $N \subseteq M$, where $N \in \boldsymbol{L}({}_RM)$, defines two preradicals α_N^M and ω_N^M by the rules:

$$\alpha_N^M(X) = \sum_{f : M \to X} f(N), \quad \omega_N^M(X) = \bigcap_{f : X \to M} f^{-1}(N),$$

for each $X \in R$-Mod. We mention the following two particular cases: every module ${}_RM$ defines the preradical r^M by $r^M(X) = \sum_{f : M \to X} Im\, f$ (i.e. $r^M = \alpha_M^M$) and the preradical r_M by $r_M(X) = \bigcap_{f : X \to M} Ker\, f$ (i.e. $r_M = \omega_0^M$). We denote by $Gen({}_RM)$ the class of modules generated by ${}_RM$.

Using the preradicals of types α_N^M and ω_N^M, in the works [1], [2] and [3] four operations in $\boldsymbol{L}({}_RM)$ were introduced and studied for an arbitrary module ${}_RM$. We remind two of these operations (α-product and ω-coproduct), which will be used in continuation.

Definition 1.1. Let $K, N \in \boldsymbol{L}({}_RM)$. The **$\alpha$-product** of K and N is defined as the following submodule of ${}_RM$:

$$K \cdot N = \alpha_K^M(N) = \sum_{f : M \to N} f(K).$$

In the next statement we give some properties of this operation ([1], [2], [3]).

Proposition 1.1. 1) *The operation of α-product is monotone in both variables:*

$$K_1 \subseteq K_2 \Rightarrow K_1 \cdot N \subseteq K_2 \cdot N, \text{ for every } N \in \boldsymbol{L}({}_RM);$$

$$N_1 \subseteq N_2 \Rightarrow K \cdot N_1 \subseteq K \cdot N_2, \text{ for every } K \in \boldsymbol{L}({}_RM).$$

2) $K \cdot N = 0 \Leftrightarrow K \subseteq \bigcap_{f : M \to N} \mathrm{Ker}\, f \;(= r_N(M))$; in particular, $0 \cdot N = 0$ and $K \cdot 0 = 0$.

3) $M \cdot N = \sum_{f : M \to N} f(M) \;(= r^M(N))$; $M \cdot N = N \Leftrightarrow N \in Gen({}_RM)$.

4) $(K \cdot N) \cdot L \subseteq K \cdot (N \cdot L)$, for every $K, N, L \in \boldsymbol{L}({}_RM)$.

5) If ${}_RM$ is a projective module, then the operation of α-product is associative, i.e. $(K \cdot N) \cdot L = K \cdot (N \cdot L)$, for every $K, N, L \in \boldsymbol{L}({}_RM)$.

6) $\Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big) \cdot N = \sum_{\alpha \in \mathfrak{A}} (K_\alpha \cdot N)$, for every $K_\alpha, N \in \boldsymbol{L}({}_RM)$.

7) If ${}_RM = {}_RR$, then the α-product of two left ideals $K, N \in \boldsymbol{L}({}_RM)$ coincides with their ordinary product in the ring R: $K \cdot N = KN$. $\square$

Now we remind the definition of ω-coproduct in $\boldsymbol{L}({}_RM)$ and some properties of this operation ([1], [2], [3]).

Definition 1.2. Let $N, K \in \boldsymbol{L}({}_RM)$. The ***ω*-coproduct** of N and K is defined as the following submodule of ${}_RM$:

$$N \odot K = \pi_N^{-1}(\omega_K^M(M/N)) = \{m \in M \mid m + N \in \bigcap_{f\,:\,M/N \to M} f^{-1}(K)\} =$$
$$= \{m \in M \mid f(m+N) \in K \quad \forall\, f : M/N \to M\},$$

where $\pi_N : M \to M/N$ is the natural morphism. Therefore:

$$(N \odot K)/N = \omega_K^M(M/N) = \bigcap_{f\,:\,M/N \to M} f^{-1}(K).$$

In other form:

$$N \odot K = \{m \in M \mid g(m) \in K \quad \forall\, g : M \to M,\ g(N) = 0\}.$$

In the next statement we enumerate some properties of ω-coproduct which are necessary for the further investigations.

Proposition 1.2. 1) $N \odot K \supseteq N$, *for every* $N, K \in \boldsymbol{L}({}_RM)$; *if* $K \in \boldsymbol{L}^{ch}({}_RM)$, *then* $N \odot K \supseteq K$.

2) $M \odot K = M$, *for every* $K \in \boldsymbol{L}({}_RM)$; $N \odot M = M$, *for every* $N \in \boldsymbol{L}({}_RM)$.

3) $0 \odot K$ *is the greatest characteristic submodule of* M *which is contained in* K; *therefore, if* $K \in \boldsymbol{L}^{ch}({}_RM)$, *then* $0 \odot K = K$.

4) $N \odot 0 = \pi_N^{-1}(\bigcap_{f\,:\,M/N \to N} \operatorname{Ker} f) = \pi_N^{-1}(r_M(M/N))$, *for every* $N \in \boldsymbol{L}({}_RM)$.

5) *The operation of* ω*-coproduct is monotone in both variables.*

6) $(N \odot K) \odot L \subseteq N \odot (K \odot L)$, *for every* $K, L, N \in \boldsymbol{L}({}_RM)$.

7) *If the module* ${}_RM$ *is injective and artinian, then the operation of* ω*-coproduct in* $\boldsymbol{L}({}_RM)$ *is associative:*
$(N \odot K) \odot L = N \odot (K \odot L)$, *for every* $K, L, N \in \boldsymbol{L}({}_RM)$.

8) $N \odot (\bigcap_{\alpha \in \mathfrak{A}} K_\alpha) = \bigcap_{\alpha \in \mathfrak{A}} (N \odot K_\alpha)$, *for every* $N, K_\alpha \in \boldsymbol{L}({}_RM)$.

9) *If* ${}_RM = {}_RR$, *then* $N \odot K = (K \odot (0 \odot N)_r)_l$, *for every left ideals* $K, N \in \boldsymbol{L}({}_RR)$. □

2. Left quotient with respect to *α*-product

Now we introduce a new operation in the lattice $\boldsymbol{L}({}_RM)$, which in some sense can be considered as an (approximation of) inverse operation for the α-product (just as the left quotient $(N : K)_l = \{a \in R \mid a\,K \subseteq N\}$ of left ideals of R can be considered as the inverse operation for the product of left ideals in R).

Definition 2.1. Let $K, N \in \boldsymbol{L}({}_RM)$. The **left quotient** of N by K with respect to α-product is defined as the greatest among submodules $L_\alpha \in \boldsymbol{L}({}_RM)$ with the property $L_\alpha \cdot K \subseteq N$. We denote this submodule by $N \mathbin{/_{\!\bullet}} K$ and observe that it is defined by the conditions:

a) $(N \mathbin{/_{\!\bullet}} K) \cdot K \subseteq N$;

b) if $L \cdot K \subseteq N$ for some $L \in \boldsymbol{L}({}_RM)$, then $L \subseteq N \mathbin{/_{\!\bullet}} K$.

The next statement is useful for applications.

Proposition 2.1. *If* $K, N, L \in \boldsymbol{L}({}_RM)$, *then:*

$$L \cdot K \subseteq N \;\Leftrightarrow\; L \subseteq N \mathbin{/_{\!\bullet}} K.$$

Proof. $(\Rightarrow)$ The condition $b)$ in Definition 2.1.

$(\Leftarrow)$ If $L \subseteq N \mathbin{/_{\!\bullet}} K$, then by the monotony of α-product and condition $a)$, we have: $L \cdot K \subseteq (N \mathbin{/_{\!\bullet}} K) \cdot K \subseteq N$. □

From the properties of α-product *the existence* of the left quotient for every pair of submodules follows.

Proposition 2.2. *For every submodules* $K, N \in \boldsymbol{L}({}_RM)$ *there exists the left quotient* $N \mathbin{/_{\!\bullet}} K$ *with respect to* α*-product and it can be represented in the form:*

$$N \mathbin{/_{\!\bullet}} K = \sum \{L_\alpha \in \boldsymbol{L}({}_RM) \mid L_\alpha \cdot K \subseteq N\}.$$

Proof. The indicated family of submodules L_α with $L_\alpha \cdot K \subseteq N$ is not empty, since it contains the submodule 0, because $0 \cdot K = 0 \subseteq N$. By the distributivity of α-product with respect to the sum of submodules (Proposition 1.1, 6)) we obtain: $\left(\sum\limits_{\alpha \in \mathfrak{A}} L_\alpha\right) \cdot K = \sum\limits_{\alpha \in \mathfrak{A}} (L_\alpha \cdot K) \subseteq N$. Therefore the submodule $\sum\limits_{\alpha \in \mathfrak{A}} L_\alpha$ satisfied the condition $a)$, and by construction it is clear that it is the greatest submodule with this property. □

In continuation we indicate other two forms of the left quotient $N \mathbin{/_{\!\bullet}} K$ with respect to α-product.

Proposition 2.3. *For every submodules* $K, N \in \boldsymbol{L}({}_RM)$ *we have:*

$$N \mathbin{/_{\!\bullet}} K = \{l \in M \mid f(l) \in N \;\; \forall f: \; M \to K\}.$$

Proof. Denote by L the right side of this relation. Then $L \in \boldsymbol{L}({}_RM)$ and since $f(L) \subseteq N$ for every $f : M \to K$, we obtain $L \cdot K = \sum\limits_{f\,:\,M \to K} f(L) \subseteq N$. Moreover, if $L_1 \cdot K \subseteq N$ for some $L_1 \in \boldsymbol{L}({}_RM)$, then $\sum\limits_{f\,:\,M \to K} f(L_1) \subseteq N$, so $f(L_1) \subseteq N$ for every $f : M \to K$. From definition of L we have $L_1 \subseteq L$, therefore L is the greatest submodule of M with $L \cdot K \subseteq N$, i.e. $L = N \,/_{\bullet}\, K$. □

Corollary 2.4. *$N \,/_{\bullet}\, K = \bigcap_{f:M\to K} f^{-1}(N\cap K)$ for every $K, N \in \boldsymbol{L}({}_RM)$.*

Proof. ($\supseteq$) If $l \in \bigcap_{f:M\to K} f^{-1}(N\cap K)$, then $f(l) \in N$ for every $f: M \to K$, so by Proposition 2.3 $l \in N \,/_{\bullet}\, K$.

($\subseteq$) If $l \in N \,/_{\bullet}\, K$, then $f(l) \in N \cap K$ for every $f: M \to K$ (Proposition 2.3), therefore $l \in f^{-1}(N\cap K)$ for every $f: M \to K$, i.e. $l \in \bigcap_{f:M\to K} f^{-1}(N\cap K)$. □

Now we will show the value of left quotient $N \,/_{\bullet}\, K$ in some particular cases.

Proposition 2.5. 1) *If $K \subseteq N$, then $N \,/_{\bullet}\, K = M$. If $K \in Gen({}_RM)$, then the inverse implication is true: $N \,/_{\bullet}\, K = M \Rightarrow K \subseteq N$. In particular, $N \,/_{\bullet}\, 0 = M$ for every $N \in \boldsymbol{L}({}_RM)$ and $M \,/_{\bullet}\, K = M$ for every $K \in \boldsymbol{L}({}_RM)$.*

2) *If $N = 0$, then $0 \,/_{\bullet}\, K = \bigcap_{f:M\to K} Ker\, f = r_K(M)$ for every $K \in \boldsymbol{L}({}_RM)$.*

3) *If $K = M$, then for every $N \in \boldsymbol{L}({}_RM)$ the left quotient $N \,/_{\bullet}\, M$ is the greatest characteristic submodule of M which is contained in N.*

Proof. 1) If $K \subseteq N$, then by Corollary 2.4

$$N \,/_{\bullet}\, K = \bigcap_{f:M\to K} f^{-1}(N\cap K) = \bigcap_{f:M\to K} f^{-1}(K) = M.$$

If $K \in Gen\,({}_RM)$, then every element $k \in K$ is of the form $k = \sum_{i=1}^{n} f_i\,(m_i)$, where $f_i : M \to K$ and $m_i \in M$. Therefore, if $N \,/_{\bullet}\, K = M$ then $f_i(m_i) \in N$, for every $i = 1, \ldots, n$, so $k \in N$, i.e. $K \subseteq N$.

2) It follows from definitions:

$$0 \,/_{\bullet}\, K = \bigcap_{f:M\to K} f^{-1}(K\cap\, 0) = \bigcap_{f:M\to K} f^{-1}(0) = \bigcap_{f:M\to K} Ker\, f = r_K(M).$$

3) If $K = M$, then by Corollary 2.4

$$L = N \,/_{\bullet}\, M = \bigcap_{f:M\to M} f^{-1}(N) \subseteq N,$$

since for $f = 1_M$ we have $f^{-1}(N) = N$.

Moreover, the submodule $L = N \mathbin{/\!.} M$ is characteristic in ${}_R M$. Indeed, for every $g : M \to M$ and $l \in L$ we have $f\big(g(l)\big) = (f\,g)(l) \in N$ for every $f : M \to M$, so $g(l) \in L$. Therefore $g(L) \subseteq L$, i.e. $L \in \boldsymbol{L}^{ch}({}_R M)$.

If $L_1 \subseteq N$ and $L_1 \in \boldsymbol{L}^{ch}({}_R M)$, then for every $f : M \to M$ and $l_1 \in L_1$ we have $f(l_1) \in L_1 \subseteq N$ and by definition of $L = N \mathbin{/\!.} M$ it follows $l_1 \in L$, i.e. $L_1 \subseteq L$. Thus L is the greatest characteristic submodule in ${}_R M$ which is contained in N. □

The next two statements show the connection between the left quotient $N \mathbin{/\!.} K$ and the partial order $(\subseteq)$ in $\boldsymbol{L}({}_R M)$.

Proposition 2.6. (Monotony in the numerator). *If* $N_1 \subseteq N_2$, *then* $N_1 \mathbin{/\!.} K \subseteq N_2 \mathbin{/\!.} K$ *for every* $K \in \boldsymbol{L}({}_R M)$.

Proof. If $N_1 \subseteq N_2$, then $(N_1 \mathbin{/\!.} K) \cdot K \subseteq N_1 \subseteq N_2$ and by the definition of left quotient it follows that $N_1 \mathbin{/\!.} K \subseteq N_2 \mathbin{/\!.} K$. □

Proposition 2.7. (Antimonotony in the denominator). *If* $K_1 \subseteq K_2$, *then* $N \mathbin{/\!.} K_2 \subseteq N \mathbin{/\!.} K_1$ *for every* $N \in \boldsymbol{L}({}_R M)$.

Proof. From $K_1 \subseteq K_2$ and the monotony of α-product it follows: $(N \mathbin{/\!.} K_2) \cdot K_1 \subseteq (N \mathbin{/\!.} K_2) \cdot K_2 \subseteq N$, therefore $N \mathbin{/\!.} K_2 \subseteq N \mathbin{/\!.} K_1$. □

Proposition 2.8. $(L \cdot N) \mathbin{/\!.} N \supseteq L$ *for every submodules* $N, L \in \boldsymbol{L}({}_R M)$.

Proof. By definition $(L \cdot N) \mathbin{/\!.} N$ is the greatest among submodules L_α with $L_\alpha \cdot N \subseteq L \cdot N$, and since L is one of such submodules, we have $L \subseteq (L \cdot N) \mathbin{/\!.} N$. □

Some properties of the left quotient $N \mathbin{/\!.} K$ with respect to α-product can be proved by assumption that the operation of α-product in $\boldsymbol{L}({}_R M)$ is *associative* (for example, it is sufficient to suppose that the module ${}_R M$ is projective, see Proposition 1.1, 5)).

Proposition 2.9. *Let* ${}_R M$ *be a module with the property that in* $\boldsymbol{L}({}_R M)$ *the operation of* α*-product is associative. Then for every submodules* $K, N, L \in \boldsymbol{L}({}_R M)$ *the following relations are true:*

1) $(N \mathbin{/\!.} K) \mathbin{/\!.} L \;=\; N \mathbin{/\!.} (L \cdot K)$;

2) $(N \mathbin{/\!.} K) \mathbin{/\!.} (L \mathbin{/\!.} K) \;\supseteq\; N \mathbin{/\!.} L$;

3) $(N \cdot K) \mathbin{/\!.} (L \cdot K) \;\supseteq\; N \mathbin{/\!.} L$;

4) $N \cdot (K \mathbin{/\!.} L) \;\subseteq\; (N \cdot K) \mathbin{/\!.} L$.

Proof. 1) ($\subseteq$) From the definition of left quotient it follows:

$$N \supseteq (N /_{\bullet} K) \cdot K, \quad N /_{\bullet} K \supseteq [(N /_{\bullet} K) /_{\bullet} L] \cdot L.$$

Multiplying on the right the last relation by K and using the monotony and associativity of α-product, we obtain:

$$N \supseteq (N /_{\bullet} K) \cdot K \supseteq \big([(N /_{\bullet} K) /_{\bullet} L] \cdot L\big) \cdot K =$$

$$= [(N /_{\bullet} K) /_{\bullet} L] \cdot (L \cdot K).$$

By definition of left quotient (or by Proposition 2.1) we have: $(N /_{\bullet} K) /_{\bullet} L \subseteq N /_{\bullet} (L \cdot K)$.

($\supseteq$) By definition of left quotient and associativity of α-product we obtain:

$$N \supseteq [N /_{\bullet} (L \cdot K)] \cdot (L \cdot K) = \big([N /_{\bullet} (L \cdot K)] \cdot L\big) \cdot K,$$

therefore $N /_{\bullet} K \supseteq [N /_{\bullet} (L \cdot K)] \cdot L$, which means that $(N /_{\bullet} K) /_{\bullet} L \supseteq N /_{\bullet} (L \cdot K)$.

2) This statement (as well as the property 3)) follows from 1), but we prefer the direct proof.

By definition $L \supseteq (L /_{\bullet} K) \cdot K$. Applying the monotony and associativity of α-product we have:

$$N \supseteq (N /_{\bullet} L) \cdot L \supseteq (N /_{\bullet} L) \cdot [(L /_{\bullet} K) \cdot K] = [(N /_{\bullet} L) \cdot (L /_{\bullet} K)] \cdot K.$$

Therefore $(N /_{\bullet} L) \cdot (L /_{\bullet} K) \subseteq N /_{\bullet} K$, thus

$$N /_{\bullet} L \subseteq (N /_{\bullet} K) /_{\bullet} (L /_{\bullet} K).$$

3) From $(N /_{\bullet} L) \cdot L \subseteq N$, associativity and monotony of α-product it follows:

$$(N /_{\bullet} L) \cdot (L \cdot K) = [(N /_{\bullet} L) \cdot L] \cdot K \subseteq N \cdot K,$$

therefore $N /_{\bullet} L \subseteq (N \cdot K) /_{\bullet} (L \cdot K)$.

4) The similar reasons as above imply $(K /_{\bullet} L) \cdot L \subseteq K$ and $[N \cdot (K /_{\bullet} L)] \cdot L = N \cdot [(K /_{\bullet} L) \cdot L] \subseteq N \cdot K$, therefore $N \cdot (K /_{\bullet} L) \subseteq (N \cdot K) /_{\bullet} L$. □

Now we will discuss the question of the relations between the left quotient $N /_{\bullet} K$ in $\boldsymbol{L}({}_R M)$ and the lattice operations of $\boldsymbol{L}({}_R M)$ (sum and intersection of submodules).

Proposition 2.10. $(N_1 \cap N_2) /_{\bullet} K = (N_1 /_{\bullet} K) \cap (N_2 /_{\bullet} K)$ *for every submodules* $N_1, N_2, K \in \boldsymbol{L}({}_R M)$.

Proof. ($\subseteq$) It follows from the monotony of left quotient in the numerator (Proposition 2.6).

($\supseteq$) We denote the right side of relation by L. Then $L \subseteq N_1 /_{\bullet} K$ and $L \subseteq N_2 /_{\bullet} K$, therefore $L \cdot K \subseteq N_1$ and $L \cdot K \subseteq N_2$, so $L \cdot K \subseteq N_1 \cap N_2$ and $L \subseteq (N_1 \cap N_2) /_{\bullet} K$. □

Corollary 2.11. $N /_{\bullet} K = (N \cap K) /_{\bullet} K$ *for every* $N, K \in \boldsymbol{L}({}_R M)$.

Proof. Since $K /_{\bullet} K = M$ (Proposition 2.5, 1)), from Proposition 2.10 it follows:

$$(N \cap K) /_{\bullet} K = (N /_{\bullet} K) \cap (K /_{\bullet} K) = (N /_{\bullet} K) \cap M = N /_{\bullet} K. \qquad \square$$

Remark. The relation of Proposition 2.10 can be obviously generalized for every family of submodules $\{N_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \boldsymbol{L}({}_R M)$:

$$\Big(\bigcap_{\alpha \in \mathfrak{A}} N_\alpha\Big) /_{\bullet} K = \bigcap_{\alpha \in \mathfrak{A}} (N_\alpha /_{\bullet} K).$$

Some more statements on this subject follow from the monotony and antimonotony of Propositions 2.6 and 2.7.

Proposition 2.12. 1) $(N_1 + N_2) /_{\bullet} K \supseteq (N_1 /_{\bullet} K) + (N_2 /_{\bullet} K)$;

2) $N /_{\bullet} (K_1 + K_2) \subseteq (N /_{\bullet} K_1) \cap (N /_{\bullet} K_2)$;

3) $N /_{\bullet} (K_1 \cap K_2) \supseteq (N /_{\bullet} K_1) + (N /_{\bullet} K_2)$. □

The next two statements show when the cancellation properties for the left quotient hold, supplementing Proposition 2.8.

Proposition 2.13. *For every submodules* $N, K \in \boldsymbol{L}({}_R M)$ *the following conditions are equivalent:*

1) $(N \cdot K) /_{\bullet} K = N$;

2) $N = L /_{\bullet} K$ *for some submodule* $L \in \boldsymbol{L}({}_R M)$.

Proof. 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). If $N = L /_{\bullet} K$, then using the inclusion $(L /_{\bullet} K) \subseteq L$ and the monotony of left quotient in the numerator, we obtain:

$$(N \cdot K) /_{\bullet} K = [(L /_{\bullet} K) \cdot K] /_{\bullet} K \subseteq L /_{\bullet} K = N.$$

By Proposition 2.8 $(N \cdot K) /_{\bullet} K \supseteq N$, therefore $(N \cdot K) /_{\bullet} K = N$. □

Proposition 2.14. *For every submodules $N, K \in \boldsymbol{L}({}_RM)$ the following conditions are equivalent:*

1) $(N /_{\bullet} K) \cdot K = N$;

2) $N = L \cdot K$ *for some submodule* $L \in \boldsymbol{L}({}_RM)$.

Proof. 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). Let $N = L \cdot K$. By definition $(N /_{\bullet} K) \cdot K \subseteq N$ and by Proposition 2.8 $(L \cdot K) /_{\bullet} K \supseteq L$. Now the monotony implies:

$$(N /_{\bullet} K) \cdot K = [(L \cdot K) /_{\bullet} K] \cdot K \supseteq L \cdot K = N,$$

therefore $(N /_{\bullet} K) \cdot K = N$. □

Finishing this section we consider the particular case when ${}_RM = {}_RR$.

Proposition 2.15. *In the lattice* $\boldsymbol{L}({}_RR)$ *of left ideals of the ring* R *the left quotient* $N /_{\bullet} K$ *of left ideals* $N, K \in \boldsymbol{L}({}_RR)$ *coincides with their ordinary left quotient in* R*:*

$$N /_{\bullet} K = (N : K)_l = \{a \in R \mid a\, K \subseteq N\}.$$

Proof. In the lattice $\boldsymbol{L}({}_RR)$ the α-product coincides with the ordinary product of left ideals in R (Proposition 1.1, 7)): $L \cdot K = L\, K$. So we have $(N : K)_l\, K \subseteq N$ and it is obvious that $(N : K)_l$ is the greatest left ideal of R with this property. □

Since the α-product ($\equiv$ product) of left ideals in $\boldsymbol{L}({}_RR)$ is associative (${}_RR$ is projective), all mentioned above properties of left quotients hold in the lattice $\boldsymbol{L}({}_RR)$.

3. Right quotient with respect to ω-coproduct

In this section we introduce and investigate the inverse operation for the ω-coproduct (see Section 1) in the lattice of submodules $\boldsymbol{L}({}_RM)$ of an arbitrary module ${}_RM \in R$-Mod.

Definition 3.1. Let $K, N \in \boldsymbol{L}({}_RM)$. The **right quotient** of K by N with respect to ω-coproduct is defined as the least submodule $L \in \boldsymbol{L}({}_RM)$ with the property $N \odot L \supseteq K$. We denote this submodule by $N \,{}_{\odot}\!\backslash\, K$. It is determined by the conditions:

a) $N \odot (N \,{}_{\odot}\!\backslash\, K) \supseteq K$;

b) if $N \odot L \supseteq K$ for some $L \in \boldsymbol{L}({}_RM)$, then $L \supseteq N \,{}_{\odot}\!\backslash\, K$.

The right quotient $N \underset{\odot}{\backslash} K$ is described by the following statement.

Proposition 3.1. *If* $K, N, L \in \boldsymbol{L}({}_R M)$, *then:*

$$K \subseteq N \odot L \quad \Leftrightarrow \quad N \underset{\odot}{\backslash} K \subseteq L.$$

Proof. ($\Rightarrow$) The condition $b)$ of Definition 3.1.
($\Leftarrow$) If $N \underset{\odot}{\backslash} K \subseteq L$, then from the condition $a)$ and the monotony of the operation $\odot$ it follows:

$$K \subseteq N \odot (N \underset{\odot}{\backslash} K) \subseteq N \odot L. \qquad \square$$

From the properties of ω-coproduct (Proposition 1.2) the *existence* of the right quotient $N \underset{\odot}{\backslash} K$ for every pair of submodules of ${}_R M$ follows.

Proposition 3.2. *For every submodules* $K, N \in \boldsymbol{L}({}_R M)$ *there exists the right quotient* $N \underset{\odot}{\backslash} K$ *with respect to* ω*-coproduct, and it can be presented in the form:*

$$N \underset{\odot}{\backslash} K = \cap \{L_\alpha \in \boldsymbol{L}({}_R M) \mid N \odot L_\alpha \supseteq K\}.$$

Proof. Since $N \odot M = M \supseteq K$, the indicated family of submodules is not empty. By Proposition 1.2, 8) we have:

$$N \odot \big(\bigcap_{\alpha \in \mathfrak{A}} L_\alpha \big) = \bigcap_{\alpha \in \mathfrak{A}} (N \odot L_\alpha) \supseteq K,$$

therefore $\bigcap_{\alpha \in \mathfrak{A}} L_\alpha$ has the property $a)$, while $b)$ follows from construction.

$\square$

Remark. For every submodules $N, K, L \in \boldsymbol{L}({}_R M)$ from the definition of $N \odot L$ it follows that:

$$N \odot L \supseteq K \Leftrightarrow f(k + N) \in L \ \ \forall\, k \in K, \ \ \forall\, f : M/N \to N \Leftrightarrow$$
$$\Leftrightarrow f\big((K + N)/N\big) \subseteq N \ \ \forall\, f : M/N \to N.$$

Now we can indicate another form of representation of the right quotient $N \underset{\odot}{\backslash} K$.

Proposition 3.3. *If* $N, K \in \boldsymbol{L}({}_R M)$ *then:*

$$N \underset{\odot}{\backslash} K = \sum_{f : M/N \to N} f\big((K + N)/N\big).$$

Proof. We denote the right side of this relation by L. Since $f\big((K + N)/N\big) \subseteq L$ for every $f : M/N \to N$, from the above remark we have $N \odot L \supseteq K$.

If $N \odot L' \supseteq K$ for some $L' \in \boldsymbol{L}({}_R M)$, then $f\big((K+N)/N\big) \subseteq L'$ for every $f : M/N \to N$ and so $L \subseteq L'$. Therefore L is the least submodule of ${}_R M$ with $N \odot L \supseteq K$, i.e. $L = N \underset{\odot}{\backslash} K$. $\square$

Proposition 3.4. *If* $K \in \boldsymbol{L}^{ch}({}_RM)$, *then* $N \underset{\odot}{\setminus} K \subseteq K$ *for every* $N \in \boldsymbol{L}({}_RM)$.

Proof. From $K \in \boldsymbol{L}^{ch}({}_RM)$ it follows that $K \subseteq N \odot K$ (Proposition 1.2, 1)), therefore by Proposition 3.1 we have $N \underset{\odot}{\setminus} K \subseteq K$. □

Now we indicate the behaviour of the right quotient with respect to the order relation $(\subseteq)$ of $\boldsymbol{L}({}_RM)$.

Proposition 3.5. (Monotony in the numerator). *If* $K_1 \subseteq K_2$, *then* $N \underset{\odot}{\setminus} K_1 \subseteq N \underset{\odot}{\setminus} K_2$ *for every* $N \in \boldsymbol{L}({}_RM)$.

Proof. By definition $N \odot (N \underset{\odot}{\setminus} K_2) \supseteq K_2 \supseteq K_1$, therefore Proposition 3.1 implies: $N \underset{\odot}{\setminus} K_2 \supseteq N \underset{\odot}{\setminus} K_1$. □

Proposition 3.6. (Antimonotony in the denominator). *If* $N_1 \subseteq N_2$, *then* $N_2 \underset{\odot}{\setminus} K \subseteq N_1 \underset{\odot}{\setminus} K$ *for every* $K \in \boldsymbol{L}({}_RM)$.

Proof. By definition of right quotient, using the inclusion $N_1 \subseteq N_2$ and the monotony of ω-coproduct, we obtain:

$$K \subseteq N_1 \odot (N_1 \underset{\odot}{\setminus} K) \subseteq N_2 \odot (N_1 \underset{\odot}{\setminus} K),$$

therefore by Proposition 3.1 $N_2 \underset{\odot}{\setminus} K \subseteq N_1 \underset{\odot}{\setminus} K$. □

Proposition 3.7. *For every submodules* $N, L \in \boldsymbol{L}({}_RM)$ *we have the relation:*

$$N \underset{\odot}{\setminus} (N \odot L) \subseteq L.$$

Proof. If we denote $K = N \odot L$, then by Proposition 3.1 from the inclusion $K \subseteq N \odot L$ it follows that $N \underset{\odot}{\setminus} K \subseteq L$. □

The next statement show the value of the right quotient $N \underset{\odot}{\setminus} K$ in some particular cases.

Proposition 3.8. 1) *If* $K \subseteq N$, *then* $N \underset{\odot}{\setminus} K = 0$. *Therefore:*

a) if $N = M$, *then* $M \underset{\odot}{\setminus} K = 0$ *for every* $K \in \boldsymbol{L}({}_RM)$;

b) if $K = 0$, *then* $N \underset{\odot}{\setminus} 0 = 0$ *for every* $N \in \boldsymbol{L}({}_RM)$;

c) if $N = K$, *then* $N \underset{\odot}{\setminus} N = 0$.

2) *If* $N = 0$, *then* $0 \underset{\odot}{\setminus} K$ *is the least characteristic submodule of* M *which contains* K; *so if* $K \in \boldsymbol{L}^{ch}({}_RM)$, *then* $0 \underset{\odot}{\setminus} K = K$.

3) *If* $K = M$, *then* $N \underset{\odot}{\setminus} M = \sum\limits_{f\,:\,M/N \to M} Im\, f \;\; (= r^{M/N}(M))$ *for every* $N \in \boldsymbol{L}({}_RM)$.

Proof. 1) Let $K \subseteq N$. Since $N \,{}_{\odot}\backslash\, K = \cap\{L_\alpha \in \boldsymbol{L}({}_RM) \mid N \odot L_\alpha \supseteq K\}$, we have $N \odot L_\alpha \supseteq N \supseteq K$ for every $L_\alpha \in \boldsymbol{L}({}_RM)$. Therefore $\cap L_\alpha = 0$, i.e. $N \,{}_{\odot}\backslash\, K = 0$.

2) If $N = 0$, then from Proposition 3.3 we obtain:

$$L = 0 \,{}_{\odot}\backslash\, K = \sum_{f\,:\,M \to M} f(K) = \alpha_K^M(M) \supseteq K.$$

Therefore $L = 0 \,{}_{\odot}\backslash\, K$ is a characteristic submodule of M containing K. If $K' \in \boldsymbol{L}^{ch}({}_RM)$ and $K \subseteq K'$, then $f(K') \subseteq K'$ for every $f : M \to M$ and so $f(K) \subseteq f(K') \subseteq K'$. Therefore $L = \sum_{f\,:\,M \to M} f(K) \subseteq K'$ and $L = 0 \,{}_{\odot}\backslash\, K$ is the least characteristic submodule of M containing K.

3) If $K = M$, then for every $N \in \boldsymbol{L}({}_RM)$ by definition of right quotient $N \,{}_{\odot}\backslash\, M = \cap\{L_\alpha \in \boldsymbol{L}({}_RM) \mid N \odot L_\alpha = M\}$. Now by definition of ω-coproduct we obtain:

$$N \odot L_\alpha = M \Leftrightarrow \omega_{L_\alpha}^M(M/N) = M/N \Leftrightarrow Im\, f \subseteq L_\alpha \;\; \forall\, f : M/N \to M \Leftrightarrow$$
$$\Leftrightarrow \sum_{f\,:\,M/N \to M} Im\, f \;\subseteq\; L_\alpha.$$

Therefore

$$N \,{}_{\odot}\backslash\, M = \cap\{L_\alpha \in \boldsymbol{L}({}_RM) \mid \sum_{f\,:\,M/N \to M} Im\, f \subseteq L_\alpha\} = \sum_{f\,:\,M/N \to M} Im\, f. \quad \square$$

Now we formulate some properties of the right quotient $N \,{}_{\odot}\backslash\, K$ which hold in the case when the operation of ω-coproduct in $\boldsymbol{L}({}_RM)$ is *associative* (Proposition 1.2, 7)).

Proposition 3.9. *Let ${}_RM$ be a module with the property that in the lattice $\boldsymbol{L}({}_RM)$ the operation of ω-coproduct is associative. Then for every submodules $K, N, L \in \boldsymbol{L}({}_RM)$ the following relations hold:*

1) $L \,{}_{\odot}\backslash\, (N \,{}_{\odot}\backslash\, K) = (N \odot L) \,{}_{\odot}\backslash\, K;$

2) $(L \,{}_{\odot}\backslash\, N) \,{}_{\odot}\backslash\, (L \,{}_{\odot}\backslash\, K) \subseteq N \,{}_{\odot}\backslash\, K;$

3) $(L \odot N) \,{}_{\odot}\backslash\, (L \odot K) \subseteq N \,{}_{\odot}\backslash\, K;$

4) $L \,{}_{\odot}\backslash\, (N \odot K) \subseteq (L \,{}_{\odot}\backslash\, N) \odot K.$

Proof. 1) $(\supseteq)$ By definition, $K \subseteq N \odot (N \,{}_{\odot}\backslash\, K)$ and $N \,{}_{\odot}\backslash\, K \subseteq L \odot [L \,{}_{\odot}\backslash\, (N \,{}_{\odot}\backslash\, K)]$. By the monotony and the associativity of ω-coproduct we obtain:

$$K \subseteq N \odot (N \,{}_{\odot}\backslash\, K) \subseteq N \odot [L \odot (L \,{}_{\odot}\backslash\, (N \,{}_{\odot}\backslash\, K))] =$$
$$= (N \odot L) \odot [L \,{}_{\odot}\backslash\, (N \,{}_{\odot}\backslash\, K)].$$

From Proposition 3.1 it follows that $(N \odot L) {}_{\odot}\backslash\, K \subseteq L {}_{\odot}\backslash\, (N {}_{\odot}\backslash\, K)$.

($\subseteq$) The inverse inclusion in 1) is obtained by the definition of right quotient and associativity of ω-coproduct:

$$K \subseteq (N \odot L) \odot [(N \odot L) {}_{\odot}\backslash\, K] = N \odot [L \odot ((N \odot L) {}_{\odot}\backslash\, K)].$$

Applying Proposition 3.1 we have $N {}_{\odot}\backslash\, K \subseteq L \odot [(N \odot L) {}_{\odot}\backslash\, K]$ and $L {}_{\odot}\backslash\, (N {}_{\odot}\backslash\, K) \subseteq (N \odot L) {}_{\odot}\backslash\, K$.

2) By definition, $N \subseteq L \odot (L {}_{\odot}\backslash\, N)$. From the monotony and associativity of ω-coproduct it follows:

$$K \subseteq N \odot (N {}_{\odot}\backslash\, K) \subseteq [L \odot (L {}_{\odot}\backslash\, N)] \odot (N {}_{\odot}\backslash\, K) =$$
$$= L \odot [(L {}_{\odot}\backslash\, N) \odot (N {}_{\odot}\backslash\, K)].$$

By Proposition 3.1 $L {}_{\odot}\backslash\, K \subseteq (L {}_{\odot}\backslash\, N) \odot (N {}_{\odot}\backslash\, K)$, therefore $(L {}_{\odot}\backslash\, N) {}_{\odot}\backslash\, (L {}_{\odot}\backslash\, K) \subseteq N {}_{\odot}\backslash\, K$.

3) By definition, $K \subseteq N \odot (N {}_{\odot}\backslash\, K)$. From the monotony and associativity of ω-coproduct it follows:

$$L \odot K \subseteq L \odot [N \odot (N {}_{\odot}\backslash\, K)] = (L \odot N) \odot (N {}_{\odot}\backslash\, K),$$

therefore $(L \odot N) {}_{\odot}\backslash\, (L \odot K) \subseteq N {}_{\odot}\backslash\, K$.

4) In a similar way we have $N \subseteq L \odot (L {}_{\odot}\backslash\, N)$ and

$$N \odot K \subseteq [L \odot (L {}_{\odot}\backslash\, N)] \odot K = L \odot [(L {}_{\odot}\backslash\, N) \odot K],$$

therefore $L {}_{\odot}\backslash\, (N \odot K) \subseteq (L {}_{\odot}\backslash\, N) \odot K$. □

Now we will indicate some relations between the right quotient with respect to ω-coproduct and the lattice operations of $\boldsymbol{L}({}_R M)$.

Proposition 3.10. *For every* $N \in \boldsymbol{L}({}_R M)$ *and every family of submodules* $\{K_\alpha \in \boldsymbol{L}({}_R M) \mid \alpha \in \mathfrak{A}\}$ *the following relation holds:*

$$N {}_{\odot}\backslash\, \Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha \Big) = \sum_{\alpha \in \mathfrak{A}} (N {}_{\odot}\backslash\, K_\alpha).$$

Proof. ($\supseteq$) It follows from the monotony of right quotient in the numerator (Proposition 3.5).

($\subseteq$) We denote $N_0 = \sum_{\alpha \in \mathfrak{A}} (N {}_{\odot}\backslash\, K_\alpha)$. Since $N {}_{\odot}\backslash\, K_\alpha \subseteq N_0$ for every $\alpha \in \mathfrak{A}$, by the definition and monotony we have:

$$K_\alpha \subseteq N \odot (N {}_{\odot}\backslash\, K_\alpha) \subseteq N \odot N_0$$

for every $\alpha \in \mathfrak{A}$. Therefore $\sum\limits_{\alpha\in\mathfrak{A}} K_\alpha \subseteq N \odot N_0$ and from Proposition 3.1 it follows that $N \,{}_{\odot}\!\setminus \Big(\sum\limits_{\alpha\in\mathfrak{A}} K_\alpha \Big) \subseteq N_0$. □

Corollary 3.11. $N \,{}_{\odot}\!\setminus K = N \,{}_{\odot}\!\setminus (K+N)$ *for every* $N, K \in \boldsymbol{L}({}_RM)$.

Proof. Since $N \,{}_{\odot}\!\setminus N = 0$ (Proposition 3.8, 1)), from Proposition 3.10 it follows:

$$N \,{}_{\odot}\!\setminus (K+N) = (N \,{}_{\odot}\!\setminus K) + (N \,{}_{\odot}\!\setminus N) = N \,{}_{\odot}\!\setminus K. \qquad \square$$

We formulate also some more relations between the right quotient and the lattice operations of $\boldsymbol{L}({}_RM)$, which immediately follow from the properties of monotony and antimonotony of Propositions 3.5 and 3.6.

Proposition 3.12. *In the lattice* $\boldsymbol{L}({}_RM)$ *the following relations hold:*

1) $N \,{}_{\odot}\!\setminus \big(\bigcap\limits_{\alpha\in\mathfrak{A}} K_\alpha \big) \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} (N \,{}_{\odot}\!\setminus K_\alpha)$;
2) $\big(\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha \big) \,{}_{\odot}\!\setminus K \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} (N_\alpha \,{}_{\odot}\!\setminus K)$;
3) $\big(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \big) \,{}_{\odot}\!\setminus K \supseteq \sum\limits_{\alpha\in\mathfrak{A}} (N_\alpha \,{}_{\odot}\!\setminus K)$. □

In the next two statements it is shown when the cancellation properties hold (see Proposition 3.7).

Proposition 3.13. *Let* $K, N \in \boldsymbol{L}({}_RM)$. *The following conditions are equivalent:*

1) $N = K \,{}_{\odot}\!\setminus (K \odot N)$;
2) $N = K \,{}_{\odot}\!\setminus L$ *for some submodule* $L \in \boldsymbol{L}({}_RM)$.

Proof. 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). Let $N = K \,{}_{\odot}\!\setminus L$, where $L \in \boldsymbol{L}({}_RM)$. By definition and monotony we have $K \odot (K \,{}_{\odot}\!\setminus L) \supseteq L$ and

$$K \,{}_{\odot}\!\setminus [K \odot (K \,{}_{\odot}\!\setminus L)] \supseteq K \,{}_{\odot}\!\setminus L.$$

From Proposition 3.7 the inverse inclusion follows and so we obtain:

$$N = K \,{}_{\odot}\!\setminus L = K \,{}_{\odot}\!\setminus [K \odot (K \,{}_{\odot}\!\setminus L)] = K \,{}_{\odot}\!\setminus (K \odot N). \qquad \square$$

Proposition 3.14. *Let* $K, N \in \boldsymbol{L}({}_RM)$. *The following conditions are equivalent:*

1) $N = K \odot (K \,{}_{\odot}\!\setminus N)$;
2) $N = K \odot L$ *for some submodule* $L \in \boldsymbol{L}({}_RM)$.

Proof. 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). Let $N = K \odot L$, where $L \in \boldsymbol{L}({}_RM)$. From Proposition 3.7 it follows that $K \,{}_{\odot}\!\backslash\, (K \odot L) \subseteq L$ and by monotony

$$K \odot [K \,{}_{\odot}\!\backslash\, (K \odot L)] \subseteq K \odot L.$$

On the other hand, from the definition the inverse inclusion follows, therefore:

$$N = K \odot L = K \odot [K \,{}_{\odot}\!\backslash\, (K \odot L)] = K \odot (K \,{}_{\odot}\!\backslash\, N). \qquad \square$$

Finally, we consider the case ${}_RM = {}_RR$ and show the form of the right quotient $N \,{}_{\odot}\!\backslash\, K$ for the left ideals of the ring R. It is known (Proposition 1.2, 9)) that for every $N, K \in \boldsymbol{L}({}_RR)$ the ω-coproduct of these left ideals is of the form:

$$N \odot K = \big(K : (0 : N)_r\big)_l = \{a \in R \mid a\, b \in K \;\; \forall\, b \in R,\; N\, b = 0\}.$$

Proposition 3.15. *$N \,{}_{\odot}\!\backslash\, K = K \cdot (0 : N)_r$, for every left ideals $K, N \in \boldsymbol{L}({}_RR)$, where $(0 : N)_r = \{b \in R \mid N\, b = 0\}$.*

Proof. Denote $L = K \cdot (0 : N)_r$, and verify the conditions a) and b) of Definition 3.1.

a) Since $N \odot L = \big(L : (0 : N)_r\big)_l$, we have:

$$N \odot L = N \odot [K \cdot (0 : N)_r] = \big([K \cdot (0 : N)_r] : (0 : N)_r\big)_l \supseteq K.$$

b) If $N \odot L_0 \supseteq K$, then $\big(L_0 : (0 : N)_r\big)_l \supseteq K$, therefore $K \cdot (0 : N)_r \subseteq L_0$, i.e. $L \subseteq L_0$. $\square$

This form of the right quotient in $\boldsymbol{L}({}_RR)$ is convenient for proving a series of properties of this operation. For example (see Proposition 3.10):

$$N \,{}_{\odot}\!\backslash\, \Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big) = \Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big) \cdot (0 : N)_r = \sum_{\alpha \in \mathfrak{A}} \big(K_\alpha \cdot (0 : N)_r\big) = \sum_{\alpha \in \mathfrak{A}} (N \,{}_{\odot}\!\backslash\, K_\alpha).$$

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