A linear algorithm of checking
of the graph connectness

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ABSTRACT. An algorithm of sorting of all simply laced graph
such that subalgorithm of checking of that the graph is connected
is of linear dependence on the number of vertices of the graph.

1. Introduction

The problem of checking the fact that given graph is connected is
at least so actual as studying of connected graphs is actual itself. This
problem naturally appears during solving problems which lead to sorting
of all the graphs and considering of those which are connected. So, in
[5] there is a list of all quivers with not more then 4 vertices, and it is
the illustration of the actuality of the problem of their description and of
considering problems of their sorting and properties. Problems which in
fact are reduced to sorting of matrices of strongly connected quivers are
considered also in [2] -[4].

Without loosing the generality assume that the graph is simply laced,
i.e. it does not have loops and multiple edges.

For convenience of further computations assume that vertices of the
graph are numerated and so we may identify the graph with its adjacency
matrix. Till the end of the work assume that $A = (\alpha_{ij})$ is adjacency matrix
of the graph under consideration and n is the number of its vertices.

Connected quivers are interesting because their adjacency matrix satisfies the condition of Frobenius theorem [1, p. 355], which describes the canonical form of nonnegative matrix.

From the other hand for checking the fact is connected the non oriented graph with given adjacency matrix there is known algorithm which have quadratic speed.

Remind this algorithm. Its idea is to check is it correct that it is possible to start from the vertex number 1 and to come to any other vertex of the graph.

Algorithm 1. *1. Consider an array M, and input into this array all the vertices of graph such that there are arrows which go into these vertices (i.e. input into the array numbers which are equal to numbers of those vertices for which some arrows go into them).*

2. Consider the following iteration process. During each iteration we will add to the array M all vertices, which are conjugate with some vertex which is in the array.

3. After making the iteration step we will check if we have added any new vertex to the array M. Is yes then come to the next iteration, if no then check are all the vertices of the graph belong to M.

4. If it will appear that all vertices of the graph belong to M then make conclusion that graph is connected, otherwise it is not connected.

Example 1. Consider the graph with n vertices as below

$$\bullet \!\!-\!\!-\!\!-\!\! \bullet \!\!-\!\!-\!\!-\!\! \ldots \!\!-\!\!-\!\!-\!\! \bullet \!\!-\!\!-\!\!-\!\! \bullet \, .$$

Calculate the necessary number of operations for answering the question on that this graph connected or not.

At the very beginning there is only number 2 in the array M. During each iteration there is exactly one number (one vertex) which is added to M and so it is necessary to do $n-1$ iterations.

As each iteration needs at least n memory accesses to matrix A, then all the algorithm needs at least n^2 memory accesses to matrix A.

That is why the speed of the algorithm is at least quadratic.

The main result of this work is proposing of such algorithm of

The main result of the work is proposing of such algorithm of sorting of non isomorphic graphs such that its subalgorithm of checking the their connectness is of linear speed.

Note that renumbering of graph vertices, specified with the permutation σ corresponds to permutating of lines and columns of the same matrix,

which is defined with the same permutation σ. The matrix which is obtained in such a way we will call the result of **an action** of permutation of the permutation σ on the former matrix.

It is known that action of the permutation on the matrix saves a lot of its properties such as characteristic polynomial and spectra.

2. Preliminary notes

Describe how we understand the algorithm of checking that graph which is given by its adjacency matrix is connected.

Under the matrix $A = (\alpha_{ij})$ with dimension n we understand an array of numbers such that for each its element are defined such actions as checking of equality (inequality) of this element and zero, going to neighbour element in the the same line or in in the same column etc. During the formulation of ideas of this article we will let us such word combinations as "move in the line" or "move in the row". The meaning of these word combinations is successive increase of the variable which notes the number of a line or a row of an element of the matrix which is under consideration.

We do not pay attention to the fact that the start of consideration of the element $\alpha_{i,j+1}$ after the element $\alpha_{i,j}$ needs less time then start of consideration of the element $\alpha_{i+1,j}$ after the element $\alpha_{i,j}$. The heart of the note is the following. Traditionally $n \times n$ matrix in programming is considered as a table with n lines and n columns. But in fact the matrix is an array (a vector) $V = (v_i)$ with n^2 elements and matrix element α_{ij} is the same as vector element $v_{n(i-1)+j}$. Of course, matrix elements $\alpha_{i,j}$ and $\alpha_{i,j+1}$ correspond to neighbour vector V elements $v_{n(i-1)+j}$ and $v_{n(i-1)+j+1}$ and in the same time analogous statement is wrong for matrix A elements $\alpha_{i,j}$ and $\alpha_{i+1,j}$. Note ones more that we do not consider and do not appreciate the mentioned difference and assume that time for consideration of some element of the matrix is independent of the elements which was under consideration before this.

Also the connectness of the graph is connected with the notion of decomposability of the matrix, which was presented in [1, p. 352]. Remind that matrix A in called decomposable if there exist a permutation σ such that the result A' of action of this permutation on the matrix A is of the form

$$A' = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}, \tag{1}$$

where B and D - are square matrices.

There is the obvious property of an adjacency matrix of a connected graph.

Lemma 1. *Graph is connected if and only if its adjacency matrix is undecomposable.*

This lemma is an immediate corollary of the definition of a connected graph and the definition of adjacency matrix.

3. Canonical form and almost canonical form of the matrix

Somehow or other but the setting of the problem which is under consideration in the work is the following. One needs to sort all non isomorphic quivers and find the answer some question (for example the number of edges, index etc.), which has algorithmically solution.

The idea of solution is to propose the canonical form of the matrix in the class of all conjugate matrices of isomorphic graphs and so consider only matrices of canonical form instead of considering matrices of all graphs.

Definition 1. Call the adjacency matrix of the graph one of **canonical form**, if it is maximal in the cense of lexicographical ordering in the set of those matrices which correspond to the same isomorphic graphs set.

Note that matrices from mentioned set can be obtained one from another by permuting of lines and columns.

Definition 2. The rectangle block of almost canonical form is such rectangle submatrix B of matrix A which is consisted from some first lines of square matrix A and satisfy the following inductive definition:

1. A line whose the first element is 0 and which contains 1 after 0 only one time is a rectangle block of almost canonical form.

2. Let some number of first lines of the matrix A is a rectangle block of almost canonical form. The rectangle matrix B_1 which is constructed by adding of one line of A to this block is a rectangle block of almost canonical form if it can not be increased in the cense of lexicographical order by acting on the matrix A with some permutation σ such that do not move lines from B_1.

Definition 3. Matrix is of the **almost canonical form**, if it is a rectangle block of almost canonical form as its submatrix.

Note that there is and unique matrix of canonical form in the those matrices which can be obtained from some given matrix with acing of all permutations. Nevertheless there can be many matrices of almost canonical form. Matrix of canonical form can be characterized with the following lemma.

Lemma 2. *The adjacency matrix A of the graph is of almost canonical from if and only if it does not contain submatrices of the form*

$$F = \begin{pmatrix} \boldsymbol{x} & \boldsymbol{x} \\ 0 & 1 \end{pmatrix}, \tag{2}$$

above its diagonal, where $\boldsymbol{x}$ is a column which starts with the first line of A all whose elements are equal to $x \in \{0, 1\}$. Also left lower 0 of F is not diagonal element of A.

Note that columns **x** in the formulation can not be just numbers.

Example 2. Give the example of matrix matrix in almost canonical form which contains the detailed submatrix of the form (2), is **x** is a number.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & \boxed{1 & 1} \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

It is obvious that matrix is of the almost canonical form.

During the solution of the given problem on the matrices ordering consider only matrices of canonical form with ordering them in lexicographic order. For each of them check its connectness.

The almost canonical form of the adjacency matrix has the following important property.

Theorem 1. *If conjugate matrix of the graph is of almost canonical from then graph is connected if and only if matrix is not of the form (1).*

Proof. The sufficient of theorem condition for the non connectness of the graph is obvious.
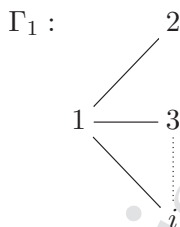
In fact the theorem states that if graph is not connected and its adjacency matrix is in almost canonical form them it is in the form (1).

As matrix A is is almost canonical form then its the first line is of the form

$$(0\,1\ldots 10\ldots 0).$$

Let i - be the number of a vertex which corresponds to the last number 1 in a line. As graph is not directed then make conclusion about possibility to connect the first vertex with all others which have numbers $2,\ldots,i$. Correspond path looks like $1 \to 2 \to 1 \to 3 \to 1 \to \ldots \to i$.

Existence of such path is equal to existence of the subgraph

Answer the question on possibility to connect the graph Γ_1 with the vertex number $i+1$.

So, the first two lines of matrix A are as following

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ * & 0 & 1 & \ldots & 1 & 0 & \ldots & 0 & ? & & \end{pmatrix},$$

where question-mark means an element $\alpha_{2\,i+1}$.

As the matrix A is of almost canonical form then for its the second line one have the following proposition. If $\alpha_{2\,i+1} = 0$ then $\alpha_{2\,k} = 0$ for all $k > i$. If it is not so (i.e. $\alpha_{2\,i+1} = 0$, but $\alpha_{2\,s} = 1$ for some s), then one can act with permutation $\tau = (i + 1, s)$ on the matrix A. After this the first line will not be changed but second line will increase in the cense of lexical ordering which contradicts to that A is in almost canonical form.

The analogous reasoning are correct for each other line of A for which $\alpha_{p\,i+1} = 0$ till this condition holds. So, consider two cases - if such p exists and if not.

It does not exist such p that $\alpha_{p\,i+1} = 0$ then matrix A is of the form (1), where B is a square matrix of dimension i, and D is square matrix of dimension $n - i$.
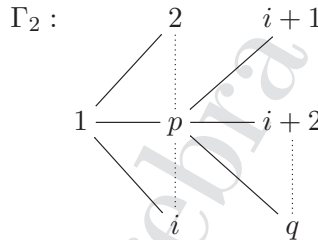
That is why consider the case when there exists a number p such that $\alpha_{p\,i+1} = 1$ and let p be the lowest number with this property (this means that $p \leq i$). In this case the matrix will look like as follows, where bold

font in separate cell is an element $\alpha_{p\,i+1}$, which is equal to 1.

$$
\begin{pmatrix}
0 & \overline{1 & \cdots & & 1} & 0 & \cdots & 0 \\
 & 0 & & & \vdots & \vdots & \cdots & \vdots \\
 & & \ddots & & & 0 & \cdots & 0 \\
 & & & 0 & & \boxed{1} & & \\
 & & & & 0 & \vdots & &
\end{pmatrix} .
$$

Like as during studying of the properties of the second column of the matrix and paying attention to the almost canonical form of the matrix make a conclusion that the line number s have the following property. After an element $\alpha_{p\,i+1}$ there is a series of 1-s and all other elements of the line number $i+1$ are equal to 0.

Let q be a maximal number such that $\alpha_{p\,q} = 1$. It is obvious that $q \geq i+1$. We have that the quiver, which is under consideration, contains a connected subquiver
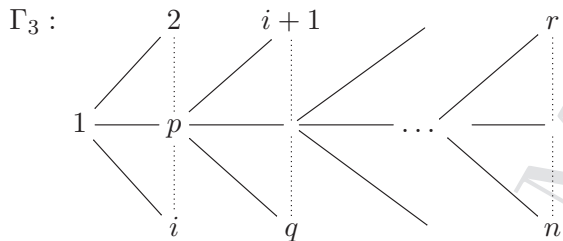

$$\Gamma_2 :$$

The existence of subquiver Γ_2 means the following form of some the first line of A

$$
\begin{pmatrix}
0 & \overline{1 & \cdots & & 1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & 0 & & & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 \\
 & & \ddots & & & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & & & 0 & & \boxed{1 & \cdots & 1} & 0 & \cdots & 0 \\
 & & & & 0 & \vdots & & & \boxed{?} &
\end{pmatrix} .
$$

With repeating if necessary (may be more then once) the same as we have made for $\alpha_{2\,i+1}$ for the element $\alpha_{p+1\,q+1}$ obtain either a graph Γ_3 as

follows

$\Gamma_3:$

or make a conclusion that matrix A is of the form (1).

The last finishes the proof. $\square$

4. The algorithm of checking the connectness

A theorem 1 which is proved (especially its proving) lets us to formulate the algorithm of checking the connectness of the graph and this algorithm will de linearly dependent on n.

We will find the path from the first vertex to the last one (n-th) which goes through all the vertices.

Algorithm 2. *1. Go through the first line of the matrix A and find the first element $\alpha_{1\,j}$ such that $\alpha_{1\,j} = 1$ and $\alpha_{1\,j+1} = 0$. If such element does not exist and $\alpha_{1n} = 1$, then graph is connected. If the necessary element does not exist and $\alpha_{1n} = 0$, then graph is not connected. Is the element α_{1j} with the property mentioned above is found then go to the next step.*

2. After finding the element which we have called α_{1j}, find the first nonzero element in the column number $j + 1$. If we have found it (call $\alpha_{k,\,j+1}$), the go through the block of ones in the line number k, find there the last 1 and repeat all the actions which we have done for α_{1j}.

3. This process will finished with either coming to the last column or with the conclusion that graph is connected. It we will not come to the last column of the matrix then conclude that graph is not connected.

As answering the question about being/unbeing equal to zero of some elements of the matrix A we mowed only right and down, then we have made not more then $2n$ checkings which gives that algorithm of checking the connectness of the graph is not quadratic but not linear.

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